General form of strongly additive $\varphi$–probability

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Abstract

In the paper the notion of strongly additive $\varphi$–probability is introduced. This notion is stronger than the notion of the $\varphi$ probability studied in [6]. The main result of this article is the general form of strongly additive $\varphi$–probabilities.

Keywords: strongly additive $\varphi$–probability, strongly additive $\varphi$–state, general form.

1 Introduction

After some construction of probability theory on IF-sets [1], [2], [3], [6], [7], an axiomatic theory has been constructed. This theory was based on the Łukasiewicz connectives $a \oplus b = (a+b) \land 1$ and $a \odot b = (a+b-1) \lor 0$, or on the min - max connectives $a \oplus b = \max(a,b)$, $a \odot b = \min(a,b)$. In [7] there are another two new types of probability on IF-sets, where instead of the Łukasiewicz sum $a \oplus b = (a+b) \land 1$ the sum $a+b = \sqrt{(a^2 + b^2)} \land 1$ is considered. There we can find some notions useful in probability theory on IF-events. Recall that IF-event is a pair $A = (\mu_A, \nu_A)$ of measurable functions $\mu_A, \nu_A : \Omega \rightarrow [0,1]$ such that $\mu_A + \nu_A \leq 1$. In this paper we examine another generalized pair of connectives on a family of all IF-events $\mathcal{F}$

$$A \hat{\otimes} B =$$

$$(\varphi^{-1}(\varphi(\mu_A) + \varphi(\mu_B))) \land 1,$$

$$(1 - \varphi^{-1}(\varphi(1-\nu_A) + \varphi(1-\nu_B))) \land 1))$$

$$A \hat{\odot} B =$$

$$((\varphi(\mu_A) + \varphi(\mu_B) - 1) \lor 0,$$

$$(1 - \varphi(1-\nu_A) + 1 - \varphi(1-\nu_B)) \land 1),$$

where $A, B \in \mathcal{F}, A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$ and $\varphi : [0,1] \rightarrow [0,1]$ is an increasing bijection such that $\varphi(u) \leq u$ for any $u \in [0,1]$.

Example 1.1 As an example we can show two following pairs of operations, where:

$\varphi_1(x) = x^n, \varphi_2(x) = 2^x - 1$, i.e.

$$A \hat{\odot} B = (\sqrt[4]{\mu_A^n + \mu_B^n} \land 1; (1 - \sqrt[4]{(1-\nu_A)^n + (1-\nu_B)^n}) \land 1)$$

$$A \hat{\odot} B = ((\mu_A^n + \mu_B^n - 1) \lor 0, (1 - (1 - \nu_A)^n + 1 - (1 - \nu_B)^n) \land 1)$$

$$A \hat{\odot} B = ((\log_2(2^{\mu_A} + 2^{\mu_B} - 2) \land 1; (1 - \log_2(2^{1-\nu_A} + 2^{1-\nu_B} - 2)) \land 1)$$

$$A \hat{\odot} B = (((2^{\mu_A} + 2^{\mu_B} - 3) \lor 0, (4 - 2^{1-\nu_A} - 2^{1-\nu_B}) \land 1)$$

In this contribution we present a general form of strongly additive $\varphi$–probability on the family $\mathcal{F}$ of all IF-events.

2 General form of strongly additive $\varphi$–probability

We are going to work with a probability space $(\Omega, S, p)$ and with the family $\mathcal{F}$ of all IF-events, i.e. pairs $(\mu_A, \nu_A)$ of measurable functions $\mu_A, \nu_A : \Omega \rightarrow [0,1]$, such that $\mu_A + \nu_A \leq 1$. In this family $\mathcal{F}$ there is already given partial ordering $\leq$ by this prescription $A \leq B \iff \mu_A \leq \mu_B; \nu_A \geq \nu_B$, and we study two binary
operations $\hat{\oplus}, \hat{\ominus}$, which has been mentioned in introduction, $\varphi : [0, 1] \rightarrow [0, 1]$ is an increasing bijection, such that $\varphi(u) \leq u$ for any $u \in [0, 1]$.

