A measure extension theorem in ℓ-groups

Alžbeta Michalíková
Faculty of Natural Sciences
Matej Bel University
Tajovského 40
974 01 Banská Bystrica
Slovakia
e-mail: michalik@fpv.umb.sk

Abstract

In the paper [2] there was proved that if H is a σ-continuous, orthomodular lattice, A is its orthocomplemented sublattice, $\mu : A \to \langle 0, 1\rangle$ is a probability measure that is subadditive and $S(A)$ is the σ-complete orthocomplemented sublattice of H generated by A, then there exist exactly one subadditive probability measure $\overline{\mu} : S(A) \to \langle 0, 1\rangle$ that is an extension of μ. In the paper we prove the theorem for ℓ-group valued measures that are subadditive. Similar problems were studied for example in [1] and [4].

Keywords: Measure, measure extension, orthomodular and orthocomplemented lattices.

1 Basic notions

In this section we introduce the basic notions which are necessary for understanding of the studying problems.

Let H be a σ-complete lattice. Then H is called a σ-continuous, if $x_n \nearrow x, y_n \nearrow y$ implies $x_n \wedge y_n \nearrow x \wedge y$ and $x_n \searrow x, y_n \searrow y$ implies $x_n \vee y_n \searrow x \vee y$.

A σ-complete lattice H with the least element 0 and the greatest element 1 is called orthocomplemented, if there is a mapping $\perp: H \to H$ such that the following properties are satisfied:

1. $(a^\perp)^\perp = a$ for each $a \in H$.
2. If $a \leq b$ then $b^\perp \leq a^\perp$.
3. $a \vee a^\perp = 1$ for each $a \in H$.

An orthocomplemented lattice is called to be an orthomodular lattice if the following condition is satisfied:

4. If $a \leq b$ then $b = a \vee (b \wedge a^\perp)$.

Two elements $a, b \in H$ are called orthogonal if $a \leq b^\perp$ or $b \leq a^\perp$.

A subset A of an orthocomplemented lattice H is called an orthocomplemented sublattice of H if $a, b \in A$ implies $a \vee b \in A,\ a^\perp \in A$.

Let G be a complete ℓ-group, i.e. a structure $(G, +, \leq)$ such that $(G, +)$ is an Abelian group, $(G, \leq)$ is a complete lattice (i.e. any upper bounded subset of G has the supremum) and $a \leq b$ implies $a + c \leq b + c$ for any $c \in G$. Let 0 be the neutral element of G (i.e. $a + 0 = a$ for any $a \in G$), $G^+ = \{a \in G; a \geq 0\}$. Denote by ∞ an ideal element and $G^* = G^+ \cup \{\infty\}$, where $a + \infty = \infty + a = \infty + \infty = \infty$ for any $a \in G$ and $a \leq \infty,\ \infty \leq \infty$ for any $a \in G$.

Let A be an orthocomplemented sublattice of an orthomodular lattice H. A mapping $\mu : A \to G^+$ is called a measure if the following properties are satisfied:

1. $\mu(0) = 0$.
2. If $a_n \in A (n = 1, 2, \ldots)$, elements a_n are

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pairwise orthogonal and $\bigvee_{n=1}^{\infty} a_n \in A$, then

$$\mu(\bigvee_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} \mu(a_n).$$

A measure $\mu : A \to G^+$ is subadditive if $\mu(a \vee b) \leq \mu(a) + \mu(b)$ for each $a, b \in A$. It is easy to prove that every measure is non-decreasing and upper continuous. Recall that a measure on an orthomodular lattice need not to be subadditive.

2 Construction

First we construct some auxiliary mappings important for the extension.

Lemma 2.1 *Let A be an orthocomplemented sublattice of an orthomodular lattice H. Let $\mu : A \to G^+$ be a measure. Let $a_n \in A$, $b_n \in A$ $(n = 1, 2, \ldots)$, $a_n \nearrow a$, $b_n \nearrow b$, $a \leq b$. Then*

$$\bigvee_{n=1}^{\infty} \mu(a_n) \leq \bigvee_{n=1}^{\infty} \mu(b_n).$$

Proof. Evidently $a_n \wedge b_m \nearrow a_n \wedge b = a_n$, hence for each $n \in N$ it holds $\mu(a_n) = \bigvee_{m=1}^{\infty} \mu(a_n \wedge b_m) \leq \bigvee_{m=1}^{\infty} \mu(b_m)$ and therefore $\bigvee_{n=1}^{\infty} \mu(a_n) \leq \bigvee_{n=1}^{\infty} \mu(b_n)$.

