Maličký-Riečan entropy on IF-dynamical systems

Marek Ďurica
Department of Mathematics
Faculty of Natural Sciences
Matej Bel University
Tajovského 40
SK-974 01 Banská Bystrica
durica@fpv.umb.sk

Abstract

In this paper we study dynamical systems based on IF-events (see [1]). We define a special type of the notion of the entropy on this systems and its Maličký-Riečan modification (see [12]).

Keywords: IF-event, IF-dynamical system, IF-partition, Maličký-Riečan entropy.

1 Introduction

We start with classical dynamical systems $(\Omega, \mathcal{S}, P, T)$, where $(\Omega, \mathcal{S}, P)$ is a probability space and $T : \Omega \to \Omega$ is a measure preserving map, i.e. $T^{-1}(A) \in \mathcal{S}$ and $P(T^{-1}(A)) = P(A)$ for any $A \in \mathcal{S}$. The entropy of the dynamical system is defined as follows (see [11], [12]). Consider measurable partition $\mathcal{A} = \{A_1, ..., A_k\}$, where $A_i \in \mathcal{S}; i = 1, ..., k, A_i \cap A_j = \emptyset; i \neq j, \bigcup_{i=1}^{k} A_i = \Omega$. Its entropy is the number

$$H(\mathcal{A}) = \sum_{i=1}^{k} \varphi(P(A_i)),$$

where $\varphi(x) = -x \log x$, if $x > 0$, and $\varphi(0) = 0$. If $\mathcal{A} = \{A_1, ..., A_k\}$ and $\mathcal{B} = \{B_1, ..., B_l\}$ are two measurable partitions, then $T^{-1}(\mathcal{A}) = \{T^{-1}(A_1), ..., T^{-1}(A_k)\}$ and $\mathcal{A} \vee \mathcal{B} = \{A \cap B; A \in \mathcal{A}, B \in \mathcal{B}\}$ are measurable partitions, too. It can be proved that there exists

$$h(\mathcal{A}, T) = \lim_{n\to\infty} H\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{A})\right).$$

The Kolmogorov-Sinaj entropy $h(T)$ of $(\Omega, \mathcal{S}, P, T)$ is defined as the supremum

$$h(T) = \sup\{h(\mathcal{A}, T); \mathcal{A} \text{ is a measurable partition}\}.$$

The aim of the Kolmogorov-Sinaj entropy was to distinguish non-isomorphic dynamical systems. Two dynamical systems with different entropies cannot be isomorphic.

The notion of the entropy has been extended using fuzzy partitions instead of set partitions (see [11], [12], [2], [3]). Let $\mathcal{T}$ be a tribe of fuzzy sets on Ω, $m : \mathcal{T} - [0, 1]$ is a state on this tribe and a mapping $\tau : \mathcal{T} \to \mathcal{T}$ is given satisfying the following conditions:

(i) If $f \in \mathcal{T}$, then $\tau(f) \in \mathcal{T}$ and $m(f) = m(\tau(f))$.

(ii) If $f, g \in \mathcal{T}$ and $f + g \leq 1$, then $\tau(f + g) = \tau(f) + \tau(g)$.

Then a triplet $(\mathcal{T}, m, \tau)$ is called fuzzy dynamical system. Fuzzy partition is a set of functions $\mathcal{A} = \{f_1, ..., f_k\} \subset \mathcal{T}$ such that $\sum_{i=1}^{k} f_i = 1$. Then we define its entropy

$$H(\mathcal{A}) = \sum_{i=1}^{k} \varphi(m(f_i)) \quad (1)$$

and the conditional entropy

$$H(\mathcal{A}|\mathcal{B}) = \sum_{i=1}^{k} \sum_{j=1}^{l} m(g_j)\varphi\left(\frac{m(f_i g_j)}{m(g_j)}\right), \quad (2)$$

where $\mathcal{B} = \{g_1, ..., g_l\}$ is a fuzzy partition, too. Further

$$h(\mathcal{A}, \tau) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right),$$

and, if $G \subset \mathcal{T}$ is an arbitrary non-empty set, then

$$h_G(\tau) = \sup\{h(\mathcal{A}, \tau); \mathcal{A} \text{ is a fuzzy partition}, \mathcal{A} \subset G\}$$

is an entropy (Kolmogorov-Sinaj type) of the fuzzy dynamical system $(\mathcal{T}, m, \tau)$.

