General form of strongly additive $\varphi-$probability

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Abstract

In the paper the notion of strongly additive $\varphi-$probability is introduced. This notion is stronger than the notion od the φ probability studied in [6]. The main result of this article is the general form of strongly additive $\varphi-$probabilities.

Keywords: strongly additive $\varphi-$probability, strongly additive $\varphi-$state, general form.

1 Introduction

After some construction of probability theory on IF-sets [1], [2], [3], [6], [7], an axiomatic theory has been constructed. This theory was based on the Łukasiewicz connectives $a \oplus b = (a+b) \wedge 1$ and $a \odot b = (a+b-1) \vee 0$, or on the min - max connectives $a \oplus b = \max(a,b)$, $a \odot b = \min(a,b)$. In [**?**] there are another two new types of probability on IF - sets, where instead of the Łukasiewicz sum $a \oplus b = (a+b) \wedge 1$ the sum $a \widehat{+} b = \sqrt{(a^2+b^2)} \wedge 1$ is considered. There we can find some notions useful in probability theory on IF-events. Recall that IF-event is a pair $A = (\mu_A, \nu_A)$ of measurable functions $\mu_A, \nu_A : \Omega \to [0,1]$ such that $\mu_A + \nu_A \leq 1$. In this paper we examine another generalized pair of connectives on a family of all IF-events $\mathcal{F}$

$$A \widehat{\oplus} B =$$

$$(\varphi^{-1}(\varphi(\mu_A) + \varphi(\mu_B)) \wedge 1,$$

$$1 - (\varphi^{-1}(\varphi(1-\nu_A) + \varphi(1-\nu_B)) \wedge 1))$$

$$A \widehat{\odot} B =$$

$$((\varphi(\mu_A) + \varphi(\mu_B) - 1) \vee 0,$$

$$(1 - \varphi(1-\nu_A) + 1 - \varphi(1-\nu_B)) \wedge 1),$$

where $A, B \in \mathcal{F}, A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ and $\varphi : [0,1] \to [0,1]$ is an increasing bijection such that $\varphi(u) \leq u$ for any $u \in [0,1]$.

Example 1.1 *As an example we can show two following pairs of operations, where:* $\varphi_1(x) = x^n, \varphi_2(x) = 2^x - 1$, *i.e.*
$A \widehat{\oplus}_1 B = (\sqrt[n]{\mu_A{}^n + \mu_B{}^n} \wedge 1; (1 - \sqrt[n]{(1-\nu_A)^n + (1-\nu_B)^n}) \wedge 1)$
$A \widehat{\odot}_1 B = ((\mu_A{}^n + \mu_B{}^n - 1) \vee 0, (1 - (1-\nu_A)^n + 1 - (1-\nu_B)^n) \wedge 1)$
$A \widehat{\oplus}_2 B = (log_2(2^{\mu_A} + 2^{\mu_B} - 2) \wedge 1; (1 - log_2(2^{(1-\nu_A)} + 2^{(1-\nu_B)} - 2)) \wedge 1)$
$A \widehat{\odot}_2 B = ((2^{\mu_A} + 2^{\mu_B} - 3) \vee 0, (4 - 2^{1-\nu_A} - 2^{1-\nu_B}) \wedge 1)$

In this contribution we present a general form of strongly additive $\varphi-$probability on the family $\mathcal{F}$ of all IF-events.

2 General form of strongly additive $\varphi-$ probability

We are going to work with a probability space $(\Omega, \mathcal{S}, p)$ and with the family $\mathcal{F}$ of all IF-events, i.e. pairs (μ_A, ν_A) of measurable functions $\mu_A, \nu_A : \Omega \to [0,1]$, such that $\mu_A + \nu_A \leq 1$. In this family $\mathcal{F}$ there is already given partial ordering $\leq$ by this prescription $A \leq B \Leftrightarrow \mu_A \leq \mu_B; \nu_A \geq \nu_B$, and we study two binary

L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, pp. 1671–1674
Torremolinos (Málaga), June 22–27, 2008

operations $\widehat{\oplus}, \widehat{\odot}$, which has been mentioned in introduction, $\varphi : [0,1] \to [0,1]$ is an increasing bijection, such that $\varphi(u) \le u$ for any $u \in [0,1]$.