Now put

$$A^+ = \{b \in H; \exists a_n \in A, a_n \nearrow b\}. \qquad (1)$$

Using preceding lemma we can define a mapping $\mu^+ : A^+ \to G^*$ by the formula

$$\mu^+(b) = \bigvee_{n=1}^{\infty} \mu(a_n),\ a_n \nearrow b. \qquad (2)$$

Since A is an orthocomplemented sublattice of H, then $1 \in A$ and for each $x \in H$ there exists $b \in A^+$ such that $b \geq x$. Therefore we can define a mapping $\mu^* : H \to G^*$ by the formula

$$\mu^*(x) = \bigwedge\{\mu^+(b); b \in A^+, b \geq x, x \in H\}. \qquad (3)$$

Similarly there can be defined A^-, μ^-, μ_*.

The last step of our construction is the set

$$L = \{x \in H;\ \mu_*(x) = \mu^*(x)\}. \qquad (4)$$

Later we will prove that $L \supset S(A)$ and $\mu^*|S(A)$ is the asked extension.

Remark 2.2 It is easy to prove that μ^+, μ^- are extensions of μ, μ^+ is upper continuous, non-decreasing and subadditive. Further μ^* is an extension of μ^+ and it is a non-decreasing, subadditive and $\mu^*(x) \geq \mu_*(x)$ for each $x \in H$.

3 Main theorem

First we prove some propositions which are necessary to prove the main theorem.

Proposition 3.1 *Let $x \in H$, $y \in L$, $y \leq x$. Then $\mu^*(x) = \mu^*(y) + \mu^*(x \wedge y^\perp)$.*

Proof. Let us divide the proof to the following steps:

1. We prove that if $a \in A$, $b \in A^+$, $a \leq b$, then $\mu^+(b) = \mu(a) + \mu^+(b \wedge a^\perp)$.
Because $a \leq a_n \nearrow b$, $a_n \in A$, then $\mu(a_n) = \mu(a) + \mu(a_n \wedge a^\perp)$. Since $a_n \nearrow b$ then $a_n \wedge a^\perp \nearrow b \wedge a^\perp$ and therefore $\mu^+(b) = \mu(a) + \mu^+(b \wedge a^\perp)$.
2. If $b, d \in A^+$, $d \leq b$, then $\mu^+(b) \geq \mu^+(d) + \mu^*(b \wedge d^\perp)$.
Indeed, $d_n \nearrow d$, $d_n \in A$ and 1. imply $\mu^+(b) = \mu(d_n) + \mu^+(b \wedge d_n^\perp) \geq \mu(d_n) + \mu^*(b \wedge d^\perp)$, which gives $\mu^+(b) \geq \mu^+(d) + \mu^*(b \wedge d^\perp)$.
3. If $b \in A^+$, $c \in A^-$, $c \leq b$, then $\mu^+(b) \geq \mu^-(c) + \mu^+(b \wedge c^\perp)$.
Take $c_n \in A$, $c_n \searrow c$. Since $b \wedge c_n \in A^+$, $b \wedge c_n \leq b$ we have by 2. $\mu^+(b) \geq \mu^+(b \wedge c_n) + \mu^*(b \wedge (b \wedge c_n)^\perp) \geq \mu^+(b \wedge c_n) + \mu^*(b \wedge c_n^\perp) = \mu^+(b \wedge c_n) + \mu^+(b \wedge c_n^\perp)$. Taking $n \to \infty$ we obtain $\mu^+(b) \geq \bigvee_{n=1}^{\infty} \mu^+(b \wedge c_n) + \bigvee_{n=1}^{\infty} \mu^+(b \wedge c_n^\perp) \geq \mu^-(c) + \mu^+(b \wedge c^\perp)$.
4. Let $x \in H$, $c \in A^-$, $c \leq x$. Then $\mu^*(x) \geq \mu^-(c) + \mu^*(x \wedge c^\perp)$.
If $b \in A^+$, $b \geq x$, then $\mu^+(b) \geq \mu^-(c) + \mu^+(b \wedge c^\perp) \geq \mu^-(c) + \mu^*(x \wedge c^\perp)$ therefore $\mu^*(x) \geq \mu^-(c) + \mu^*(x \wedge c^\perp)$.
5. Finally we prove the statement of the Proposition.
Let $x \in H$, $y \in L$, $y \leq x$. Take $c \leq y$, $c \in A^-$. By 4. we have $\mu^*(x) \geq \mu^-(c) + \mu^*(x \wedge c^\perp) \geq$

$\mu^-(c)+\mu^*(x\wedge y^\perp)$, hence $\mu^*(x)-\mu^*(x\wedge y^\perp) \geq \mu^-(c)$. Therefore $\mu^*(x)-\mu^*(x\wedge y^\perp) \geq \mu_*(y) = \mu^*(y)$.
The opposite inequality follows from the subadditivity of μ^*.