In [5] there was defined a special type of the entropy of dynamical systems based on IF-events. We start with a measurable space $(\Omega, \mathcal{S})$. By an IF-event (see [1]) we consider a pair $A = (\mu_A, \nu_A)$ of $\mathcal{S}$-measurable functions $\mu_A, \nu_A : \Omega \to [0, 1]$, such that $\mu_A + \nu_A \le 1$. Denote by $\mathcal{F}$ the family of all IF-events. On $\mathcal{F}$ we define partial binary operation $\oplus$ and binary operation $\odot$. Namely

$$A \oplus B = (\mu_A, \nu_A) \oplus (\mu_B, \nu_B) =$$

$$= (\mu_A + \mu_B, \nu_A + \nu_B - 1),$$

whenever $\mu_A + \mu_B \le 1$ and $0 \le \nu_A + \nu_B - 1$, and

$$A \odot B = (\mu_A, \nu_A) \odot (\mu_B, \nu_B) =$$

$$= (\mu_A.\mu_B, \nu_A + \nu_B - \nu_A.\nu_B).$$

Further

$$A_n \nearrow A \Longleftrightarrow \mu_{A_n} \nearrow \mu_A, \nu_{A_n} \searrow \nu_A,$$

where $A = (\mu_A, \nu_A), A_n = (\mu_{A_n}, \nu_{A_n}) \in \mathcal{F}$ $(n = 1, 2, ...)$.

Definition 1.1 A mapping $m : \mathcal{F} \to [0, 1]$ is called a state on the family of all IF-events, if the following conditions are satisfied:

(i) $m((1, 0)) = 1, m((0, 1)) = 0$;

(ii) If $A, B, C \in \mathcal{F}$ and $A \oplus B = C$, then $m(A) + m(B) = m(C)$;

(iii) If $A_n \in \mathcal{F}(n = 1, 2, ...), A_n \nearrow A$, then $m(A_n) \nearrow m(A)$.

Theorem 1.2 To any state $m : \mathcal{F} \to [0, 1]$ there exists $\alpha \in [0, 1]$ and a probability measure $P : \mathcal{S} \to [0, 1]$ such that

$$m_\alpha(A) = m(A) =$$

$$= (1 - \alpha) \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$$

for any $A = (\mu_A, \nu_A) \in \mathcal{F}$.

Proof. See [8].

Definition 1.3 Let $m : \mathcal{F} \to [0, 1]$ be a state on the family of all IF-events $\mathcal{F}$ and $\tau : \mathcal{F} \to \mathcal{F}$ be a mapping satisfying the following conditions:

(I) If $A \in \mathcal{F}$, then $\tau(A) \in \mathcal{F}$ and $m(A) = m(\tau(A))$.

(II) If $A, B, C \in \mathcal{F}$ and $A \oplus B = C$, then $\tau(C) = \tau(A) \oplus \tau(B)$.

Then a triplet $(\mathcal{F}, m, \tau)$ is called an IF-dynamical system.

Let a mapping $\tau : \mathcal{F} \to \mathcal{F}$ be defined by $\tau(A) = \tau((\mu_A, \nu_A)) = (\mu_A \circ T, \nu_A \circ T)$. Then $(\mathcal{F}, m_\alpha, \tau)$ is an IF-dynamical system.

2 Entropy of IF-partitions

We shall consider a family of all couples of fuzzy sets

$$\mathcal{M} = \{(f, g); f, g : \Omega \to [0, 1] \text{ are } \mathcal{S}\text{-measurable}\}.$$

We extend the definition of the operation $\oplus$ and $\odot$ from $\mathcal{F}$ to $\mathcal{M}$. Recall that

$$\bigoplus_{i=1}^{k} (\mu_{A_i}, \nu_{A_i}) =$$

$$= \left(\sum_{i=1}^{k} \mu_{A_i}, \left(\sum_{i=1}^{k} \nu_{A_i}\right) - (n - 1)\right)$$

and operations $\oplus, \odot$ fulfill the commutative, associative and distributive law.

Definition 2.1 By an IF-partition we shall mean a finite collection $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\} \subset \mathcal{M}$ such that

$$\bigoplus_{i=1}^{k} (\mu_{A_i}, \nu_{A_i}) = (1, 0).$$

If $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ and $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ..., (\mu_{B_l}, \nu_{B_l})\}$ are two IF-partitions, then we define

$$\mathcal{A} \vee \mathcal{B} = \{(\mu_{A_i}, \nu_{A_i}) \odot (\mu_{B_j}, \nu_{B_j});$$
$$i = 1, ..., k, j = 1, ..., l\}$$

and we write $\mathcal{B} \geq \mathcal{A}$ (and say $\mathcal{B}$ is a refinement of $\mathcal{A}$), if there exists a partition $\{I(1), ..., I(k)\}$ of the set $\{1, 2, ..., l\}$ such that

$$(\mu_{A_i}, \nu_{A_i}) = \bigoplus_{j \in I(i)} (\mu_{B_j}, \nu_{B_j})$$

for every $i = 1, ..., k$.

