General form of M-probabilities on IF-events

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Abstract

In M-probability theory the additivity is considered with respect to the Gödel operations (maximum, minimum). The representation theorem is proved under the assumption that the M-probability depends on the integrals of the membership function and the nonmembership function.

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1 IF-events

Consider a classical probability space \((\Omega, \mathcal{S}, P)\). An IF-event is a pair \(A = (\mu_A, \nu_A)\) of \(\mathcal{S}\)-measurable real functions \(\mu_A, \nu_A : \Omega \to [0, 1]\) such that

\[ \mu_A + \nu_A \leq 1. \]

There is a very suitable terminology: \(\mu_A\) is called the membership function, \(\nu_A\) the nonmembership function. If \(f : \Omega \to [0, 1]\) is an \(\mathcal{S}\)-measurable fuzzy set, then the pair \((f, 1 - f)\) is an IF-event, of course IF-events present a larger family.

Denote by \(\mathcal{F}\) the family of all IF-events. There are many possibilities how to define a state \(m : \mathcal{F} \to [0, 1]\). First in [3 - 5] the additivity was studied with respect to Łukasiewicz connectives. In [4] general form of states (with respect to the connectives) was presented and in [5] the theory was imbedded to the MV-algebra probability theory [7].

Of course, in [2] the Gödel connectives were introduced instead of Łukasiewicz ones. Some basic results in M-probability theory (using the Gödel connectives) has been summarized in [1] and [6]. In this communication we present a general form of M-states.

The Gödel connectives are defined in the following way:

\[ A \lor B = (\mu_A \lor \mu_B, \nu_A \land \nu_B), \]
\[ A \land B = (\mu_A \land \mu_B, \nu_A \lor \nu_B), \]

where

\[ f \lor g = \max(f, g), f \land g = \min(f, g) \]

Recall that

\[ A \leq B \iff \mu_A \leq \mu_B, \nu_A \geq \nu_B. \]

An additive M-state is a mapping \(m : \mathcal{F} \to [0, 1]\) such that

(i) \(m((0, 1)) = 0, m((1, 0)) = 1, \)

(ii) \(m(A) + m(B) = m(A \lor B) + m(A \land B)\)

for any \(A, B \in \mathcal{F}\).

An additive M-state is called an M-state, if it is continuous, i.e.

(iii) \(A_n \nearrow A, B_n \searrow B \implies m(A_n) \nearrow m(A), m(B_n) \searrow m(B).\)

2 Representation theorem

Theorem. Let \(m : \mathcal{F} \to [0, 1]\) be an additive state, \(m(A) = f(\int_{\Omega} \mu_A dP, \int_{\Omega} \nu_A dP)\).
Then there are functions
\[ \varphi : [0,1] \to [0,1], \psi : [0,1] \to [0,1] \]
such that \( \varphi \) is non-decreasing, \( \psi \) is non-increasing,
\[ \varphi(0) = \psi(0) = 1, \varphi(0) + \psi(1) = 1, \]
and
\[ m(A) = \varphi(\int_{\Omega} \mu_A dP) + \psi(\int_{\Omega} \nu_A dP) - 1. \]

If \( m : \mathcal{F} \to [0,1] \) is an \( M \)-state, then \( \varphi, \psi \) are continuous.

**Example 1.** Choose \( \alpha \in [0,1] \) and put \( \varphi(x) = (1-\alpha)x + \alpha, \psi(y) = -\alpha y + 1. \) Then
\[ m(A) = (1-\alpha) \int_{\Omega} \mu_A dP + \alpha(1 - \int_{\Omega} \nu_A dP), \]
hence by [4] (see also [6]), any \( L \)-state is an \( M \)-state.

**Example 2.** Put \( \varphi(x) = \frac{x^2}{2} + \frac{1}{2}, \psi(y) = 1 - \frac{y^2}{2}. \) Then
\[ m(A) = \frac{1}{2} (\int_{\Omega} \mu_A^2 dP + 1 - \int_{\Omega} \nu_A^2 dP). \]
The mapping \( m : \mathcal{F} \to [0,1] \) is an example of an \( M \)-state that is not an \( L \)-state.

**Proof of Theorem.** First by the formula
\[ \bar{m}((\mu_A, \nu_A)) = m((\mu_A,0)) + m((0,\nu_A)) - m((0,0)) \]
an extension \( \bar{m} : \mathcal{M} \to [0,1] \) can be constructed, where
\[ \mathcal{M} = \{(\mu_A, \nu_A) : \mu_A, \nu_A : \Omega \to [0,1], \mu_A, \nu_A \text{ are S-measurable } \}. \]

It is easy to see that \( \bar{m} \) is an additive \( M \)-probability, and \( \bar{m} \) is continuous, if \( m \) is continuous. Moreover, if \( (\mu_A, \nu_A) \in \mathcal{F} \), then
\[ (0,0) \lor (\mu_A, \nu_A) = (\mu_A \lor 0, \nu_A \lor 0) = (\mu_A, 0), \]
\[ (0,0) \land (\mu_A, \nu_A) = (\mu_A \land 0, \nu_A \land 1) = (0, \nu_A), \]
hence
\[ m((0,0)) + m((\mu_A, \nu_A)) = m((\mu_A, 0)) + m((0, \nu_A)), \]
and therefore
\[ m((\mu_A, \nu_A)) = m((\mu_A, 0)) + m((0, \nu_A)) - m((0,0)) = \bar{m}((\mu_A, \nu_A)). \]
\( \bar{m} \) is an extension of \( m \). It is unique, because if \( s \) is any extension of \( m \), then again
\[ s((\mu_A, \nu_A)) = s((\mu_A, 0)) + s((0, \nu_B)) - s((0,0)) = m((\mu_A, 0)) + m((0, \nu_A)) - m((0,0)) = \bar{m}((\mu_A, \nu_A)). \]

Put now
\[ x = \int_{\Omega} \mu_A dP, y = \int_{\Omega} \nu_A dP, \]
hence
\[ m(A) = f(x,y). \]

Define
\[ \varphi(x) = f(x,0) = m(\int_{\Omega} \mu_A dP), \]
\[ \psi(y) = f(1,y) = m(1, \int_{\Omega} \nu_A dP), \]
hence \( \varphi : [0,1] \to [0,1] \) is non-decreasing, \( \psi : [0,1] \to [0,1] \) is non-increasing. Since
\[ (\mu_A, 0) \lor (1, \nu_A) = (1, 0), \]
\[ (\mu_A, 0) \land (1, \nu_A) = (\mu_A, \nu_A), \]
we have
\[ m((\mu_A, \nu_A)) + m((0, \nu_A)) = m((\mu_A, 0)) + m((1, 0)), \]
\[ f(x,0) + f(1,y) = f(x,y) + f(1,0), \]
\[ \varphi(x) + \psi(y) = f(x,y) + 1, \]
\[ f(x,y) = \varphi(x) + \psi(y) - 1. \]

We have
\[ m(A) = f \left( \int_{\Omega} dP, \int_{\Omega} \nu_A dP \right) = \varphi \left( \int_{\Omega} \mu_A dP \right) + \psi \left( \int_{\Omega} \nu_A dP \right) - 1. \]

Since
\[ (0,0) \lor (1,1) = (1,0), \]
\[ (0,1) \land (1,1) = (0,1) \]
we have
\[ f(0,0) + f(1,1) = f(1,0) + f(0,1) = 1, \]
\[ \varphi(0) + \psi(1) = 1. \]

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References