Proposition 3.2 *If $y \in L$, then $y^\perp \in L$.*

Proof. For each $b \in A^+$ it holds $\mu^+(b) + \mu^-(b^\perp) = \mu^*(b \vee b^\perp) = \mu(1)$. Let $b \geq y$. Then $b^\perp \leq y^\perp$ and therefore $\mu(1) = \mu^+(b) + \mu^-(b) \leq \mu^+(b)+\mu_*(y^\perp)$. Since this inequality holds for arbitrary $b \in A^+$, then it holds also for infimum and therefore $\mu(1) - \mu_*(y^\perp) \leq \mu^*(y)$. If we put $x = 1$ into the Proposition 3.1, then we get $\mu(1) = \mu^*(1) = \mu^*(y) + \mu^*(1 \wedge y^\perp) = \mu^*(y) + \mu^*(y^\perp)$. Hence $\mu^*(y) + \mu_*(y^\perp) \geq \mu(1) = \mu^*(y) + \mu^*(y^\perp)$, which implies $\mu_*(y^\perp) \geq \mu^*(y^\perp)$. Opposite inequality follows from Remark 2.2.

Definition 3.3 *An ℓ-group G is called a Dedekind complete if any upper bounded sequence of elements from G has a supremum in G.*

Definition 3.4 *Dedekind complete ℓ-group G is called to be of countable type, if to any bounded set $A \subset G$ there exists such a countable subset $B \subset A$ that*

$$\bigwedge A = \bigwedge B.$$

Proposition 3.5 *Let G be a Dedekind complete ℓ-group of countable type. Let μ^* be the subadditive mapping defined on the lattice H generated by μ^+. Then for each $x \in H$ there exists bounded double sequence $a_{i,j} \searrow 0$, $(j \to \infty, i = 1, 2, \ldots)$ such that for each $\varphi : N \to N$ there exists $b \in A^+$, $b \geq x$ and it holds*

$$\mu^*(x) + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \geq \mu^+(b).$$

Proof. Let G be a Dedekind complete ℓ-group of countable type. Then there exists a sequence $(b_n)_{n=1}^{\infty}$ of elements of the set A^+ such that $\mu^*(x) = \bigwedge_{n=1}^{\infty}\{\mu^+(b_n), b_n \geq x, b_n \in A^+, n = 1, 2, \ldots\}$.

Put $c_1 = b_1$, $c_2 = b_1 \wedge b_2$, $c_n = \bigwedge_{i=1}^{n} b_i$ then $c_n \geq c_{n+1}$, $c_n \in A^+$ and $b_n \geq c_n \geq x$. Therefore $\mu^+(b_n) \geq \mu^+(c_n) \geq \mu^*(x)$. Hence $\mu^+(b_n) - \mu^*(x) \geq \mu^+(c_n) - \mu^*(x) \geq 0$. Define $a_{i,j} = \mu^+(c_j) - \mu^*(x)$, $(j \to \infty, i = 1, 2, \ldots)$, then $0 = \bigvee_j\{\mu^+(b_j) - \mu^*(x)\} \geq \bigvee_j\{\mu^+(c_j) - \mu^*(x)\}$. Therefore $a_{i,j} \searrow 0$ $(j \to \infty, i = 1, 2, \ldots)$. Let $\varphi : N \to N$. Then $\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \geq a_{i,\varphi(i)} = \mu^+(c_{\varphi(i)}) - \mu^*(x)$. Put $b = c_{(\varphi(i))}$ then holds following inequality $\mu^*(x)+\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \geq \mu^+(b)$.

Proposition 3.6 *Let G be Dedekind complete ℓ-group. Then the following assertion holds:*
Let for each $n \in N$ exists double bounded sequence $(a_{n,i,j})_{i,j}$ of elements of the set G such that $a_{n,i,j} \searrow 0$, $(j \to \infty, i = 1, 2, \ldots)$. Then for each $a \in G$, $a > 0$ there exists double bounded sequence $(a_{i,j})_{i,j}$ such that sequence $a_{n,i,j} \searrow 0$, $(j \to \infty, i = 1, 2, \ldots)$ and at the same time $a \wedge (\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(n+i)}) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ for each $\varphi : N \to N$.

Proposition 3.6 is called Fremlin Theorem. The proof can be find in [3] (see Proposition 3.2.4).