Proposition 2.2 If $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ and $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ..., (\mu_{B_l}, \nu_{B_l})\}$ are two IF-partitions, then $\tau(\mathcal{A}) = \{\tau((\mu_{A_1}, \nu_{A_1})), ..., \tau((\mu_{A_k}, \nu_{A_k}))\}$ and $\mathcal{A} \vee \mathcal{B}$ are IF-partitions, too. Further $\mathcal{A} \vee \mathcal{B} \geq \mathcal{A}$.

Proof. Since

$$\bigoplus_{i=1}^{k} \tau((\mu_{A_i}, \nu_{A_i})) = \bigoplus_{i=1}^{k} (\mu_{A_i} \circ T, \nu_{A_i} \circ T) =$$

$$= \left(\left(\sum_{i=1}^{k} \mu_{A_i} \right) \circ T, \left(\sum_{i=1}^{k} \nu_{A_i} - (n-1) \right) \circ T \right) =$$

$$= (1 \circ T, 0 \circ T) = (1, 0),$$

so $\tau(\mathcal{A}) = \{\tau((\mu_{A_1}, \nu_{A_1})), ..., \tau((\mu_{A_k}, \nu_{A_k}))\}$ is an IF-partition. Further $\mathcal{A} \vee \mathcal{B} = \{(\mu_{A_i}, \nu_{A_i}) \odot (\mu_{B_j}, \nu_{B_j}); i = 1, ..., k, j = 1, ..., l\}$. Therefore

$$\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{l} (\mu_{A_i}, \nu_{A_i}) \odot (\mu_{B_j}, \nu_{B_j}) =$$

$$= \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{l} (\mu_{A_i} \mu_{B_j}, \nu_{A_i} + \nu_{B_j} - \nu_{A_i} \nu_{B_j}) =$$

$$= \left(\sum_{i=1}^{k} \sum_{j=1}^{l} \mu_{A_i} \mu_{B_j}, \sum_{i=1}^{k} \sum_{j=1}^{l} \nu_{A_i} + \sum_{i=1}^{k} \sum_{j=1}^{l} \nu_{B_j} - \right.$$

$$\left. - \sum_{i=1}^{k} \sum_{j=1}^{l} \nu_{A_i} \nu_{B_j} - (kl - 1) \right) =$$

$$= \left(\left(\sum_{i=1}^{k} \mu_{A_i} \right) \left(\sum_{j=1}^{l} \mu_{B_j} \right), \right.$$

$$\sum_{j=1}^{l} \left(\sum_{i=1}^{k} \nu_{A_i} \right) + \sum_{i=1}^{k} \left(\sum_{j=1}^{l} \nu_{B_j} \right) -$$

$$\left. - \left(\sum_{i=1}^{k} \nu_{A_i} \right) \left(\sum_{j=1}^{l} \nu_{B_j} \right) - (kl - 1) \right) =$$

$$= (1, k(l-1) + l(k-1) -$$

$$-(l-1)(k-1) - (kl-1)) = (1, 0).$$

Finally, let us mention that $\mathcal{A} \vee \mathcal{B}$ is indexed by $\{(i, j); i = 1, ..., n; j = 1, ..., m\}$. Therefore, if we put $I(i) = \{(i, 1), ..., (i, m)\}$, then by the equalities

$$(1, 0) = \bigoplus_{j=1}^{l} (\mu_{B_j}, \nu_{B_j})$$

we obtain

$$(\mu_{A_i}, \nu_{A_i}) = (\mu_{A_i}, \nu_{A_i}) \odot (1, 0) =$$

$$= (\mu_{A_i}, \nu_{A_i}) \odot \left(\bigoplus_{j=1}^{l} (\mu_{B_j}, \nu_{B_j}) \right) =$$

$$= \bigoplus_{j=1}^{l} \left((\mu_{A_i}, \nu_{A_i}) \odot (\mu_{B_j}, \nu_{B_j}) \right) =$$

$$= \bigoplus_{(k,j) \in I(i)} ((\mu_{A_k}, \nu_{A_k}) \odot (\mu_{B_j}, \nu_{B_j}))$$

for every $i = 1, ..., k$. It follows $\mathcal{A} \vee \mathcal{B} \geq \mathcal{A}$.

