The problem of distributivity between binary operations in bifuzzy set theory

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Abstract

In this work we study the functional equation given by the following condition

$$\mathcal{F}(x, \mathcal{G}(y, z)) = \mathcal{G}(\mathcal{F}(x, y), \mathcal{F}(x, z))$$

for all $x, y, z \in L^*$, i.e. distributivity equation.
The problem of distributivity is of great interest both for particular as well as fundamental reasons. This relates for instance to the theory of binary operations like triangular norms, triangular conorms etc. In this contribution we consider the distributivity between the operations on bifuzzy set theory. Mainly, we consider the decomposable operations (Theorem 1). In this class of operations the distributivity equation is equivalent to the distributivity between underlying operations. So, in this case we should consider the distributivity in the class of operations on the unit interval.

Keywords: Bifuzzy sets, interval valued fuzzy sets, L-fuzzy sets, t-norms, distributivity, decomposable operations.

1 Introduction

In this paper we study the distributivity equation given by the following condition

$$\underset{x,y,z \in L^*}{\forall} \mathcal{F}(x, \mathcal{G}(y, z)) = \mathcal{G}(\mathcal{F}(x, y), \mathcal{F}(x, z)).$$

The problem of distributivity has been posed many years ago (cf. Aczél [1], pp. 318-319). A new direction of investigations is mainly concerned of distributivity between triangular norms and triangular conorms ([11] p.17, [25]), aggregation functions ([5]), fuzzy implications ([4], [23], [24]), uninorms and nullnorms ([17], [20], [21], [22]).

In Section 2 we put the definitions of a fuzzy set, a bifuzzy set (an intuitionistic fuzzy set), an interval valued fuzzy set and an L-fuzzy set. Next, we recall relationships between them.
In Sections 3 we put properties of binary operations and a description of decomposable operations.
In Section 4 we recall the definition of left and right distributivity. Next, solutions of distributivity equations from described families are characterized.

2 Bifuzzy sets

First we put basic definitions.

Definition 1 ([27])**.** A fuzzy set A in a universe X is a mapping

$$A : X \to [0, 1].$$

Example 1. The mapping $A : \mathbb{R} \to [0, 1]$ given by following formula

$$A(x) = \frac{\text{arctg} x}{\pi} + \frac{1}{2}$$

is a fuzzy set on $\mathbb{R}$.

Definition 2 (cf. [2], [3])**.** A bifuzzy set (an intuitionistic fuzzy set) A in a universe X is

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a triple

$$A = \{(x, \mu(x), \nu(x)) : x \in X\}$$

where $\mu, \nu : X \to [0,1]$ and $\mu(x) + \nu(x) \le 1$, $x \in X$.

Example 2. The triple

$$A = \{(x, \mu(x), \nu(x)) : x \in \mathbb{R}\}$$

where

$$\mu(x) = \begin{cases} \frac{1}{2}x & if\ x \in [0,2], \\ 0 & otherwise, \end{cases}$$

$$\nu(x) = \begin{cases} \frac{2}{3} - \frac{1}{3}x & if\ x \in [-1,2], \\ 0 & otherwise \end{cases}$$

is a bifuzzy set on $\mathbb{R}$.

We use the name bifuzzy set instead of the intuitionistic fuzzy set, because there is no terminological difficulties with this name (cf. [14]) and in fact, a bifuzzy set is described by two fuzzy sets μ and ν.

Definition 3 (cf. [9])**.** An interval valued fuzzy set A in a universe X is a mapping $A : X \to Int([0,1])$, where $Int([0,1])$ denotes the set of all closed subintervals of $[0,1]$, i.e. a mapping which assigns to each element $x \in X$ the interval $[\underline{A}(x), \overline{A}(x)]$, where $\underline{A}(x), \overline{A}(x) \in [0,1]$ such that $\underline{A}(x) \le \overline{A}(x)$.

Definition 4 ([13])**.** An L-fuzzy set A in a universe X is a function $A : X \to L$ where L is a lattice.

It was shown in [7] that bifuzzy sets, interval valued fuzzy sets and L^*-fuzzy sets are equivalent, where

$$L^* = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \le 1\},$$

$(x_1, x_2) \le (y_1, y_2) \Leftrightarrow x_1 \le y_1$ and $x_2 \ge y_2$ for all $(x_1, x_2), (y_1, y_2) \in L^*$.

The operation $\wedge$ and $\vee$ on L^* are defined as follows

$$(x_1, x_2) \wedge (y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2)),$$

$$(x_1, x_2) \vee (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2)).$$

The greatest element in L^* is $1_{L^*} = (1,0)$. The least element in L^* is $0_{L^*} = (0,1)$.