Definition 3.7 *Dedekind complete ℓ-group G is called to be weakly σ-distributive if for any bounded double sequence $(a_{i,j})$ such that $a_{i,j} \searrow 0$ $(j \to \infty, i = 1, 2, \ldots)$ it is*

$$\bigwedge_{\varphi \in N \to N} \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} = 0.$$

Proposition 3.8 *Let G be a Dedekind complete, weakly σ-distributive ℓ-group of countable type. Let H be a σ-complete lattice. Let $\mu^* : H \to G^*$ be a mapping satisfying the condition (3). Let L be the set satisfying the condition (4). Let $z_n \in L$ $(n = 1, 2, \ldots)$, $z_n \nearrow z$, (or $z_n \searrow z$ resp.), $z \in H$. Then $z \in L$ and $\mu^*(z_n) \nearrow \mu^*(z)$.*

Proof. Let $z_n \nearrow z$. Then $\bigvee_{n=1}^{\infty} \mu^*(z_n) \leq \mu^*(z)$ and for each $z \in H$ there exists $y \in A^+$ such that $y \geq z$ and certainly $\mu^+(y) \geq \mu^*(z)$.

Put $z_0 = 0$. By Proposition 3.1 $\mu^*(z_n) - \mu^*(z_{n-1}) = \mu^*(z_n \wedge z_{n-1}^{\perp})$, $n = 1, 2, \ldots$. From Proposition 3.5 follows that for every $n \in N$ there exists bounded double sequence $(a_{n,i,j})_{i,j}$ such that for each $\varphi : N \to N$ there exists $b_n \in A^+$, $b_n \geq z_n \wedge z_{n-1}^{\perp}$ and $\mu^*(z_n \wedge z_{n-1}^{\perp}) \geq \mu^+(b_n) - \bigvee_{i=1}^{\infty} a_{n,i,\varphi(n+i)}$. Put $y_n = y \wedge (\bigvee_{i=1}^{n} b_i)$. Then $y_n \in A^+$, $y_n \leq y_{n+1}$ and $y_n \leq y$.

Therefore $\mu^*(z_1) - \mu^*(z_0) = \mu^*(z_1 \wedge z_0^{\perp}) \geq \mu^+(y_1) - \bigvee_{i=1}^{\infty} a_{1,i,\varphi(i+1)}$. Similarly $\mu^*(z_2) = \mu^*(z_1) + \mu^*(z_2 \wedge z_1^{\perp}) \geq \mu^+(y_1) - \bigvee_{i=1}^{\infty} a_{1,i,\varphi(i+1)} + \mu^+(y_2) - \bigvee_{i=1}^{\infty} a_{2,i,\varphi(i+2)}$.

By induction we get:

$$\mu^*(z_n) \geq \sum_{k=1}^{n} \mu^+(y_k) - \sum_{k=1}^{n} \bigvee_{k=1}^{\infty} a_{k,i,\varphi(i+k)}.$$

Hence

$$\bigvee_{n=1}^{\infty} \mu^*(z_n) \geq \bigvee_{n=1}^{\infty} \sum_{k=1}^{n} \mu^+(y_k) -$$

$$- \bigvee_{n=1}^{\infty} \sum_{k=1}^{n} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(n+i)} \geq$$

$$\geq \mu^+(\bigvee_{n=1}^{\infty} y_n) - \sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(n+i)}.$$

We assumed that there exists $y \in A^+$ such that $y \geq \bigvee_{n=1}^{\infty} y_n$. Than $\mu^+(y) \geq \mu^+(\bigvee_{n=1}^{\infty} y_n)$ and also $\mu^+(y) \geq \mu^+(\bigvee_{n=1}^{\infty} y_n) - \bigvee_{n=1}^{\infty} \mu^*(z_n)$. Then by Proposition 3.6

$$\mu^+(\bigvee_{n=1}^{\infty} y_n) - \bigvee_{n=1}^{\infty} \mu^*(z_n) \leq$$

$$\leq \mu^+(y) \wedge \sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(n+i)} \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for each $\varphi : N \to N$. But $\bigvee_{n=1}^{\infty} y_n \geq z$ and either $\mu^+(\bigvee_{n=1}^{\infty} y_n) \geq \mu^*(z)$. Since G is weakly σ-distributive ℓ-group, it holds:

$$\mu^*(z) - \bigvee_{n=1}^{\infty} \mu^*(z_n) \leq \bigwedge_{\varphi:N\to N} \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} = 0.$$

Therefore $\mu^*(z_n) \nearrow \mu^*(z)$. Further $\mu_*(z) \leq \mu^*(z) = \bigvee_{n=1}^{\infty} \mu^*(z_n) = \bigvee_{n=1}^{\infty} \mu_*(z_n) \leq \mu_*(z)$, hence $z \in L$.

