On the structure of uninorms on L^*

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Abstract

Uninorms are an important generalization of triangular norms and triangular conorms. Uninorms allow the neutral element to lie anywhere in the unit interval rather than at zero or one as in the case of a t-norm and a t-conorm.

Since interval valued fuzzy sets, bifuzzy sets (intuitionistic fuzzy sets) and L^*-fuzzy sets are equivalent, therefore in this paper we describe a generalization of uninorms on L^*. For example, we discuss the possible values of the zero element for uninorms and of the neutral element for t-representable uninorms.

Keywords: Intuitionistic fuzzy sets, L-fuzzy sets, interval valued fuzzy sets, t-norms, uninorms.

1 Introduction

The triangular norms and conorms play an important role in the fuzzy set theory. They are used for the generalization of intersection and union of fuzzy sets, for extension the composition of fuzzy relations and for many other concepts.

Uninorms are important generalizations of triangular norms and triangular conorms. Uninorms allow a neutral element to lie anywhere in the unit interval rather than at zero or one as in the case of a t-norm and a t-conorm.

Bifuzzy sets are generalization of fuzzy sets. Because of this we consider a generalization of uninorm on the field of bifuzzy set theory.

In Section 2, we put the definition of a fuzzy set, a bifuzzy set (an intuitionistic fuzzy set), an interval valued fuzzy set and L-fuzzy set. Next, we recall the relationship between them. In Sections 3, we recall the properties of uninorms in $[0,1]$. In section 4, the definition and properties of uninorms on L^* are given. Additionally, we put the description of t-representable uninorms and we discuss the possible values of the neutral element and zero element for these uninorms.

2 Basic definition

First we put the basic definition

Definition 1 (cf. [1])**.** A bifuzzy set (an intuitionistic fuzzy set) A in a universe X is a triple

$$A = \{(x, \mu(x), \nu(x)) : x \in X\}$$

where $\mu, \nu : X \to [0,1]$ and $\mu(x) + \nu(x) \leq 1$, $x \in X$.

We use the name bifuzzy set instead of the intuitionistic fuzzy set, because there is no terminological difficulties with this name (cf. [11]) and in fact, a bifuzzy set is described by two fuzzy sets μ and ν.

Definition 2 (cf. [6])**.** An interval valued fuzzy set A in a universe X is a mapping $A : X \to Int([0,1])$, where $Int([0,1])$ denotes the set of all closed subintervals of $[0,1]$, i.e. a mapping which assigns to each element $x \in X$

L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, pp. 1795–1800
Torremolinos (Málaga), June 22–27, 2008

the interval $[\underline{A}(x), \overline{A}(x)]$, where $\underline{A}(x), \overline{A}(x) \in [0,1]$ such that $\underline{A}(x) \leq \overline{A}(x)$.

Definition 3 ([10])**.** An L-fuzzy set A in a universe X is a function $A : X \to L$ where L is a lattice.

It was shown in [4] that bifuzzy sets, interval valued fuzzy sets and L^*-fuzzy sets are equivalent, where

$$L^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$ for all $(x_1, x_2), (y_1, y_2) \in L^*$.

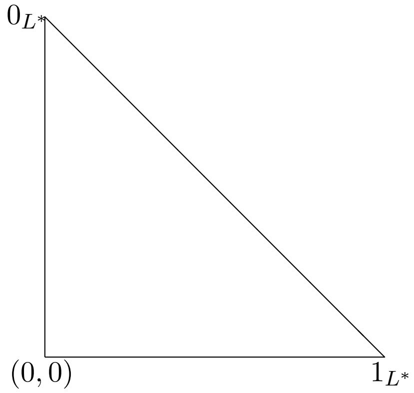

Figure 1: Lattice L^*

Remark 1. *The greatest element in* L^* *is* $1_{L^*} = (1, 0)$. *The least element in* L^* *is* $0_{L^*} = (0, 1)$.

Definition 4 ([12])**.** A triangular norm T is an increasing, commutative, associative operation $T : [0,1]^2 \to [0,1]$ with a neutral element 1.
A triangular conorm S is an increasing, commutative, associative operation $S : [0,1]^2 \to [0,1]$ with a neutral element 0.