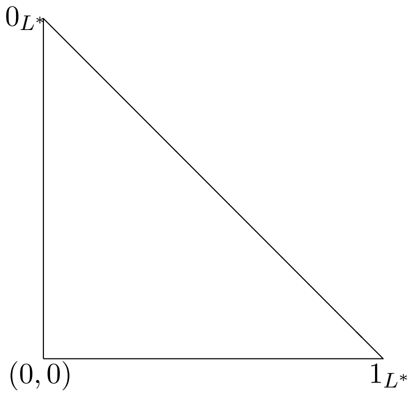

Figure 1: Lattice L^*

3 Binary operation

Since bifuzzy sets, interval valued fuzzy sets and L^*-fuzzy set are equivalent, in this paper we will consider only the binary operations $\mathcal{F} : (L^*)^2 \to L^*$. First we recall some basic properties of binary operations.

Definition 5 (c.f. [12])**.** A binary operation $\mathcal{F}$ is idempotent in L^* if

$$\underset{x \in L^*}{\forall} \mathcal{F}(x, x) = x. \tag{1}$$

It is associative if

$$\underset{x,y,z \in L^*}{\forall} \mathcal{F}(x, \mathcal{F}(y, z)) = \mathcal{F}(\mathcal{F}(x, y), z). \tag{2}$$

It is commutative if

$$\underset{x,y \in L^*}{\forall} \mathcal{F}(x, y) = \mathcal{F}(y, x). \tag{3}$$

It has a neutral element $e \in L^*$ if

$$\underset{x \in L^*}{\forall} \mathcal{F}(x, e) = \mathcal{F}(e, x) = x. \tag{4}$$

The operation $\mathcal{F}$ is increasing in $(L^*, \le)$ if

$$\underset{x,y,z \in L^*}{\forall} (x \le y) \Rightarrow \quad (\mathcal{F}(x, z) \le \mathcal{F}(y, z), \mathcal{F}(z, x) \le \mathcal{F}(z, y)). \tag{5}$$

Lemma 1 ([6])**.** *Let $\mathcal{F} : (L^*)^2 \to L^*$ be an increasing operation. If the operation $\mathcal{F}$ is idempotent, then*

$$\wedge \le \mathcal{F} \le \vee. \tag{6}$$

Lemma 2 ([6])**.** *Let $\mathcal{F} : (L^*)^2 \to L^*$ be an increasing operation. If the operation $\mathcal{F}$ has a neutral element $e = 1_{L^*}$ ($e = 0_{L^*}$), then $\mathcal{F} \le \wedge$ ($\mathcal{F} \ge \vee$).*

Corollary 1. Let $\mathcal{F} : (L^*)^2 \to L^*$ be an increasing operation. If the operation $\mathcal{F}$ is idempotent and has a neutral element $e = 1_{L^*}$ ($e = 0_{L^*}$), then $\mathcal{F} = \wedge$ ($\mathcal{F} = \vee$).

Definition 6. An operation $\mathcal{F} : (L^*)^2 \to L^*$ is called decomposable if there exist operations $F_1, F_2 : [0,1]^2 \to [0,1]$ such that for all $x, y \in L^*$

$$\mathcal{F}(x,y) = (F_1(x_1,y_1), F_2(x_2,y_2)),$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$.

Lemma 3. *Increasing operations $F_1, F_2 : [0,1]^2 \to [0,1]$ lead to the decomposable operation $\mathcal{F}$ if and only if $F_1 \leq F_2^*$ where F_2^* is a dual operation to the operation F_2, i.e. $F_2^*(x,y) = 1 - F_2(1-x, 1-y)$.*

Lemma 4. *Let $\mathcal{F} : (L^*)^2 \to L^*$ be a decomposable operation. The operation $\mathcal{F}$ is idempotent if and only if underlying operations $F_1, F_2 : [0,1]^2 \to [0,1]$ are idempotent.*

Definition 7 (c.f. [8], [15])**.** A triangular norm $\mathcal{T}$ on L^* is an increasing, commutative, associative operation $\mathcal{T} : (L^*)^2 \to L^*$ with a neutral element 1_{L^*}.
A triangular conorm $\mathcal{S}$ on L^* is an increasing, commutative, associative operation $\mathcal{S} : (L^*)^2 \to L^*$ with a neutral element 0_{L^*}.

