Probability on pseudo-MV-algebras

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Abstract

The concept of Intuitionistic Fuzzy Sets (IFS) was proposed by K. Atanassov in 1983. It is an extension of the well-known notion of fuzzy set defined by L. Zadeh. Every element a of an IFS set has a degree of membership $\mu : a \mapsto \langle 0,1\rangle$ and a degree of non-membership $\nu : a \mapsto \langle 0,1\rangle$. The sum of the two degrees have to be less than 1: $\mu(a) + \nu(a) \leq 1$. It is well known that every IF set can be embeded in MV-algebra. The natural noncomnutative generalization of MV-algebra is the pseudo-MV-algebra. The aim of this paper is to show that some results holding in B-structures are used for pseudo-MV-algebras.

Keywords: Pseudo-MV-algebra, B-structures, probability.

1 Basic notions

Definition 1.1 *A B-structure is a system* $(B, \hat{\oplus}, \leq, 0_B, 1_B)$ *such that:*

(i) $\hat{\oplus}$ *is a partial binary operation on* B*;*

(ii) $\leq$ *is a partial ordering on* B*;*

(iii) 0_B *is the smallest,* 1_B *is the largest element in* $(B, \leq)$.

Definition 1.2 *A state on* B *is a mapping* $m : B \to [0,1]$ *satisfying the following conditions:*

(I) $m(1_B) = 1$, $m(0_B) = 0$

(II) if $a = b\hat{\oplus}c$, *then* $m(a) = m(b) + m(c)$

(III) if $a_n \nearrow a$, *then* $m(a_n) \nearrow m(a)$.

In the next part we define a pseudo-MV-algebras and the states on them. For the first time they were introduced in [6] and [7]. This algebraic structure is known under the name generalized MV-algebras, too.

Definition 1.3 *A pseudo-MV-algebra is a system* $M = (M, \oplus, \odot, *, ', 0_M, 1_M)$, *where* $\oplus, \odot$ *are binary operations,* $*, '$ *are the unary operations and* $0, 1$ *are the elements of* M *such that for all* a, b, c *in* M *the following identities are satisfied:*

(i) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$*;*

(ii) $a \oplus 0_M = 0_M \oplus a = a$*;*

(iii) $a \oplus 1_M = 1_M \oplus a = 1_M$*;*

(iv) $(a^*)' = a$*;*

(v) $1_M^* = 0_M; 1_M' = 0_M$*;*

(vi) $(b^* \oplus a^*)' = (b' \oplus a')^*$*;*

(vii) $a \odot b = (a^* \oplus b^*)'$*;*

(viii) $a \oplus (a' \odot b) = b \oplus (b' \odot a) = (a \odot b^*) \oplus b = (b \odot a^*) \oplus a$*;*

(ix) $a \odot (a^* \oplus b) = (a \oplus b') \odot b$.

We shall assume that $0 \neq 1$. We shall define a partial order $\leq$: $a \leq b$ if and only if $a^* \oplus$

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$b = 1$. Then we can say that $(M, \leq)$ is a distributive lattice with operations supremum and infimum on M:

$a \vee b = a \oplus a' \odot b$,

$a \wedge b = a \odot (a^* \oplus b)$.

If for the operation $\oplus$ on pseudo-MV-algebra M the commutative law holds then M is MV-algebra.

We define a partial binary operation $+$ on M:

$a + b$ is defined iff $a \leq b^*$ and then holds: $a + b := a \oplus b$.

It is clear that $a \leq b^*$ holds if and only if $b \odot a = 0$.

Definition 1.4 *A state on M is a mapping $m : M \to [0, 1]$ satisfying the folowing conditions:*

(I) $m(1_M) = 1$, $m(0_M) = 0$

(II) $m(a + b) = m(a) + m(b)$ whenewer $a + b$ is defined in M

(III) if $a_n \nearrow a$, then $m(a_n) \nearrow m(a)$.

In the next we shall assume that the state exists. In generally it can happen that pseudo-MV-algebras has no states, which is shown in [8].

Now we take a structure $(M, +, \leq, 0_M, 1_M)$ with $+, \leq$, which we defined in a previous text. About this system we can say that it is a B-structure and a state from Definition 2. on B is corresponding with a state on pseudo-MV-algebras M. In the next text we shall work with the B-structure, which we get from the pseudo-MV-algebras. We define a probability on it and we get some results like a Central limit theorem for this B-structure. By reason that the states on M and the new system B coincides where $(B, \oplus, \leq, 0_B, 1_B) = (M, +, \leq, 0_M, 1_M)$ we can use these results for the pseudo-MV-algebras.