Let us define some notions and propositions, first.

**Definition 2.1** Let $\mathcal{F}$ be the family of all IF-events, $\mathcal{J}$ be the family of all compact subintervals of the unit interval $[0, 1]$. By a strongly additive $\varphi-$probability we understand any mapping $\mathcal{P} : \mathcal{F} \rightarrow \mathcal{J}$ satisfying the following conditions:

(i) $\mathcal{P}(\{1, 0\}) = [1, 1], \mathcal{P}(\{0, 1\}) = [0, 0];$

(ii) $A \hat{\ominus} B = (0, 1)$

$\Rightarrow \mathcal{P}(A \hat{\ominus} B) = \mathcal{P}(A) + \mathcal{P}(B);$

(iii) $A_n \not\in A \Rightarrow \mathcal{P}(A_n) \not\in \mathcal{P}(A)$.

(Here $a_n, b_n \in [a, b], a_n \not\in a, b_n \not\in b$)

**Definition 2.2** A mapping $m : \mathcal{F} \rightarrow [0, 1]$ is called a strongly additive $\varphi-$state, if the following conditions are satisfied:

(i) $m((1, 0)) = 1, m((0, 1)) = 0;$

(ii) $A \hat{\otimes} B = (0, 1)$

$\Rightarrow m(A \hat{\otimes} B) = m(A) + m(B);$

(iii) $A_n \not\in A \Rightarrow m(A_n) \not\in m(A)$.

**Remark 2.3** If instead of $A \hat{\otimes} B$ the operation $A \ominus B = ((\mu_A + \mu_B - 1) \vee 0, (1 - \nu_A + 1 - \nu_B) \land 1)$ is considered, then we obtain the notion of a $\varphi-$probability, $\varphi-$state. Since $A \ominus B = (0, 1)$ implies $A \hat{\otimes} B = (0, 1)$ then any strongly additive $\varphi-$state is a $\varphi-$state. Also if $\hat{\oplus}$ is considered instead of $\hat{\otimes}$, then we obtain the notion of IF-state.

**Example 2.4** We show some examples of strongly additive $\varphi-$states in probability space $(\Omega, \mathcal{S}, p)$ :

(i) $m_1((\mu_A, \nu_A)) = \int_{\Omega} \mu_A^n dp,$

for fixed $n \in N$,

(ii) $m_2((\mu_A, \nu_A)) = \int_{\Omega} (a^{\mu_A} - 1) dp,$

for fixed $a \in R, a > 0$.

(iii) $m_3((\mu_A, \nu_A)) = \int_{\Omega} \log \mu_A dp.$

Sincerely as in [5], [8] we shall present also the representation theorem for $\varphi-$ state.

**Theorem 2.5** To any strongly additive $\varphi-$ state $m : \mathcal{F} \rightarrow [0, 1]$ there exists $\alpha \in [0, 1]$ and a probability measure $p : \mathcal{S} \rightarrow [0, 1]$ such that

$m(A) = (1 - \alpha) \int_{\Omega} \nu_A dp + \alpha \int_{\Omega} (1 - \nu_A) dp$

for any $A = (\mu_A, \nu_A) \in \mathcal{F}$.

**Proof.** Let $m$ be a given strongly additive $\varphi-$ state. We should define a new function $\tilde{m} : \mathcal{F} \rightarrow [0, 1]$ by the formula

$\tilde{m}((\mu, \nu)) = m((\varphi^{-1}(\mu), 1 - \varphi^{-1}(1 - \nu)),$

i.e.

$m((f, g)) = \tilde{m}((\varphi(f), 1 - \varphi(g))).$

We should show, that this new $\tilde{m}$ is an IF-state. So we should show all 3 conditions from Definition 2.2.