Definition 5 ([5])**.** A triangular norm $\mathcal{T}$ on L^* is an increasing, commutative, associative operation $\mathcal{T} : (L^*)^2 \to L^*$ with a neutral element 1_{L^*}.
A triangular conorm $\mathcal{S}$ on L^* is an increasing, commutative, associative operation $\mathcal{S} : (L^*)^2 \to L^*$ with a neutral element 0_{L^*}.

Definition 6 ([5])**.** A t-norm $\mathcal{T}$ on L^* is called t-representable if there exist a t-norm T and t-conorm S such that for all $x, y \in L^*$

$$\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2)).$$

Example 1. The operation

$$\mathcal{T} = (x_1 y_1, \min(1, x_2 + y_2))$$

is a t-representable t-norm with the product t-norm T and the Łukasiewicz t-conorm S.

Example 2. The Łukasiewicz t-norm $\mathcal{T}_W = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1))$ is not t-representable.

3 Uninorms

In this section we put the definition of a uninorms in $[0,1]$ and some properties of these operations.

Definition 7 ([14])**.** Operation $U : [0,1]^2 \to [0,1]$ is called a uninorm if it is commutative, associative, increasing and has a neutral element $e \in [0,1]$.

Theorem 1 ([9])**.** *If a uninorm* U *has a neutral element* $e \in (0,1)$*, then there exist a triangular norm* T *and a triangular conorm* S *such that*

$$U(x, y) = \begin{cases} T^*(x, y) & \text{if} \quad x, y \leq e, \\ S^*(x, y) & \text{if} \quad x, y \geq e, \end{cases} \tag{1}$$

where
$T^*(x, y) = \varphi^{-1}(T(\varphi(x), \varphi(y)))$,
$\varphi(x) = x/e$, $x, y \in [0, e]$,
$S^*(x, y) = \psi^{-1}(S(\psi(x), \psi(y)))$,
$\psi(x) = (x - e)/(1 - e)$, $x, y \in [e, 1]$.

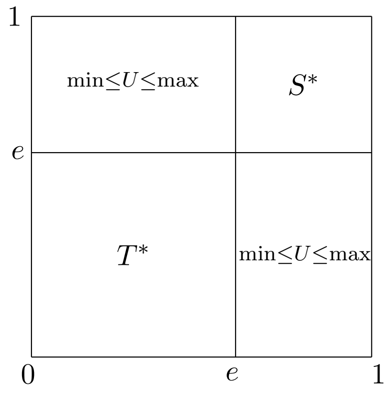

Figure 2: Structure of uninorms

Lemma 1 (cf.[9])**.** *If* U *is a uninorm with a neutral element* $e \in (0,1)$ *then for all* $x, y \in [0,1]$ *such that* $\min(x, y) \leq x, y \leq \max(x, y)$ *one has* $\min(x, y) \leq U(x, y) \leq \max(x, y)$. *Furthermore,* $U(0,1) \in \{0, 1\}$ *and* $U(0,1)$ *is the zero element of the operation* U.

If a uninorm is idempotent ($U(x,x) = x$ for every $x \in [0,1]$) then we have the following representation

Theorem 2 (c.f. [13])**.** *Let $e \in [0,1]$. Operation U is an idempotent uninorm with a neutral element e iff there exists a decreasing function $g : [0,1] \to [0,1]$ with $g(e) = e$, $g(x) = 0$ for all $x > g(0)$, $g(x) = 1$ for all $x < g(1)$, satisfying for $x \in [0,1]$*

$$\begin{aligned} inf\{y : g(y) = g(x)\} &\leq g^2(x) \\ &\leq \sup\{y : g(y) = g(x)\} \end{aligned} \tag{2}$$

and such that for all $x, y \in [0,1]$

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } \; y < g(x) \text{ or} \\ & (y = g(x) \text{ and} \\ & x < g^2(x)\ ) \\ \max(x,y) & \text{if } \; y > g(x) \text{ or} \\ & (y = g(x) \text{ and} \\ & x > g^2(x)\ ) \\ x \text{ or } y & \text{if } \; y = g(x) \text{ and} \\ & x = g^2(x) \end{cases} \tag{3}$$

and U is commutative for all points $(x, g(x))$ such that $x = g^2(x)$.