Proposition 2.3 If $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ and $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ..., (\mu_{B_l}, \nu_{B_l})\}$ are two IF-partitions, then $\mathcal{A}^\flat = \{\mu_{A_1}, ..., \mu_{A_k}\}$ and $\mathcal{A}^\sharp = \{1 - \nu_{A_1}, ..., 1 - \nu_{A_k}\}$ are fuzzy partitions, and

$$(\mathcal{A} \vee \mathcal{B})^\flat = \mathcal{A}^\flat \vee \mathcal{B}^\flat, \quad (\mathcal{A} \vee \mathcal{B})^\sharp = \mathcal{A}^\sharp \vee \mathcal{B}^\sharp,$$

$$(\tau(\mathcal{A}))^\flat = \tau(\mathcal{A}^\flat), \quad (\tau(\mathcal{A}))^\sharp = \tau(\mathcal{A}^\sharp).$$

Proof. Since $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$ and $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ... \; ..., (\mu_{B_l}, \nu_{B_l})\}$, then we have

$$\mathcal{A} \vee \mathcal{B} = \{(\mu_{A_i}, \nu_{A_i}) \odot (\mu_{B_j}, \nu_{B_j});$$

$$i = 1, ..., k, j = 1, ..., l\} =$$

$$= \{(\mu_{A_i}\mu_{B_j}, \nu_{A_i} + \nu_{B_j} - \nu_{A_i}\nu_{B_j});$$

$$i = 1, ..., k, j = 1, ..., l\}$$

and

$$\tau(\mathcal{A}) = \{(\mu_{A_i} \circ T, 1 - \nu_{A_i} \circ T); i = 1, ..., k\}.$$

By [12]

$$(\mathcal{A} \vee \mathcal{B})^\flat = \{\mu_{A_i}\mu_{B_j}; i = 1, ..., k, j = 1, ..., l\} =$$

$$= \{\mu_{A_i}; i = 1, ..., k\} \vee$$

$$\vee \{\mu_{B_j}; j = 1, ..., l\} = \mathcal{A}^\flat \vee \mathcal{B}^\flat$$

and

$$(\mathcal{A} \vee \mathcal{B})^\sharp = \{1 - \nu_{A_i} - \nu_{B_j} + \nu_{A_i}\nu_{B_j};$$

$$i = 1, ..., k, j = 1, ..., l\} =$$

$$= \{(1-\nu_{A_i})(1-\nu_{B_j}); i = 1, ..., k, j = 1, ..., l\} =$$

$$= \{1 - \nu_{A_i}; i = 1, ..., k\} \vee$$

$$\vee \{1 - \nu_{B_j}; j = 1, ..., l\} = \mathcal{A}^\sharp \vee \mathcal{B}^\sharp.$$

Finally we have

$$(\tau(\mathcal{A}))^\flat = \{\mu_{A_i} \circ T; i = 1, ..., k\} =$$

$$= \tau(\{\mu_{A_i}; i = 1, ..., k\}) = \tau(\mathcal{A}^\flat)$$

and

$$(\tau(\mathcal{A}))^\sharp = \{1 - \nu_{A_i} \circ T; i = 1, ..., k\} =$$

$$= \tau(\{1 - \nu_{A_i}; i = 1, ..., k\}) = \tau(\mathcal{A}^\flat).$$

Definition 2.4 If $\mathcal{A}$ is an IF-partition, then we define its entropy (with respect to a given state m_α)

$$H_\alpha(\mathcal{A}) = (1 - \alpha)H(\mathcal{A}^\flat) + \alpha H(\mathcal{A}^\sharp),$$

where H is the entropy of the fuzzy partition (see equation (1)).

Proposition 2.5 If $\mathcal{A} = \{A_1, ..., A_k\}$ and $\mathcal{B} = \{B_1, ..., B_l\}$ are two IF-partitions, then

$$H_\alpha(\mathcal{A} \vee \mathcal{B}) \leq H_\alpha(\mathcal{A}) + H_\alpha(\mathcal{B}).$$

Proof. Put for fixed $i \in \{1, ..., k\}$ and for all $j \in \{1, ..., l\}$

$$\lambda_j = m_\alpha(B_j), \quad x_j = \frac{m_\alpha(A_i \odot B_j)}{m_\alpha(B_j)},$$

where $m_\alpha(B_j) > 0$ ($j = 1, ..., l$). Since

$$\sum_{j=1}^{l} \lambda_j = \sum_{j=1}^{l} m_\alpha(B_j) =$$

$$= m_\alpha\left(\bigoplus_{j=1}^{l} B_j\right) = m_\alpha((1, 0)) = 1$$

and φ is a concave function, we have

$$\varphi(m_\alpha(A_i)) = \varphi(m_\alpha(A_i \odot (1, 0))) =$$

$$= \varphi\left(m_\alpha\left(A_i \odot \left(\bigoplus_{j=1}^{l} B_j\right)\right)\right) =$$

$$= \varphi\left(m_\alpha\left(\bigoplus_{j=1}^{l} A_i \odot B_j\right)\right) =$$

$$= \varphi\left(\sum_{j=1}^{l} m_\alpha(A_i \odot B_j)\right) =$$

$$= \varphi\left(\sum_{j=1}^{l} m_\alpha(B_j)\frac{m_\alpha(A_i \odot B_j)}{m_\alpha(B_j)}\right) =$$

