General form of M-probabilities on IF-events

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Abstract

In M-probability theory the additivity is considered with respect to the Gödel operations (maximum, minimum). The representation theorem is proved under the assumption that the M-probability depends on the integrals of the membership function and the nonmembership function.

Keywords: IF-events, t-norms, probability.

1 IF-events

Consider a classical probability space $(\Omega, \mathcal{S}, P)$. An IF-event is a pair $A = (\mu_A, \nu_A)$ of $\mathcal{S}$-measurable real functions $\mu_A, \nu_A : \Omega \to [0, 1]$ such that

$$\mu_A + \nu_A \leq 1.$$

There is a very suitable terminology: μ_A is called the membership function, ν_A the nonmembership function. If $f : \Omega \to [0, 1]$ is an $\mathcal{S}$-measurable fuzzy set, then the pair $(f, 1-f)$ is an IF-event, of course IF-events present a larger family.

Denote by $\mathcal{F}$ the family of all IF-events. There are many possibilities how to define a state $m : \mathcal{F} \to [0, 1]$. First in [3 - 5] the additivity was studied with respect to Lukasiewicz connectives. In [4] general form of states (with respect to the connectives) was presented and in [5] the theory was imbedded to the MV-algebra probability theory [7].

Of course, in [2] the Gödel connectives were introduced instead of Lukasiewicz ones. Some basic results in M-probability theory (using the Gödel connectives) has been summarized in [1] and [6]. In this communication we present a general form of M-states.

The Gödel connectives are defined in the following way:

$$\begin{aligned} A \vee B &= (\mu_A \vee \mu_B, \nu_A \wedge \nu_B), \\ A \wedge B &= (\mu_A \wedge \mu_B, \nu_A \vee \nu_B), \end{aligned}$$

where

$$f \vee g = \max(f, g), f \wedge g = \min(f, g)$$

Recall that

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

An additive M-state is a mapping $m : \mathcal{F} \to [0, 1]$ such that

(i) $m((0, 1)) = 0, m((1, 0)) = 1$,

(ii) $m(A) + m(B) = m(A \vee B) + m(A \wedge B)$ for any $A, B \in \mathcal{F}$.

An additive M-state is called an M-state, if it is continuous, i.e.

(iii) $A_n \nearrow A, B_n \searrow B \Longrightarrow$ $m(A_n) \nearrow m(A), m(B_n) \searrow m(B)$.

2 Representation theorem

Theorem. Let $m : \mathcal{F} \to [0, 1]$ be an additive state, $m(A) = f(\int_\Omega \mu_A dP, \int_\Omega \nu_A dP)$.

L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, pp. 1675–1677
Torremolinos (Málaga), June 22–27, 2008

Then there are functions

$$\varphi : [0,1] \to [0,1], \psi : [0,1] \to [0,1]$$

such that φ is non-decreasing, ψ is non-increasing,

$$\varphi(1) = \psi(0) = 1, \varphi(0) + \psi(1) = 1,$$

and

$$m(A) = \varphi(\int_\Omega \mu_A dP) + \psi(\int_\Omega \nu_A dP) - 1.$$

If $m : \mathcal{F} \to [0,1]$ is an M-state, then φ, ψ are continuous.

Example 1. Choose $\alpha \in [0,1]$ and put $\varphi(x) = (1-\alpha)x + \alpha, \psi(y) = -\alpha y + 1$. Then

$$m(A) = (1-\alpha)\int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP),$$

hence by [4] (see also [6]), any L-state is an M-state.

Example 2. Put $\varphi(x) = \frac{x^2}{2} + \frac{1}{2}, \psi(y) = 1 - \frac{y^2}{2}$. Then

$$m(A) = \frac{1}{2}(\int_\Omega \mu_A^2 dP + 1 - \int_\Omega \nu_A^2 dP).$$

The mapping $m : \mathcal{F} \to [0,1]$ is an example of an M-state that is not an L-state.

Proof of Theorem. First by the formula

$$\bar{m}((\mu_A, \nu_A)) = m((\mu_A, 0)) + m((0, \nu_A)) - m((0,0))$$

an extension $\bar{m} : \mathcal{M} \to [0,1]$ can be constructed, where

$\mathcal{M} = \{(\mu_A, \nu_A); \mu_A, \nu_A : \Omega \to [0,1], \mu_A, \nu_A$ are $\mathcal{S}$-measurable $\}$.

It is easy to see that $\bar{m}$ is an additive M-probability, and $\bar{m}$ is continuous, if m is continuous. Moreover, if $(\mu_A, \nu_A) \in \mathcal{F}$, then

$$(0,0) \vee (\mu_A, \nu_A) = (\mu_A \vee 0, \nu_A \wedge 0) = (\mu_A, 0),$$

$$(0,0) \wedge (\mu_A, \nu_A) = (\mu_A \wedge 0, \nu_A \vee 1) = (0, \nu_A),$$

hence

$$m((0,0)) + m((\mu_A, \nu_A)) = m((\mu_A, 0)) + m((0,0)),$$

and therefore

$$m((\mu_A, \nu_A)) = m((\mu_A, 0)) + m((0, \nu_A)) - m((0,0)) = \bar{m}((\mu_A, \nu_A)),$$

$\bar{m}$ is an extension of m. It is unique, because if s is any extension of m, then again

$$s((\mu_A, \nu_A)) = s((\mu_A, 0)) + s((0, \nu_B)) - s((0,0)) = m((\mu_A, 0)) + m((0, \nu_A)) - m((0,0)) = \bar{m}((\mu_A, \nu_A)).$$

Put now

$$x = \int_\Omega \mu_A dP, y = \int_\Omega \nu_A dP,$$

hence

$$m(A) = f(x, y).$$

Define

$$\varphi(x) = f(x, 0) = m(\int_\Omega \mu_A, 0),$$

$$\psi(y) = f(1, y) = m(1, \int_\Omega \nu_A dP),$$

hence $\varphi : [0,1] \to [0,1]$ is non-decreasing, $\psi : [0,1] \to [0,1]$ is non-increasing. Since

$$(\mu_A, 0) \vee (1, \nu_A) = (1, 0),$$
$$(\mu_A, 0) \wedge (1, \nu_A) = (\mu_A, \nu_A),$$

we have

$$m(\mu_A, 0) + m(1, \nu_A) = m(\mu_A, \nu_A) + m(1, 0),$$
$$f(x, 0) + f(1, y) = f(x, y) + f(1, 0),$$
$$\varphi(x) + \psi(y) = f(x, y) + 1,$$
$$f(x, y) = \varphi(x) + \psi(y) - 1.$$

We have

$$m(A) = f\left(\int_\Omega dP, \int_\Omega \nu_A dP\right) = = \varphi\left(\int_\Omega \mu_A dP\right) + \psi\left(\int_\Omega \nu_A dP\right) - 1.$$

Since

$$(0,0) \vee (1,1) = (1,0),$$
$$(0,1) \wedge (1,1) = (0,1)$$

we have

$$f(0,0) + f(1,1) = f(1,0) + f(0,1) = 1,$$
$$\varphi(0) + \psi(1) = 1.$$

Acknowledgements

Supported by Grant VEGA 1/0539/08.

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