2 Probability on pseudo-MV-algebras

Let (Ω, S, P) be a probability space. A random variable on it is a mapping $\xi : \Omega \to R$ such that

$$A \in B(R) \Rightarrow \xi^{-1}(A) \in S.$$

If we define

$$x : A \to \xi^{-1}(A)$$

then we obtain a σ-homomorphism

$$x : B(R) \to S.$$

Instead of point mappings $\xi : \Omega \to R$ we can work with σ-homomorphisms from the σ-algebra $B(R)$ of Borel sets to given structure. This approach has been used in quantum structures, where the corresponding σ-homomorphism is called observable.

Definition 2.1 *Let $M = (M, \oplus, \odot, *, ', 0_M, 1_M)$ be a pseudo-MV-algebra. An observable of M is a mapping $x : \beta(R) \to M$ satisfying the following conditions:*

(i) $x(R) = 1_M$, $x(\emptyset) = 0_M$;

(ii) if $A, B \in \beta(R)$ and $A \cap B = \emptyset$, then $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \in \beta(R)$, $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

The next theorem is proved in [1].

Theorem 2.2 *Let be an observable $x : \beta(R) \to M$ and $m : M \to [0, 1]$ be a state. Then the composite map $m \circ x = m_x : \beta(R) \to [0, 1]$ is a probability measure.*

Let we have the B-structure $(M, +, \leq, 0_M, 1_M)$. When we use the previous definition and theorem on this structure we get the probability measure on the pseudo-MV-algebras.

Definition 2.3 *The expected value $E(x)$ of the observable x is defined by the formula*

$$E(x) = \int_R t dm_x(t),$$

if the integral exists.

In the following text under the B-structure we mean the system $(M, +, \leq, 0_M, 1_M)$, when we got from the pseudo-MV-algebra. We denote the remembered B-structure by the letter M.

Definition 2.4 *Let $g : R \to R$ be a Borel function (i.e. $A \in B(R) \Rightarrow g^{-1}(A) \in B(R)$), $x : B(R) \to M$ be an observable. Then we define $g \circ x : B(R) \to M$ by the formula*

$$g \circ x(A) = x\left(g^{-1}(A)\right).$$

Theorem 2.5 *The mapping $g \circ x : B(R) \to M$ is an observable and*

$$E(g \circ x) = \int_R g dm_x,$$

if the integral exists.

Proof: First, by the definition

$$y(A) = g \circ x(A) = x\left(g^{-1}(A)\right) \in M,$$

because of $g^{-1}(A) \in B(R)$. Secondly recall the integral transformation theorem

$$\int_{\varphi^{-1}(A)} f \circ \varphi dP = \int_A f dP_\varphi.$$

Put $A = R, \varphi : R \to R, \varphi(t) = g(t), f(u) = u$. Then $P_\varphi(A) = P_g(A) = P\left(g^{-1}(A)\right) = m_x\left(g^{-1}(A)\right) = m\left(x\left(g^{-1}(A)\right)\right) = m(y(A)) = m_y(A)$. Further $(f \circ \varphi)(t) = g(t)$

$$\int_R g dm_x = \int_R t dP_g(t) = \int_R t dm_y(t) = E(y) = E(g \circ x).$$

Corollary 1 $D(x) = \int_R (t - E(\xi))^2 dm_x(t)$

We have shown the way how to define and compute moments of observables. Now we define the sum of observables. If $\xi, \eta : \Omega \to R$ are random variables and $T = (\xi, \eta) : \Omega \to R^2$ is the corresponding random vector, then we can express the sum by the help of the function $g : R^2 \to R, g(u, v) = u + v$:

$$\xi + \eta = g(\xi, \eta) = g \circ T.$$

Therefore

$$(\xi + \eta)^{-1}(A) = T^{-1}\left(g^{-1}(A)\right).$$

$T^{-1} : B\left(R^2\right) \to S$ is now a σ-homomorphism such that

$$T^{-1}(C \times D) = \xi^{-1}(C) \cap \eta^{-1}(D); \quad C, D \in B(R).$$

Generally we are no able to construct a σ-homomorphism from $B\left(R^2\right) \to M$ connected with x, y. Different situation occurs if $(\xi + \eta)$ are independent:

$$P\left(T^{-1}(C \times D)\right) = P\left(\xi^{-1}(C) \cap \eta^{-1}(D)\right) = P\left(\xi^{-1}(C)\right) \cdot \left(\eta^{-1}(D)\right).$$

This approach can be realized also in the general case.

Definition 2.6 *Two observables $x, y : B(R) \to M$ are independent, if there exists a mapping $h : B\left(R^2\right) \to M$ satisfying the following conditions:*

(i) $h\left(R^2\right) = 1, h(\emptyset) = 0$;

(ii) $A \cap B = \emptyset \Rightarrow h(A \cup B) = h(A) + h(B)$;

(iii) $A_n \nearrow A \Rightarrow h(A_n) \nearrow (A)$;

(iv) $m(h(C \times D)) = m(x(C)) \cdot m(y(D))$, $C, D \in B(R)$.