$$= \varphi\left(\sum_{j=1}^{l} \lambda_j x_j\right) \geq \sum_{j=1}^{l} \lambda_j \varphi(x_j) =$$

$$= \sum_{j=1}^{l} m_\alpha(B_j)\varphi(x_j)$$

for all $i \in \{1, ..., k\}$. Therefore

$$H_\alpha(\mathcal{A}) = \sum_{i=1}^{k} \varphi(m_\alpha(A_i)) \geq$$

$$\geq \sum_{i=1}^{k}\sum_{j=1}^{l} m_\alpha(B_j)\varphi(x_j) =$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{l} m_\alpha(A_i \odot B_j) \log \frac{m_\alpha(A_i \odot B_j)}{m_\alpha(B_j)} =$$

$$= \sum_{i=1}^{k}\sum_{j=1}^{l} \varphi(m_\alpha(A_i \odot B_j)) -$$

$$- \sum_{j=1}^{l} \varphi(m_\alpha(B_j)) \sum_{i=1}^{k} m_\alpha(A_i \odot B_j) =$$

$$= H_\alpha(\mathcal{A} \vee \mathcal{B}) - H_\alpha(\mathcal{B}).$$

So we have $H_\alpha(\mathcal{A} \vee \mathcal{B}) \leq H_\alpha(\mathcal{A}) + H_\alpha(\mathcal{B})$.

3 Conditional entropy

Definition 3.1 If $\mathcal{A}$ and $\mathcal{B}$ are two IF-partitions, then we define the conditional entropy (with respect to a given state m_α)

$$H_\alpha(\mathcal{A}|\mathcal{B}) = (1-\alpha)H(\mathcal{A}^\flat|\mathcal{B}^\flat) + \alpha H(\mathcal{A}^\sharp|\mathcal{B}^\sharp),$$

where H is the conditional entropy of fuzzy partitions (see equation (2)).

Proposition 3.2 If $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are IF-partitions, then the following properties are satisfied:

(i) If $\mathcal{B} \leq \mathcal{C}$, then $H_\alpha(\mathcal{A}|\mathcal{C}) \leq H_\alpha(\mathcal{A}|\mathcal{B})$;

(ii) $H_\alpha(\mathcal{B} \vee \mathcal{C}|\mathcal{A}) = H_\alpha(\mathcal{B}|\mathcal{A}) + H_\alpha(\mathcal{C}|\mathcal{B} \vee \mathcal{A})$.

Proof. Let $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_k}, \nu_{A_k})\}$, $\mathcal{B} = \{(\mu_{B_1}, \nu_{B_1}), ..., (\mu_{B_l}, \nu_{B_l})\}$ and $\mathcal{C} = \{(\mu_{C_1}, \nu_{C_1}), ..., (\mu_{C_m}, \nu_{C_m})\}$. Since $\mathcal{B} \leq \mathcal{C}$, there exists a partition $\{I(1), ..., I(l)\}$ of the set $\{1, ..., m\}$ such that

$$(\mu_{B_j}, \nu_{B_j}) = \bigoplus_{t \in I(j)} (\mu_{C_t}, \nu_{C_t}) =$$

$$= \left(\sum_{t \in I(j)} \mu_{C_t}, \sum_{t \in I(j)} \nu_{C_t} - (|I(j)| - 1) \right)$$

for every $j = 1, ..., l$. Therefore

$$\mu_{B_j} = \sum_{t \in I(j)} \mu_{C_t}$$

and

$$1 - \nu_{B_j} = 1 - \sum_{t \in I(j)} \nu_{C_t} + |I(j)| - 1 =$$

$$= \sum_{t \in I(j)} (1 - \nu_{C_t})$$

for every $j = 1, ..., l$. So we obtain

$$\mathcal{B}^\flat = \{\mu_{B_1}, ..., \mu_{B_l}\} \leq \{\mu_{C_1}, ..., \mu_{C_m}\} = \mathcal{C}^\flat$$

and

$$\mathcal{B}^\sharp = \{1 - \nu_{B_1}, ..., 1 - \nu_{B_l}\} \leq$$
$$\leq \{1 - \nu_{C_1}, ..., 1 - \nu_{C_m}\} = \mathcal{C}^\sharp.$$

By [12]

$$H(\mathcal{A}^\flat|\mathcal{C}^\flat) \leq H(\mathcal{A}^\flat|\mathcal{B}^\flat) \quad \text{and}$$
$$H(\mathcal{A}^\sharp|\mathcal{C}^\sharp) \leq H(\mathcal{A}^\sharp|\mathcal{B}^\sharp)$$

and then

$$H_\alpha(\mathcal{A}|\mathcal{C}) = (1-\alpha)H(\mathcal{A}^\flat|\mathcal{C}^\flat) + \alpha H(\mathcal{A}^\sharp|\mathcal{C}^\sharp) \leq$$
$$\leq (1-\alpha)H(\mathcal{A}^\flat|\mathcal{B}^\flat) + \alpha H(\mathcal{A}^\sharp|\mathcal{B}^\sharp) = H_\alpha(\mathcal{A}|\mathcal{B}).$$