Let us define some notions and propositions, first.

Definition 2.1 *Let $\mathcal{F}$ be the family of all IF-events, $\mathcal{J}$ be the family of all compact subintervals of the unit interval $[0,1]$. By a strongly additive $\varphi-$probability we understand any mapping $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ satisfying the following conditions:*

(i) $\mathcal{P}((\mathbf{1}, \mathbf{0})) = [1,1], \mathcal{P}((\mathbf{0}, \mathbf{1})) = [0,0]$;

(ii) $A\widehat{\odot}B = (0,1)$
$\Rightarrow \mathcal{P}(A\widehat{\oplus}B) = \mathcal{P}(A) + \mathcal{P}(B)$;

(iii) $A_n \nearrow A \Rightarrow \mathcal{P}(A_n) \nearrow \mathcal{P}(A)$.

(Here $[a_n, b_n] \nearrow [a,b]$, if $a_n \nearrow a, b_n \nearrow b$.)

Definition 2.2 *A mapping $m : \mathcal{F} \to [0,1]$ is called a strongly additive $\varphi-$state, if the following conditions are satisfied:*

(i) $m((\mathbf{1},\mathbf{0}))=1$, $m((\mathbf{0},\mathbf{1}))=0$;

(ii) $A\widehat{\odot}B = (0,1)$
$\Rightarrow m(A\widehat{\oplus}B) = m(A) + m(B)$;

(iii) $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$.

Remark 2.3 If instead of $A\widehat{\odot}B$ the operation $A \odot B = ((\mu_A + \mu_B - 1) \vee 0, (1 - \nu_A + 1 - \nu_B) \wedge 1)$ is considered, then we obtain the notion of a $\varphi-$probability, $\varphi-$state. Since $A \odot B = (0,1)$ implies $A\widehat{\odot}B = (0,1)$ then any strongly additive $\varphi-$state is a $\varphi-$state. Also if $\oplus$ is considered instead of $\widehat{\oplus}$, then we obtain the notion of IF-state.

Example 2.4 *We show some examples of strongly additive $\varphi-$states in probability space $(\Omega, \mathcal{S}, p)$:*

(i) $m_1((\mu_A, \nu_A)) = \int_\Omega {\mu_A}^n dp$,
for fixed $n \in N$,

(ii) $m_2((\mu_A, \nu_A)) = \int_\Omega (a^{\mu_A} - 1) dp$,
for fixed $a \in R, a > 0$.

(iii) $m_3((\mu_a, \nu_A)) = \int_\Omega \log \mu_A dp$.

Sincerely as in [5], [8] we shall present also the representation theorem for $\varphi-$ state.

Theorem 2.5 *To any strongly additive $\varphi-$ state $m : \mathcal{F} \to [0,1]$ there exists $\alpha \in [0,1]$ and a probability measure $p : \mathcal{S} \to [0,1]$ such that*

$$m(A) = (1-\alpha) \int_\Omega \varphi(\mu_A) dp + \alpha \int_\Omega \varphi(1 - \nu_A) dp$$

for any $A = (\mu_A, \nu_A) \in \mathcal{F}$.

Proof. Let m be a given strongly additive $\varphi-$state. We should define a new function $\widehat{m} : \mathcal{F} \to [0,1]$ by the formula

$$\widehat{m}((\mu, \nu)) = m((\varphi^{-1}(\mu), 1 - \varphi^{-1}(1 - \nu))),$$

i.e.