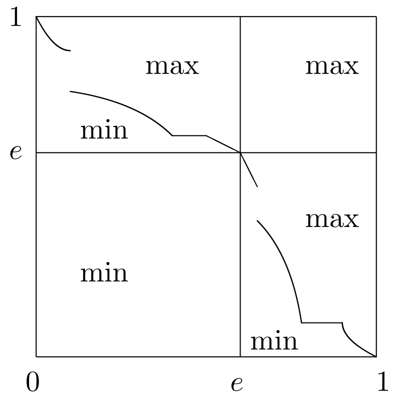

Figure 3: Example of idempotent uninorm and function g.

4 Uninorms on L^*

In this section we present results relating to uninorms on the lattice L^*.

Definition 8 ([6])**.** Operation $\mathcal{U} : (L^*)^2 \to L^*$ is called a uninorm if it is commutative, associative, increasing and has a neutral element $e \in L^*$.

In order to obtain similar results to the ones for uninorms in $[0,1]$ we divide lattice into a few parts and show connections between them. First, we define the following sets

$$E_e = \{x \in L^* : x \leq e\},$$

$$E'_e = \{x \in L^* : x \geq e\},$$

$$D = \{x \in L^* : x_1 + x_2 = 1\}.$$

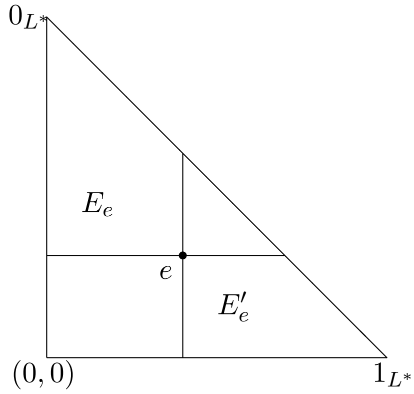

Figure 4: Lattice L^* and sets E_e and E'_e

Theorem 3 ([6])**.** *Let $e \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$. If $e \notin D$, then there does not exist an increasing bijection Φ_e from L^* to E_e such that Φ_e^{-1} is increasing and there does not exist an increasing bijection Ψ_e from L^* to E'_e such that Ψ_e^{-1} is increasing.*

Lemma 2 ([6])**.** *Let $e \in D \setminus \{0_{L^*}, 1_{L^*}\}$. The mapping $\Phi_e : L^* \to L^*$ defined by $\Phi_e(x) = (e_1x_1, 1 - e_1(1 - x_2))$ for all $x \in L^*$ is an increasing bijection from L^* to E_e such that Φ_e^{-1} is increasing.*
The mapping $\Psi_e : L^* \to L^*$ defined by $\Psi_e(x) = (e_1 + x_1 - e_1x_1, (1 - e_1)x_2)$ for all $x \in L^*$ is an increasing bijection from L^* to E'_e such that Ψ_e^{-1} is increasing.

Theorem 4 ([6])**.** *If a uninorm $\mathcal{U}$ has the neutral element $e \in D \setminus \{0_{L^*}, 1_{L^*}\}$, then there exist a t-norm $\mathcal{T}$ and a t-conorm $\mathcal{S}$ such that*

$$\mathcal{U}(x,y) = \begin{cases} \mathcal{T}^*(x,y) \text{ if } x, y \leq e, \\ \mathcal{S}^*(x,y) \text{ if } x, y \geq e, \end{cases} \tag{4}$$

where

$$\begin{cases} \mathcal{T}^*(x,y) = & \Phi_e\left(\mathcal{T}\left(\Phi_e^{-1}(x), \Phi_e^{-1}(y)\right)\right), \\ & x, y \in E_e \\ \mathcal{S}^*(x,y) = & \Psi_e\left(\mathcal{S}\left(\Psi_e^{-1}(x), \Psi_e^{-1}(y)\right)\right), \\ & x, y \in E'_e \end{cases} . \tag{5}$$

Lemma 3. *If $\mathcal{U}$ is a uninorm with a neutral element $e \in L^*$, then for all $x, y \in L^*$ such that $x \leq e \leq y$ one has*

$$x \leq \mathcal{U}(x, y) \leq y.$$

Proof. Let $x, y \in L^*$ and $x \leq e \leq y$. Thus $x = \mathcal{U}(x, e) \leq \mathcal{U}(x, y) \leq \mathcal{U}(e, y) = y$. □