Finally, since

$$H(\mathcal{B}^\flat \vee \mathcal{C}^\flat|\mathcal{A}^\flat) = H(\mathcal{B}^\flat|\mathcal{A}^\flat) + H(\mathcal{C}^\flat|\mathcal{B}^\flat \vee \mathcal{A}^\flat)$$

and

$$H(\mathcal{B}^\sharp \vee \mathcal{C}^\sharp|\mathcal{A}^\sharp) = H(\mathcal{B}^\sharp|\mathcal{A}^\sharp) + H^\sharp(\mathcal{C}^\sharp|\mathcal{B}^\sharp \vee \mathcal{A}^\sharp)$$

we have

$$H_\alpha(\mathcal{B} \vee \mathcal{C}|\mathcal{A}) =$$
$$= (1-\alpha)H(\mathcal{B}^\flat \vee \mathcal{C}^\flat|\mathcal{A}^\flat) + \alpha H(\mathcal{B}^\sharp \vee \mathcal{C}^\sharp|\mathcal{A}^\sharp) =$$
$$= (1-\alpha)H(\mathcal{B}^\flat|\mathcal{A}^\flat) + (1-\alpha)H(\mathcal{C}^\flat|\mathcal{B}^\flat \vee \mathcal{A}^\flat) +$$
$$+ \alpha H(\mathcal{B}^\sharp|\mathcal{A}^\sharp) + \alpha H^\sharp(\mathcal{C}^\sharp|\mathcal{B}^\sharp \vee \mathcal{A}^\sharp) =$$
$$= H_\alpha(\mathcal{B}|\mathcal{A}) + H_\alpha(\mathcal{C}|\mathcal{B} \vee \mathcal{A}).$$

4 Entropy on IF-dynamical systems

Proposition 4.1 For any IF-partition $\mathcal{A}$ there exists

$$\lim_{n\to\infty} \frac{1}{n} H_\alpha\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right).$$

Proof. By Proposition 2.5 $H_\alpha(\mathcal{B} \vee \mathcal{C}) \leq H_\alpha(\mathcal{B}) + H_\alpha(\mathcal{C})$ for any IF-partitions $\mathcal{B}$ and $\mathcal{C}$. Put $a_n = H_\alpha\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right)$ for any $n \in \mathbf{N}$. Then $a_{n+m} \leq a_n + a_m$ for every $n, m \in \mathbf{N}$ and this property guarantees the existence of limit

$$\lim_{n\to\infty} \frac{1}{n} a_n = \lim_{n\to\infty} \frac{1}{n} H_\alpha\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right).$$

Definition 4.2 For every IF-partition $\mathcal{A}$ we define

$$h_\alpha(\mathcal{A},\tau) = \lim_{n\to\infty} \frac{1}{n} H_\alpha\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right)$$

and, if $G \subset \mathcal{M}$ is an arbitrary set, then the entropy of IF-dynamical system $(\mathcal{F}, m_\alpha, \tau)$ is

$${}_Gh_\alpha(\tau) = \sup\{h_\alpha(\mathcal{A},\tau); \mathcal{A} \text{ is an IF-partition}, \mathcal{A} \subset G\}.$$

Example 4.3 Let $(\Omega, \mathcal{S}, P, T)$ be a dynamical system, $\tau((\mu_A,\nu_A)) = (\mu_A \circ T, \nu_A \circ T)$, $G = \{(\chi_A, 1-\chi_A); A \in \mathcal{S}\}$. Then the entropy of IF-dynamical system $(\mathcal{F}, m_\alpha, \tau)$ ${}_Gh_\alpha(\tau) = h(T)$ is the Kolmogorov-Sinaj entropy.

Since

$$h(\mathcal{A}^\flat,\tau) = \lim_{n\to\infty} \tfrac{1}{n} H\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A}^\flat)\right) \quad \text{and}$$
$$h(\mathcal{A}^\sharp,\tau) = \lim_{n\to\infty} \tfrac{1}{n} H\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A}^\sharp)\right),$$

then we have

$$h_\alpha(\mathcal{A},\tau) = (1-\alpha)h(\mathcal{A}^\flat,\tau) + \alpha h(\mathcal{A}^\sharp,\tau).$$

Theorem 4.4 Let $\mathcal{C} = \{C_1, ..., C_t\}$ be a measurable partition of Ω being a generator, i.e. $\sigma\left(\bigcup_{i=0}^{\infty} \tau^i(\mathcal{C})\right) = \mathcal{S}$. Then for every IF-partition $\mathcal{A} = \{(\mu_{A_1},\nu_{A_1}), ..., (\mu_{A_k},\nu_{A_k})\}$ there holds

$$h_\alpha(\mathcal{A},\tau) \le h_\alpha(\mathcal{C},\tau) +$$
$$+\int_\Omega \left(\sum_{i=1}^{k} (1-\alpha)\varphi(\mu_{A_i}) + \alpha\varphi(1-\nu_{A_i})\right) dP.$$

Proof. See [5].