Example 3. The following are examples of t-norms on L^*

$$\begin{aligned} \inf(x,y) = &\ (\min(x_1,y_1), \max(x_2,y_2)), \\ \mathcal{T}(x,y) = &\ (\max(0, x_1+y_1-1), \\ &\ \min(1, x_2+y_2)), \end{aligned}$$

and t-conorm on L^*

$$\sup(x,y) = (\max(x_1,y_1), \min(x_2,y_2)).$$

Remark 1 (c.f. [8])**.** *A decomposable t-norm $\mathcal{T}$ on L^* is also called a t-representable t-norm. In this case there exist a t-norm T and t-conorm S on $[0,1]$ such that for all $x, y \in L^*$*

$$\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2)).$$

Example 4. The operation

$$\mathcal{T}(x,y) = (\max(x_1+y_1-1, 0), x_2+y_2-x_2y_2)$$

is a decomposable t-norm, with the Łukasiewicz t-norm T and the product t-conorm S.
However, the Łukasiewicz t-norm

$$\begin{aligned} \mathcal{T}_W(x,y) &= (\max(0, x_1+y_1-1), \\ &\quad \min(1, x_2+1-y_1, y_2+1-x_1)) \end{aligned}$$

is not decomposable.

Remark 2 (c.f. [8])**.** *A decomposable t-conorm $\mathcal{S}$ on L^* is also called t-representable t-conorm. In this case there exist a t-conorm S and t-norm T on $[0,1]$ such that for all $x, y \in L^*$*

$$\mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2)).$$

Definition 8 (c.f. [9], [26])**.** An operation $\mathcal{U} : (L^*)^2 \to L^*$ is called a uninorm if it is commutative, associative, increasing and has a neutral element $e \in L^*$.

Remark 3 (c.f. [9])**.** *A decomposable uninorm $\mathcal{U}$ on L^* is also called t-representable uninorm. In this case there exist uninorms U_1 and U_2 on $[0,1]$ such that for all $x, y \in L^*$*

$$\mathcal{U}(x,y) = (U_1(x_1,y_1), U_2(x_2,y_2)).$$

4 Distributivity equation

Now we consider the distributivity between operations on L^* (left distributivity). The considerations for right distributivity are similar.

Definition 9 ([12])**.** An operation $\mathcal{F}$ is left distributive over an operation $\mathcal{G}$ in L^* if

$$\underset{x,y,z \in L^*}{\forall} \mathcal{F}(x, \mathcal{G}(y,z)) = \mathcal{G}(\mathcal{F}(x,y), \mathcal{F}(x,z)). \tag{7}$$

An operation $\mathcal{F}$ is right distributive over an operation $\mathcal{G}$ in L^* if

$$\underset{x,y,z \in L^*}{\forall} \mathcal{F}(\mathcal{G}(y,z), x) = \mathcal{G}(\mathcal{F}(y,x), \mathcal{F}(z,x)). \tag{8}$$

First we consider the distributivity equation in the class of decomposable operations

Theorem 1. *Let $\mathcal{F}, \mathcal{G} : (L^*)^2 \to L^*$ be two decomposable binary operations such that $\mathcal{F} = (F_1, F_2)$, $\mathcal{G} = (G_1, G_2)$. Operation $\mathcal{F}$ is left (right) distributive over the operation $\mathcal{G}$ if and only if operation F_1 is left (right) distributive over the operation G_1 and operation F_2 is left (right) distributive over the operation G_2.*

Proof. Let operation $\mathcal{F}$ be left distributive over the operation $\mathcal{G}$. Then

$$\begin{aligned}
&(F_1(x_1, G_1(y_1, z_1)), F_2(x_2, G_2(y_2, z_2))) \\
=\;& \mathcal{F}((x_1, x_2), (G_1(y_1, z_1), G_2(y_2, z_2))) \\
=\;& \mathcal{F}((x_1, x_2), \mathcal{G}((y_1, y_2), (z_1, z_2))) \\
=\;& \mathcal{F}(x, \mathcal{G}(y, z)) \\
=\;& \mathcal{G}(\mathcal{F}(x, y), \mathcal{F}(x, z)) \\
=\;& \mathcal{G}(\mathcal{F}((x_1, x_2), (y_1, y_2)), \\
& \mathcal{F}((x_1, x_2), (z_1, z_2))) \\
=\;& \mathcal{G}((F_1(x_1, y_1), F_2(x_2, y_2)), \\
& (F_1(x_1, z_1), F_2(x_2, z_2))) \\
=\;& (G_1(F_1(x_1, y_1), F_1(x_1, z_1)), \\
& G_2(F_2(x_2, y_2), F_2(x_2, z_2))).
\end{aligned}$$