Directly from the above results we obtain

Lemma 4. *If $\mathcal{U}$ is a uninorm with a neutral element $e \in L^*$, then for all $x, y \in L^*$ such that $x \leq e \leq y$ or $y \leq e \leq x$ one has*

$$\min(x, y) \leq \mathcal{U}(x, y) \leq \max(x, y).$$

Remark 2. *The analogous lemma to Lemma 1 for uninorms in L^* does not exist, i.e. $\min(x, y) \leq \mathcal{U}(x, y) \leq \max(x, y)$ does not hold for all x, y such that $\min(x, y) \leq e \leq \max(x, y)$ but it holds only for all x, y such that $x \leq e \leq y$ or $y \leq e \leq x$.*

Lemma 5 ([6])**.** *If $\mathcal{U}$ is a uninorm with a neutral element $e \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, then for all $x \in L^*$ one has*

$$\mathcal{U}(0_{L^*}, 1_{L^*}) = \mathcal{U}(\mathcal{U}(0_{L^*}, 1_{L^*}), x).$$

Lemma 6 ([6])**.** *If $\mathcal{U}$ is a uninorm with a neutral element $e \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ then $\mathcal{U}(0_{L^*}, 1_{L^*}) = 0_{L^*}$ or $\mathcal{U}(0_{L^*}, 1_{L^*}) = 1_{L^*}$ or $\mathcal{U}(0_{L^*}, 1_{L^*}) \| e$ (i.e. elements $\mathcal{U}(0_{L^*}, 1_{L^*})$ and e are not comparable).*

If $\mathcal{U}(0_{L^*}, 1_{L^*}) = 0_{L^*}$, then $\mathcal{U}$ is called a conjunctive uninorm on L^*. If $\mathcal{U}(0_{L^*}, 1_{L^*}) = 1_{L^*}$, then $\mathcal{U}$ is called a disjunctive uninorm on L^*.

Example 3. Let U_{e_1} be the uninorm given by

$$U_{e_1}(x, y) = \begin{cases} \max(x, y) & \text{if } x, y \in [e_1, 1], \\ \min(x, y) & \text{else,} \end{cases}$$

then for the uninorm

$$\mathcal{U}(x, y) = (U_{e_1}(x_1, y_1), U_{1-e_1}(x_2, y_2))$$

one has $\mathcal{U}(0_{L^*}, 1_{L^*}) = (0, 0)$ and $\mathcal{U}$ is neither conjunctive nor disjunctive.

Definition 9 ([6])**.** A uninorm $\mathcal{U}$ is called t-representable if there exist uninorms U_1 and U_2 such that for all $x, y \in L^*$

$$\mathcal{U}(x, y) = (U_1(x_1, y_1), U_2(x_2, y_2)).$$

Example 4. The uninorm from Example 3 is a t-representable uninorm. Let U be an arbitrary uninorm. Operation $\mathcal{U}(x, y) = (\min(U(x_1, 1-y_2), U(y_1, 1-x_2)), U(1-x_2, 1-y_2))$ is not t-representable.

Theorem 5 ([6])**.** *If $\mathcal{U}$ is a t-representable uninorm with a neutral element $e \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, then e is of the form $e = (e_1, e_2)$ where e_1 is a neutral element of the uninorm U_1 and e_2 is a neutral element of the uninorm U_2.*

Lemma 7. *If $\mathcal{U}$ is a t-representable uninorm with a neutral element $e \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$, then $\mathcal{U}(0_{L^*}, 1_{L^*}) = 0_{L^*}$ or $\mathcal{U}(0_{L^*}, 1_{L^*}) = 1_{L^*}$ or $\mathcal{U}(0_{L^*}, 1_{L^*}) = (0, 0)$.*