Of course this IF-entropy has the following defect.

Proposition 4.5 Let $G = \{(\mu, 1-\mu); \mu(\omega) = c \in [0,1] \text{ for all } \omega \in \Omega\} \subset \mathcal{F}$, then

$${}_Gh_\alpha(\tau) = \infty.$$

Proof. Put $\mathcal{A} = \{(\frac{1}{k}, 1-\frac{1}{k}), ..., (\frac{1}{k}, 1-\frac{1}{k})\}$, where $k \in \mathbf{N}$. Then $\mathcal{A}^\flat = \mathcal{A}^\sharp = \{1/k, ..., 1/k\}$ and

$$\mathcal{A}^\flat \vee \tau(\mathcal{A}^\flat) = \mathcal{A}^\sharp \vee \tau(\mathcal{A}^\sharp) = \{1/k^2, ..., 1/k^2\},$$

hence

$$H(\mathcal{A}^\flat \vee \tau(\mathcal{A}^\flat)) = H(\mathcal{A}^\sharp \vee \tau(\mathcal{A}^\sharp)) =$$
$$= -\sum_{i=1}^{k^2} \frac{1}{k^2} \log \frac{1}{k^2} = 2\log k,$$

and

$$H_\alpha(\mathcal{A} \vee \tau(\mathcal{A})) = 2\log k.$$

Similarly

$$H_\alpha\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right) = n\log k,$$

hence

$$h_\alpha(\mathcal{A},\tau) = \log k.$$

Since $k \in \mathbf{N}$ was arbitrary, we obtain ${}_Gh_\alpha(\tau) \ge \log k$ for every k. Therefore ${}_Gh_\alpha(\tau) = \infty$.

To eliminate this defect we used the Maličký-Riečan modification of the notion of entropy (see [12]).

5 Maličký-Riečan entropy on IF-dynamical systems

Definition 5.1 Let $\mathcal{A}$ be an IF-partition. Then we define its Maličký-Riečan entropy by the formula

$$H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^k(\mathcal{A})) =$$
$$= \inf\{H_\alpha(\mathcal{C}); \mathcal{C} \ge \mathcal{A}, \mathcal{C} \ge \tau(\mathcal{A}), ..., \mathcal{C} \ge \tau^k(\mathcal{A})\}.$$

Proposition 5.2 There exists

$$\lim_{n\to\infty} \frac{1}{n} H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{n-1}(\mathcal{A})).$$

Proof. Put

$$a_n = H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{n-1}(\mathcal{A})).$$

Then

$$a_{n+m} = H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{n+m-1}(\mathcal{A})) \le$$
$$\le H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{n-1}(\mathcal{A})) +$$
$$+H_\alpha(\tau^n(\mathcal{A}), \tau^{n+1}(\mathcal{A}), ..., \tau^{n+m-1}(\mathcal{A})) =$$
$$H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{n-1}(\mathcal{A})) +$$

$$+H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{m-1}(\mathcal{A})) = a_n + a_m.$$

This property guarantees existence of

$$\lim_{n\to\infty} \frac{1}{n} H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{n-1}(\mathcal{A})).$$

Definition 5.3 For an IF-partition $\mathcal{A}$ define the entropy

$$\overline{h_\alpha}(\mathcal{A}, \tau) = \lim_{n\to\infty} \frac{1}{n} H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{n-1}(\mathcal{A})),$$

and for arbitrary $G \subset \mathcal{F}$ the Maličký-Riečan entropy of an IF-dynamical system $(\mathcal{F}, m_\alpha, \tau)$ by the formula

$$_G\overline{h_\alpha}(\tau) = \sup\{\overline{h_\alpha}(\mathcal{A}, \tau); \mathcal{A} \text{ is an IF-partition}, \mathcal{A} \subset G\}.$$

Proposition 5.4 It holds $h(T) \le {}_G\overline{h_\alpha}(\tau) \le {}_G h_\alpha(\tau)$ if $G = \{(\chi_E, 1 - \chi_E); E \in \mathcal{S}\}$.