The second part of Proposition (for non-increasing sequences) follows from Proposition 3.2 and the first part of proof.

Proposition 3.9 *Let G be a Dedekind complete, weakly σ-distributive ℓ-group of countable type. Let $\mu^* : H \to G^*$ be a mapping satisfying the condition* (3)*. Let L be the set satisfying the condition* (4)*. Then $\overline{\mu} = \mu^*|L$ is additive mapping, i.e. for each $x, y \in L$ holds $\mu^*(x \vee y) = \mu^*(x) + \mu^*(y)$.*

Proof. First we prove that for each $c, d \in A^-$ it hold $\mu^-(c \vee d) = \mu^-(c) + \mu^-(d)$. By Proposition 3.1 it holds: $1 - \mu^-(d) = \mu^+(d^{\perp}) = \mu^*(d^{\perp}) = \mu^-(c) + \mu^*(d^{\perp} \wedge c^{\perp}) = \mu^-(c) + \mu^+((d \vee c)^{\perp}) = \mu^-(c) + 1 - \mu^-(d \vee c)$.

Now let $x, y \in H$, $x \leq y^{\perp}$. Since G is Dedekind complete, weakly σ-distributive ℓ-group of countable type, then there exist $a_{i,j} \searrow 0$, $b_{i,j} \searrow 0$ $(j \to \infty, i = 1, 2, \ldots)$ such that for each $\varphi : N \to N$ there exist $c, d \in A^-$, $c \leq x$, $d \leq y$ and it hold $\mu_*(x) - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} < \mu^-(c)$, $\mu_*(y) - \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} < \mu^-(d)$.

Because $c \leq x \leq y^{\perp} \leq d^{\perp}$, then $\mu^*(x \vee y) \leq \mu^*(x) + \mu^*(y) = \mu_*(x) + \mu_*(y) < \mu^-(c) + \mu^-(d) + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} = \mu^-(c \vee d) + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \leq \mu_*(x \vee y) + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$.

Proposition 3.10 *Let G be a Dedekind complete, weakly σ-distributive ℓ-group of countable type. Let A be an orthocomplemented sublattice of an orthomodular σ-complete lattice H. Let $S(A)$ be the σ-complete orthocomplemented lattice generated by A, $M(A)$ be the least set over A closed under monotone sequences. Then $S(A) = M(A)$.*

Proof. Since $S(A)$ is σ-complete orthocomplemented lattice then $M(A) \subset S(A)$. On the other side, let $x \in A$ be an arbitrary but fixed element. Denote $P = \{y \in M, x \vee y \in M\}$.

Evidently $A \subset P$ and P is closed under monotone sequences, therefore $M \subset P$. Hence for each $x \in A$ and each $y \in M$ it hold $x \vee y \in M$. Now let us take a fixed $y \in M$ and put $R = \{x \in M, x \vee y \in M\}$. Then also R is closed under monotone sequences, $A \subset R$ and $M \subset R$ therefore M is closed under the operation $\vee$. Because for each $x \in A$ holds that $x^{\perp} \in A$, it is easy to prove, that if $y \in M$ also $y^{\perp} \in M$. Hence $M(A)$ is the σ-complete orthocomplemented lattice generated by A and $S(A) \subset M(A)$.

Theorem 3.11 *Let G be a Dedekind complete, weakly σ-distributive ℓ-group of countable type. Let H be a σ-continuous, orthomodular lattice, A its orthocomplemented sublattice and $\mu : A \to G^{+}$ a subadditive measure. Let $S(A)$ be the σ-complete orthocomplemented sublattice generated by A. Then there is exactly one subadditive measure $\overline{\mu} : S(A) \to G^{*}$ that is an extension of μ.*

Proof. Existence. Evidently $S(A) = M(A) \subset L$. Put $\overline{\mu} = \mu^{*}|S(A)$. By Propositions 3.8 and 3.9 $\overline{\mu}$ is a measure.
Uniqueness. Let $\nu : S(A) \to G^{*}$ be a measure, $\nu|A = \mu$. Put $K = \{x \in S(A); \overline{\mu}(x) = \nu(x)\}$. Evidently $A \subset K$, K is closed under monotone sequences. Therefore $S(A) = M(A) \subset K$.

4 Conclusion

We proved the extension theorem for ℓ-group valued measures being moreover subadditive and that are defined on an orthocomplemented sublattice of σ-continuous orthomodular lattice.

Another approach has been realized by Vrábelová in [5], where the assumption that μ is σ- additive is substituted by assumption that μ is exhaustive.

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