Proof. If $\mathcal{U}$ is a t-representable uninorm, then there exist uninorms U_1 and U_2 such that, for all $x, y \in L^*$ $\mathcal{U}(x, y) = (U_1(x_1, y_1), U_2(x_2, y_2))$. Thus, we have four possibilities:
(*a*) $U_1(0, 1) = 0$, $U_2(0, 1) = 0$, then $\mathcal{U}(0_{L^*}, 1_{L^*}) = (U_1(0, 1), U_2(1, 0)) = (0, 0)$
(*b*) $U_1(0, 1) = 0$, $U_2(0, 1) = 1$, then $\mathcal{U}(0_{L^*}, 1_{L^*}) = (U_1(0, 1), U_2(1, 0)) = (0, 1) = 0_{L^*}$
(*c*) $U_1(0, 1) = 1$, $U_2(0, 1) = 0$, then $\mathcal{U}(0_{L^*}, 1_{L^*}) = (U_1(0, 1), U_2(1, 0)) = (1, 0) = 1_{L^*}$
(*d*) $U_1(0, 1) = 1$, $U_2(0, 1) = 1$, then $\mathcal{U}(0_{L^*}, 1_{L^*}) = (U_1(0, 1), U_2(1, 0)) = (1, 1) \notin L^*$. As a result, this case can not hold. □

Corollary 1. We cannot use two disjunctive uninorms for construction of a t-representable uninorm.

The following open problem arises

Problem 1. Find a counter-example or prove the following:
If $\mathcal{U}$ is a uninorm with a neutral element $e \in L^*$, then $\mathcal{U}(0_{L^*}, 1_{L^*}) = 0_{L^*}$ or $\mathcal{U}(0_{L^*}, 1_{L^*}) = 1_{L^*}$ or $\mathcal{U}(0_{L^*}, 1_{L^*}) = (0, 0)$.

Lemma 8. *If $\mathcal{U}$ is a t-representable uninorm with a neutral element $e = (e_1, e_2)$ then $e_1 + e_2 \geq 1$.*

Proof. Let $\mathcal{U}$ be a t-representable uninorm with a neutral element e, then there exist uninorms U_1 and U_2 with neutral elements e_1, e_2 respectively, such that $e = (e_1, e_2)$ and $\mathcal{U}(x, y) = (U_1(x_1, y_1), U_2(x_2, y_2))$. Taking $x = (e_1, 1 - e_1)$, $y = (1 - e_2, e_2)$ one has $\mathcal{U}(x, y) = \mathcal{U}((e_1, 1 - e_1), (1 - e_2, e_2)) = (U_1(e_1, 1-e_2), U_2(1-e_1, e_2)) = (1-e_2, 1-e_1)$. Since $\mathcal{U}(x, y) \in L^*$, then $1 - e_2 + 1 - e_1 \leq 1$. Thus, we have $e_1 + e_2 \geq 1$. □

Theorem 6. *If $\mathcal{U}$ is a t-representable uninorm with a neutral element $e = (e_1, e_2)$, then $e \in D$.*

Proof. Directly from the previous lemma one has $e_1 + e_2 \geq 1$. Since $e = (e_1, e_2) \in L^*$, then $e_1 + e_2 \leq 1$ and consequently $e_1 + e_2 = 1$, i.e. $e \in D$. □

The following open problem arises

Problem 2. Find a counter-example or prove the following:
If $\mathcal{U}$ is a uninorm with a neutral element $e \in L^*$, then $e \in D$.

Definition 10. A uninorm $\mathcal{U}$ is idempotent if $\mathcal{U}(x, x) = x$ for all $x \in L^*$.

It is easy to see that

Theorem 7. *Let $\mathcal{U}$ be a t-representable uninorm given by $\mathcal{U}(x, y) = (U_1(x_1, y_1), U_2(x_2, y_2))$, where U_1 and U_2 are uninorm on $[0, 1]$. A uninorm $\mathcal{U}$ is idempotent iff U_1 and U_2 are idempotent uninorm.*

Example 5. Let

$$U(x, y) = \begin{cases} \min(x, y), & \text{if } x, y \in [0, \frac{1}{2}], \\ \max(x, y) & \text{elsewhere.} \end{cases}$$

Operation $\mathcal{U}(x, y) = (U(x_1, y_1), U(x_2, y_2))$ is not a uninorm.

The following open problem arises

Problem 3. Find the relationship between the functions g_1 and g_2 (c.f. Theorem 2) connected with idempotent uninorms U_1, U_2, which allow to obtain t-representable idempotent uninorm.

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