So, we have

$$F_1(x_1, G_1(y_1, z_1)) = G_1(F_1(x_1, y_1), F_1(x_1, z_1))$$

and

$$F_2(x_2, G_2(y_2, z_2)) = G_2(F_2(x_2, y_2), F_2(x_2, z_2))$$

which means that F_1 is left distributive over the G_1 and F_2 is left distributive over G_2.
Conversely, if F_1 is left distributive over the G_1 and F_2 is left distributive over G_2 then

$$\begin{aligned}
&\mathcal{F}(x, \mathcal{G}(y, z)) \\
=\;& \mathcal{F}((x_1, x_2), \mathcal{G}((y_1, y_2), (z_1, z_2))) \\
=\;& \mathcal{F}((x_1, x_2), (G_1(y_1, z_1), G_2(y_2, z_2))) \\
=\;& (F_1(x_1, G_1(y_1, z_1)), F_2(x_2, G_2(y_2, z_2))) \\
=\;& (G_1(F_1(x_1, y_1), F_1(x_1, z_1)), \\
& G_2(F_2(x_2, y_2), F_2(x_2, z_2))) \\
=\;& \mathcal{G}((F_1(x_1, y_1), F_2(x_2, y_2)), \\
& (F_1(x_1, z_1), F_2(x_2, z_2))) \\
=\;& \mathcal{G}(\mathcal{F}((x_1, x_2), (y_1, y_2)), \\
& \mathcal{F}((x_1, x_2), (z_1, z_2))) \\
=\;& \mathcal{G}(\mathcal{F}(x, y), \mathcal{F}(x, z))
\end{aligned}$$

which means that operation $\mathcal{F}$ is left distributive over the operation $\mathcal{G}$.
The proof for right distributivity is analogous. □

So, we may use the results from the papers about the distributivity equation for t-norms, t-conorms, uninorms and another operations on $[0, 1]$. Below we mention some of them and their application to the distributivity between operations on L^*.

Lemma 5 ([21]). *Let T be a t-norm and S be a t-conorm on $[0, 1]$. The operation T is left (or right) distributive over the operation S if and only if $S = \max$.*

Lemma 6 ([21]). *Let T be a t-norm and S be a t-conorm on $[0, 1]$. The operation S is left (or right) distributive over the operation T if and only if $T = \min$.*

Theorem 2. *Let $\mathcal{T}$ be a decomposable t-norm and $\mathcal{S}$ be a decomposable t-conorm on L^*. The operation $\mathcal{T}$ is left (or right) distributive over the operation $\mathcal{S}$ if and only if $\mathcal{S} = (\max, \min)$.*

Corollary 2. Every increasing decomposable operation is distributive over t-conorm $\mathcal{S} = (\max, \min)$.

Theorem 3. *Let $\mathcal{T}$ be a decomposable t-norm and $\mathcal{S}$ be a decomposable t-conorm on L^*. The operation $\mathcal{S}$ is left (or right) distributive over the operation $\mathcal{T}$ if and only if $\mathcal{T} = (\min, \max)$.*

Corollary 3. Every increasing decomposable operation is distributive over t-norm $\mathcal{T} = (\min, \max)$.

5 Distributivity in the class of nondecomposable operations

In this section we consider the distributivity equation in the class of more general operations, i.e. we omit the assumption that the operations are decomposable. Our consideration leads to the similar results as for operations on the unit interval.

Lemma 7 (c.f. [20]). *Let $\mathcal{F} : (L^*)^2 \to L^*$ has a neutral element e in a subset $Y \subset L^*$ (i.e. $\forall_{x \in Y} \mathcal{F}(e, x) = \mathcal{F}(x, e) = x$). If the operation $\mathcal{F}$ is left (or right) distributive over an operation $\mathcal{G} : (L^*)^2 \to L^*$ fulfilling $\mathcal{G}(e, e) = e$, then $\mathcal{G}$ is idempotent in Y.*

Lemma 8 (c.f. [10]). *If an operation $\mathcal{F}$ with a neutral element $s \in L^*$ is left (or right) distributive over an operation $\mathcal{G}$ such that $\mathcal{G}(s, s) = s$, then the operation $\mathcal{G}$ is idempotent.*

Directly from Lemma 8 and Corollary 1 we obtain

Theorem 4 ([10], Theorem 5). *If an operation $\mathcal{F}$ with a neutral element $s \in L^*$ is* *left (or right) distributive over an operation $\mathcal{G}$ with neutral element $e = 0$ ($e = 1$) and $\mathcal{G}(s,s) = s$, then $\mathcal{G} = \vee$ ($\mathcal{G} = \wedge$).*

Corollary 4. If a t-norm $\mathcal{T}$ is left (or right) distributive over a t-conorm $\mathcal{S}$, then $\mathcal{S} = \vee$.