While h is not uniquely determined, any two mappings h_1 and h_2 satisfying the condition in the above definition satisfy $m \circ h_1 = m \circ h_2$ automatically.

Definition 2.7 *Let $x, y : B(R) \to M$ are independent observables and $g : R^2 \to R$ be a Borel function. Then the mapping $z = g(x, y) : B(R) \to M$ defined by equality*

$$z = h \circ g^{-1}$$

is a observable, where $h : B\left(R^2\right) \to M$ is a joint observable of x, y.

Definition 2.8 *Let $(x_i)_1^\infty$ be an independent sequence of observables in an B-structure M with a state m. Let $C = \left\{\pi_n^{-1}(M), M \in B(R^n), n \in N\right\}$ be a set of all cylinders, where the function $\pi_n : R^N \to R^n$ defined by $\pi_n\left((u_i)_1^\infty\right) = (u_1, ..., u_n)$ is called the n-th coordinate random vector. The* *infinite product* $\boldsymbol{P}$ *of the measures* $m \circ x_i, i = 1, 2, ...$ *on the space* $\left(R^N, \sigma(C)\right)$ *is defined by*

$$\boldsymbol{P}\left\{(u_i)_1^\infty \in R^N; u_1 \in A_1, ..., u_n \in A_n\right\} = (m \circ x_1)(A_1) \cdot ... \cdot (m \circ x_n)(A_n)$$

for every $n \in N$ *and every* $A_1, ..., A_n \in \mathcal{B}(R)$.

For each $n \in N$ *the function* $\xi_n : R^N \to R$ *given by* $\xi_n((u_i)_1^\infty) = u_n$ *is called n-th coordinate random variable of* $\left(R^N, \sigma(C), \boldsymbol{P}\right)$.

Theorem 2.9 *Let* $(x_i)_1^\infty$ *be an independent sequence of observables in an B-structure* M *with a state* m. *Let* $g_n : R^N \to R$ *be a Borel measurable function and* $\xi_n : R^N \to R$ *n-th coordinate random variable of* $\left(R^N, \sigma(C), \boldsymbol{P}\right)$, *where* $\boldsymbol{P}$ *is infinite product of* $m \circ x_i$. *Then*

$$\boldsymbol{P}\{(u_i)_1^\infty; g_n(\xi_1((x_i)_1^\infty), ..., \xi_n((x_i)_1^\infty))) \in C =$$
$$= (m \circ g_n(x_1, ..., x_n))(C).$$

Proof: Since the sequence $(x_i)_1^\infty$ is independent, then there exists n-dimensional observable $h_n : B\left(R^N\right) \to M$ such that $m \circ h_n = m_{x_1} \times ... \times m_{x_n}$. But $g_n(x_1, ..., x_n) = h_n \circ g_n^{-1}$. Hence

$$(m \circ g_n(x_1, ..., x_n))(C) = (m \circ h_n \circ g_n^{-1})(C) = (m \circ h_n)(g_n^{-1}(C)) =$$
$$= ((m \circ x_1) \times ... \times (m \circ x_n))(g_n^{-1}(C)) =$$
$$= (\mathbf{P} \circ \pi_n^{-1})(g_n^{-1}(C)) =$$
$$= \mathbf{P}(\{(u_i)_1^\infty; (u_1, ..., u_n) \in g_n^{-1}(C)\}) =$$
$$= \mathbf{P}(\{(u_i)_1^\infty; g_n(u_1, ..., u_n) \in C\}) =$$
$$= \mathbf{P}(\{(u_i)_1^\infty; g_n(\xi_1((u_i)_1^\infty), ..., \xi_n((u_i)_1^\infty))) \in C\}.$$

Theorem 2.10 *(Central limit theorem) Let* M *be a pseudo-MV-algebra and* m *be a state on* M. *Let* $(x_i)_{i=1}^\infty$ *be an independent sequence of square integrable observables having the same probability distribution* $m \circ x_1 = m \circ x_2 = \ldots$ *such that* $E(x_n) = a$, $\sigma^2(x_n) = \sigma^2$ $(n = 1, 2, \ldots)$ *and* $y_n = \frac{\sqrt{n}}{\sigma}\left(\frac{1}{n}\sum_{i=1}^n x_i - a\right)$. *Then for all* $t \in R$ $\lim_{n\to\infty}(m \circ y_n)((-\infty, t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t e^{-\frac{u^2}{2}}\mathrm{d}u$.

This theorem is proved for every B-structure in [1].

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