Let

$C = (\mu_C, \nu_C), D = (\mu_D, \nu_D) \in \mathcal{F},$

$C \otimes D = (0, 1).$

It follows

$\mu_C + \mu_D \leq 1, \nu_C + \nu_D \geq 1.$

Define

$A = (\mu_A, \nu_A), B = (\mu_B, \nu_B)$

by the following formulas

$\mu_A = \varphi^{-1}(\mu_C), \nu_A = 1 - \varphi^{-1}(1 - \nu_C),$

$\mu_B = \varphi^{-1}(\mu_D), \nu_B = 1 - \varphi^{-1}(1 - \nu_D).$
Then
\[ A \hat{\otimes} B = \]
\[ = ((\varphi(\varphi^{-1}(\mu_C)) + \varphi(\varphi^{-1}(\mu_D)) - 1) \lor 0, \]
\[ 1 - (\varphi(\varphi^{-1}(1 - \nu_C)) + \varphi(\varphi^{-1}(1 - \nu_D)) \land 1) \]
\[ = ((\mu_C + \mu_D - 1) \lor 0, (\nu_C + \nu_D) \land 1) \]
\[ = (0, 1). \]

It follows
\[ m(A \hat{\otimes} B) = m(A) + m(B). \]

\[ \hat{m}(C \oplus D) = \]
\[ = \hat{m}((\mu_C + \mu_D, \nu_C + \nu_D - 1)) = \]
\[ = \hat{m}((\varphi(\mu_A) + \varphi(\mu_B), \]
\[ 1 - (\varphi(1 - \nu_A) + 1 - \varphi(1 - \nu_B) - 1)) = \]
\[ = m((\varphi^{-1}(\varphi(\mu_A)) + \varphi(\mu_B)), \]
\[ 1 - \varphi^{-1}(\varphi(1 - \nu_A) + \varphi(1 - \nu_B))) = \]
\[ = m(A \hat{\otimes} B) = m(A) + m(B) = \]
\[ = \hat{m}(\varphi(\mu_A), \]
\[ 1 - \varphi(1 - \nu_A)) + \hat{m}((\varphi(\mu_B), 1 - \varphi(1 - \nu_B))) \]
\[ = \hat{m}(\mu_C, \nu_C) + \hat{m}(\mu_D, \nu_D) = \]
\[ = \hat{m}(C) + \hat{m}(D). \]

Next, we can use know results about \( \hat{m}. \) Hence there exists \( \alpha \in [0, 1] \) and a probability \( p : S \rightarrow [0, 1] \) such that for any \( C \in \mathcal{F} \)
\[ \hat{m}(C) = (1 - \alpha) \int_{\Omega} \mu_C dp + \alpha(1 - \int_{\Omega} \nu_C dp), \]
finally, we should show how we can translate obtained formula to \( m \) and hence
\[ m(A) = \hat{m}((\varphi(\mu_A), 1 - \varphi(1 - \nu_A))) = \]
\[ = (1 - \alpha) \int_{\Omega} \varphi(\mu_A) dp + \alpha \int_{\Omega} \varphi(1 - \nu_A) dp. \]

A direct corollary of Theorem 2.5 is the following assertion.

**Theorem 2.6** To any strongly additive \( \varphi \)-probability \( P : \mathcal{F} \rightarrow \mathcal{F} \) there exist \( \alpha, \beta \in [0, 1] \) such that \( \alpha \leq \beta \) and
\[ P(A) = \]
\[ = [(1 - \alpha) \int_{\Omega} \varphi(\mu_A) dp + \alpha(1 - \int_{\Omega} \varphi(\nu_A) dp), \]
\[ (1 - \beta) \int_{\Omega} \varphi(\mu_A) dp + \beta(1 - \int_{\Omega} \varphi(\nu_A) dp)]. \]

### 3 Conclusion

There are various notions of probability on IF-events ([3], [8], [10]). In [6] so-called \( \varphi \)-probability has been studied what is a generalization of IF-probability ([4], [7], [9]). In the paper we studied a special case of \( \varphi \)-probability, so-called strongly additive probability. We have shown the general form of strongly additive probability (Theorem 2.6). The problem of a general form of arbitrary \( \varphi \)-probability is open.

### References


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References