Corollary 5. If a t-conorm $\mathcal{S}$ is left (or right) distributive over a t-norm $\mathcal{T}$, then $\mathcal{T} = \wedge$.

6 Conclusion

In this paper we present the problem of distributivity between binary operations on L^*. The main result is presented in Theorem 1. Using this theorem we may transform all results concerning the problem of distributivity equation for operations on the unit interval into the field of operations on L^*.

References

[1] Aczél J. (1966). Lectures on Functional Equations and their Applications, Acad. Press, New York, .

[2] Atanassov K.T. (1986). Intuitionistic fuzzy sets. *Fuzzy Sets Syst.*, 20, 87–96.

[3] Atanassov K.T. (1999). Intuitionistic Fuzzy Sets. Springer-Verlag, Heidelberg.

[4] Baczyński M. (2001). On a class of distributive fuzzy implications *Internat. J. Uncertainty, Fuzzines Knowledge-Based Syst.* 9, 229-238.

[5] Calvo T. (1999), On some solutions of the distributivity equation, *Fuzzy Sets Syst.* 104, 85–96.

[6] Czogała E., Drewniak J. (1984), Associative monotonic operations in fuzzy set theory, *Fuzzy Sets Syst.* 12, 249–269.

[7] Deschrijver G., Kerre E.E. (2003). On the relationship beetwen some extensions of fuzzy set theory. *Fuzzy Sets Syst.*, 133, 227–235.

[8] Deschrijver G., Cornelis C., Kerre E.E. (2004). On the Representation of Intuitonistic Fuzzy t-Norms and t-Conorms. *IEEE Transactions on Fuzzy Syst.*, 12, 45–61.

[9] Deschrijver G., Kerre E.E. (2004). Uninorms in L^*-fuzzy set theory. *Fuzzy Sets Syst.* 148, 243–262.

[10] Drewniak J. (1983). Binary operations on fuzzy sets. *BUSEFAL* 14, 69–74.

[11] Fodor J., Roubens M. (1994). Fuzzy Preference Modeling and Multicriteria Decision Support, Kluwer Acad. Publ., New York.

[12] Fuchs L. (1963), Partially Ordered Algebraic Systems, Pergamon Press, Oxford.

[13] Goguen A. (1967). L-fuzzy sets. *J. Math. Anal. Appl.* 18, 145–174.

[14] Grzegorzewski P., Mrówka E. (2005). Some notes on (Atanassov's) intuitionistic fuzzy sets. *Fuzzy Sets Syst.* 156, 492–495.

[15] Klement E.P., Mesiar R., Pap E. (2000). Triangular norms. Kluwer Acad. Publ., Dordrecht.

[16] Klir G.J., Yuan B. (1995). Fuzzy Sets and Fuzzy Logic, Theory and Application, Prentice Hall PTR, Upper Saddle River, New Jersey.

[17] Mas M., Mayor G., Torrens J. (2002). The distributivity condition for uninorms and t-operators. *Fuzzy Sets Syst.* 128, 209-225.

[18] Moore R.E. (1966). Interval Analysis, Prentice-Hall, Englewood Cliffs, N. J.

[19] Pękala B. (2007). Operations on interval matrices, in: M. Kryszkiewicz et al. (eds.), Rough Sets and Intelligent Systems Paradigms, Springer-Verlag, Berlin, 613–620.

[20] Rak E. (2005). Distributivity equation for nullnorms. *J. Electrical Engin.* 56, 12/s, 53-55.

[21] Rak E., Drygaś P. (2006). Distributivity between uninorms. *J. Electrical Engin.* 57, 7/s, 35-38.

[22] Ruiz D., Torrens J. (2003). Distributive idempotent uninorms. *Internat. J. Uncertainty, Fuzzines Knowledge-Based Syst.* 11, 413-428.

[23] Ruiz-Aguilera D., Torrens J. (2006). Distributivity of strong implications over conjunctive and disjunctive uninorms. *Kybernetika* 42, 319-336.

[24] Ruiz-Aguilera D., Torrens J. (2007). Distributivity of residual implications over conjunctive and disjunctive uninorms. *Fuzzy Sets Syst.* 158, 23-37.

[25] Sander W., Siedekum J. (2005). Multiplication, Distributivity and Fuzzy-Integral I, II, III *Kybernetika* 41, 397-422; 469-496; 497-518.

[26] Yager R., Rybalov A. (1996). Uninorm aggregation operators. *Fuzzy Sets Syst.*, 80, 111–120.

[27] Zadeh L.A. (1965). Fuzzy sets. *Inform. and Control*, 8, 338–353.