$$m((f,g)) = \widehat{m}((\varphi(f), 1 - \varphi(g))).$$

We should show, that this new $\widehat{m}$ is an IF-state. So we should show all 3 conditions from Definition 2.2.
Let

$$C = (\mu_C, \nu_C), D = (\mu_D, \nu_D) \in \mathcal{F},$$

$$C \odot D = (0,1).$$

It follows

$$\mu_C + \mu_D \le 1, \nu_C + \nu_D \ge 1.$$

Define

$$A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$$

by the following formulas

$$\mu_A = \varphi^{-1}(\mu_C), \nu_A = 1 - \varphi^{-1}(1 - \nu_C),$$

$$\mu_B = \varphi^{-1}(\mu_D), \nu_B = 1 - \varphi^{-1}(1 - \nu_D).$$

Then

$$A\widehat{\odot}B =$$

$$= ((\varphi(\varphi^{-1}(\mu_C)) + \varphi(\varphi^{-1}(\mu_D) - 1) \vee 0,$$
$$1-(\varphi(\varphi^{-1}(1-\nu_C)+1-\varphi(\varphi^{-1}(1-\nu_D))))\wedge 1))$$
$$= ((\mu_C + \mu_D - 1) \vee 0, (\nu_C + \nu_D) \wedge 1)$$

$$= (0, 1).$$

It follows

$$m(A\widehat{\odot}B) = m(A) + m(B).$$

$$\widehat{m}(C \oplus D) =$$

$$= \widehat{m}((\mu_C + \mu_D, \nu_C + \nu_D - 1)) =$$
$$= \widehat{m}((\varphi(\mu_A) + \varphi(\mu_B),$$
$$1 - \varphi(1 - \nu_A) + 1 - \varphi(1 - \nu_B) - 1)) =$$
$$= m((\varphi^{-1}(\varphi(\mu_A) + \varphi(\mu_B)),$$
$$1 - \varphi^{-1}(\varphi(1 - \nu_A) + \varphi(1 - \nu_B)))) =$$
$$= m(A\widehat{\oplus}B) = m(A) + m(B) =$$
$$= \widehat{m}((\varphi(\mu_A),$$
$$1 - \varphi(1 - \nu_A))) + \widehat{m}((\varphi(\mu_B), 1 - \varphi(1 - \nu_B)))$$
$$= \widehat{m}((\mu_C, \nu_C)) + \widehat{m}((\mu_D, \nu_D)) =$$
$$= \widehat{m}(C) + \widehat{m}(D).$$

Next, we can use know results about $\widehat{m}$. Hence there exists $\alpha \in [0, 1]$ and a probability $p : \mathcal{S} \to [0, 1]$ such that for any $C \in \mathcal{F}$

$$\widehat{m}(C) = (1 - \alpha) \int_\Omega \mu_C dp + \alpha(1 - \int_\Omega \nu_C dp),$$

finally, we should show how we can translate obtained formula to m and hence

$$m(A) = \widehat{m}((\varphi(\mu_A), 1 - \varphi(1 - \nu_A))) =$$

$$= (1 - \alpha) \int_\Omega \varphi(\mu_A)dp + \alpha \int_\Omega \varphi(1 - \nu_A)dp.$$

$\square$

A direct corollary of Theorem 2.5 is the following assertion.

Theorem 2.6 *To any strongly additive $\varphi-$probability $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ there exist $\alpha, \beta \in [0, 1]$ such that $\alpha \leq \beta$ and*

$$\mathcal{P}(A) =$$

$$= [(1 - \alpha) \int_\Omega \varphi(\mu_A)dp + \alpha(1 - \int_\Omega \varphi(\nu_A)dp),$$

$$(1 - \beta) \int_\Omega \varphi(\mu_A)dp + \beta(1 - \int_\Omega \varphi(\nu_A)dp)].$$

3 Conclusion

There are various notions of probability on IF-events ([3], [8], [10],). In [6] so-called $\varphi-$probability has been studied what is a generalization of IF-probability ([4], [7], [9]). In the paper we studied a special case of $\varphi-$probability, so-called strongly additive probability. We have shown the general form of strongly additive probability (*Theorem* 2.6). The problem of a general form of arbitrary $\varphi-$ probability is open.

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Acknowledgements

The paper was supported by Grant VEGA 1/0539/08.

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