Proof. If $\mathcal{A}$ is an IF-partition, then by Proposition 2.2

$$\mathcal{A} \le \bigvee_{i=0}^{n-1} \tau^i(\mathcal{A}), \quad \tau(\mathcal{A}) \le \bigvee_{i=0}^{n-1} \tau^i(\mathcal{A}), \quad ...$$

$$... \quad , \tau^{n-1}(\mathcal{A}) \le \bigvee_{i=0}^{n-1} \tau^i(\mathcal{A}),$$

hence $H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{n-1}(\mathcal{A})) \le H_\alpha\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right)$ and ${}_G\overline{h_\alpha}(\tau) \le {}_G h_\alpha(\tau)$. If $G = \{(\chi_E, 1 - \chi_E); E \in \mathcal{S}\}$, then for every crisp partition $\mathcal{A}$ the relations $\mathcal{A} \le \mathcal{C}, \tau(\mathcal{A}) \le \mathcal{C}, ..., \tau^{n-1}(\mathcal{A}) \le \mathcal{C}$ imply $\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A}) \le \mathcal{C}$. Hence $H_\alpha\left(\bigvee_{i=0}^{n-1} \tau^i(\mathcal{A})\right) \le H_\alpha(\mathcal{C})$, and $h_\alpha(\mathcal{A}, \tau) \le \overline{h_\alpha}(\mathcal{A}, \tau)$, and $h(T) \le {}_G\overline{h_\alpha}(\tau)$ (see Example 4.3).

Theorem 5.5 Let G consists of all IF-events of the form $\sum_{i=1}^{n} a_i(\chi_{E_i}, 1 - \chi_{E_i})$, where $\{E_1, ..., E_n\}$ is a set partition of Ω and $a_i \in [0, 1] \cap \mathbf{Q}$. Then $\overline{h}_G(\tau) = h(T)$.

Proof. It suffices to prove $\overline{h}_G(\tau) \le h(T)$. Let $\mathcal{A} = \{(\mu_{A_1}, \nu_{A_1}), ..., (\mu_{A_m}, \nu_{A_m})\}$ be an IF-partition. Every $(\mu_{A_j}, \nu_{A_j}); j = 1, 2, ..., m$ is of the form

$$\sum_{i=1}^{n_j} a_{ij}(\chi_{E_i}, 1 - \chi_{E_i}),$$

where $a_{ij} \in [0, 1] \cap \mathbf{Q}$ and $\mathcal{B} = \{E_1, ..., E_n\}$ is a set partition. There are natural s_{ij} and integers $p_{ij} \in \{0, 1, ..., s_{ij}\}$ such that $a_{ij} = p_{ij}/s_{ij}$. Let s be the smallest common multiple of all $s_{ij}; i = 1, ..., n$ and $j = 1, ..., m$. There are integers $r_{ij} \in \{0, 1, ..., s\}$ such that $a_{ij} = r_{ij}/s$. Denote by $\mathcal{B}_n$ the set partition

$$\mathcal{B} \vee T^{-1}(\mathcal{B}) \vee ... \vee T^{-(n-1)}(\mathcal{B}),$$

which consists of some measurable sets $\{U_1, ..., U_k\}$. Let A_{ij} be an IF-event defined by the formula

$$A_{ij} = \frac{1}{s}(\chi_{E_i}, 1-\chi_{E_i}); i = 1, ..., n, j = 1, ..., m.$$

If $\mathcal{A}_n = \{A_{ij}; i = 1, ..., n, j = 1, ..., m\}$, then $\mathcal{A}_n \ge \tau^i(\mathcal{A})$ for all $i = 0, 1, ..., n - 1$. So we have

$$H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{n-1}(\mathcal{A})) \le H(\mathcal{A}_n) =$$

$$= -\sum_{i=1}^{s}\sum_{j=1}^{k} \frac{P(U_j)}{s} \log \frac{P(U_j)}{s} =$$

$$= -\sum_{j=1}^{k} s\frac{P(U_j)}{s}(\log P(U_j) - \log s) =$$

$$= -\sum_{j=1}^{k} P(U_j) \log P(U_j) + \sum_{j=1}^{k} P(U_j) \log s =$$

$$= \log s - \sum_{j=1}^{k} P(U_j) \log P(U_j) = \log s + H(\mathcal{B}_n).$$

Since s does not depend on n, we have

$$\overline{h_\alpha}(\mathcal{A}, \tau) = \lim_{n\to\infty} \frac{1}{n} H_\alpha(\mathcal{A}, \tau(\mathcal{A}), ..., \tau^{n-1}(\mathcal{A})) \le$$

$$\le \lim_{n\to\infty} \left(\frac{\log s}{n} + \frac{1}{n} H(\mathcal{B}_n)\right) =$$

$$= 0 + \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{B})\right) = h(\mathcal{B}, T).$$

This implies the inequality ${}_G\overline{h_\alpha}(\tau) \leq h(T)$ and the equality ${}_G\overline{h_\alpha}(\tau) = h(T)$.

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