Abstract

In this paper we consider a special class of t-norms $T_{1,2,t}$ on the lattice $L^I$, where $L^I$ is the underlying lattice of both intuitionistic fuzzy set theory (Atanassov, 1983) and interval-valued fuzzy set theory (Sambuc, 1975). We investigate under which conditions these t-norms are meet-morphisms. Using these results, we obtain a characterization for t-norms on $L^I$ which are both join- and meet-morphisms and which satisfy an additional condition.

Keywords: Triangular norm, join-morphism, meet-morphism, Atanassov’s intuitionistic fuzzy set.

1 Introduction

Atanassov’s intuitionistic fuzzy set theory [1, 2] is an extension of fuzzy set theory in which to each element of the universe a membership and a non-membership degree is assigned. Unlike in fuzzy set theory, the sum of these two degrees is only required to be less than or equal to 1. Interval-valued fuzzy set theory [13, 16] is another extension of fuzzy set theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. In [9] it is shown that Atanassov’s intuitionistic fuzzy set theory is equivalent to interval-valued fuzzy set theory and that both are equivalent to $L$-fuzzy set theory in the sense of Goguen [12] w.r.t. a special lattice $L^I$.

Triangular norms on the unit interval are all join- and meet-morphism, since the unit interval is a chain. On the lattice $L^I$, however, the situation is more complicated, as there exist t-norms which are not a join- or an inf-morphism. There exist several characterizations for t-norms on $L^I$ which are join-morphisms and which satisfy additional conditions (see e.g. [5, 7, 8]). In this paper we start the research on meet-morphisms. We start from the class of t-norms $T_{1,2,t}$ introduced in [8] and we investigate under which conditions the t-norms of this class are meet-morphisms. We also show that there are t-norms in this class for which the t-norms $T_1$ and $T_2$ involved in the construction are not equal to each other. Finally, we give a characterization of t-norms on $L^I$ which are join- and meet-morphisms and which satisfy an additional condition.

2 The lattice $L^I$

Definition 2.1 We define $L^I = (L^I, \leq_{L^I})$, where $L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0, 1]^2 \text{ and } x_1 \leq x_2\}$, $[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2)$, for all $[x_1, x_2], [y_1, y_2]$ in $L^I$.

Similarly as Lemma 2.1 in [9] it can be shown that $L^I$ is a complete lattice.

Definition 2.2 [13, 16] An interval-valued
fuzzy set on U is a mapping $A : U \rightarrow L^I$.

**Definition 2.3** [1, 2] An intuitionistic fuzzy set in the sense of Atanassov on U is a set

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

where $\mu_A(u) \in [0, 1]$ denotes the membership degree and $\nu_A(u) \in [0, 1]$ the non-membership degree of u in A and where for all $u \in U$, $\mu_A(u) + \nu_A(u) \leq 1$.

An intuitionistic fuzzy set in the sense of Atanassov A on U can be represented by the $L^I$-fuzzy set $A$ given by

$$A : U \rightarrow L^I : u \mapsto [\mu_A(u), 1 - \nu_A(u)].$$

In Figure 1 the set $L^I$ is shown. Note that $x = [x_1, x_2] \in L^I$ is identified with the point $(x_1, x_2) \in \mathbb{R}^2$.

![Figure 1: The grey area is $L^I$.](image)

In the sequel, if $x \in L^I$, then we denote its bounds by $x_1$ and $x_2$, i.e. $x = [x_1, x_2]$. The length $x_2 - x_1$ of the interval $x \in L^I$ is called the degree of uncertainty and is denoted by $x_T$. The smallest and the largest element of $L^I$ are given by $0_{L^I} = [0, 0]$ and $1_{L^I} = [1, 1]$. Note that, for $x, y \in L^I$, $x \leq L^I y$ is equivalent to $x \leq y$ and $x \neq y$, i.e. either $x_1 < y_1$ and $x_2 \leq y_2$, or $x_1 \leq y_1$ and $x_2 < y_2$. We define the relation $\ll L^I$ by $x \ll L^I y \iff x_1 < y_1$ and $x_2 < y_2$, for $x, y \in L^I$. We define for further use the set $D = \{[x_1, x_1] \mid x_1 \in [0, 1]\}$.

Note that for any non-empty subset $A$ of $L^I$ it holds that

$$\sup A = \left\{ \sup \{x_1 \mid x_1 \in [0, 1] \text{ and } (\exists x_2 \in [x_1, 1]) ([x_1, x_2] \in A) \} \right\},$$
$$\sup \{x_2 \mid x_2 \in [0, 1] \text{ and } (\exists x_1 \in [0, x_2]) ([x_1, x_2] \in A) \} ;$$
$$\inf A = \left\{ \inf \{x_1 \mid x_1 \in [0, 1] \text{ and } (\exists x_2 \in [x_1, 1]) ([x_1, x_2] \in A) \} \right\},$$
$$\inf \{x_2 \mid x_2 \in [0, 1] \text{ and } (\exists x_1 \in [0, x_2]) ([x_1, x_2] \in A) \} .$$

**Theorem 2.4** (Characterization of supremum in $L^I$) [7] Let $A$ be an arbitrary non-empty subset of $L^I$ and $a \in L^I$. Then $a = \sup A$ if and only if

$$(\forall x \in A)(x \leq L^I a)$$

and $(\forall \varepsilon > 0)(\exists z \in A)(z_1 > a_1 - \varepsilon_1)$

and $(\forall \varepsilon_2 > 0)(\exists z \in A)(z_2 > a_2 - \varepsilon_2)$.

**Definition 2.5** A t-norm on $L^I$ is a commutative, associative, increasing mapping $T : (L^I)^2 \rightarrow L^I$ which satisfies $T(1_{L^I}, x) = x$, for all $x \in L^I$.

A t-conorm on $L^I$ is a commutative, associative, increasing mapping $S : (L^I)^2 \rightarrow L^I$ which satisfies $S(0_{L^I}, x) = x$, for all $x \in L^I$.

**Example 2.6** [8, 10] We give some special classes of t-norms on $L^I$. Let $T, T_1$ and $T_2$ be t-norms on $([0, 1], \leq)$ such that $T_1(x_1, y_1) \leq T_2(x_1, y_1)$ for all $x_1, y_1 \in [0, 1]$, and let $t \in [0, 1]$. Then we have the following classes:

- t-representable t-norms: $T_{T_1, T_2}(x, y) = [T_1(x_1, y_1), T_2(x_2, y_2)]$, for all $x, y \in L^I$;
- pseudo-t-representable t-norms: $T_T(x, y) = [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))]$, for all $x, y \in L^I$;
- $T_{T,T}(x, y) = [T(x_1, y_1), \max(\min(T(x_1, y_2), T(x_2, y_1)), T(x_2, y_2))$, for all $x, y \in L^I$;
- $T_{T_T,T}(x, y) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(x_2, y_1))$, for all $x, y \in L^I$.
Let $T$ be a t-norm on $L^I$. We say that

1. $T$ is a join-morphism if $T(x, \sup(y, z)) = \sup(T(x, y), T(x, z))$, for all $x, y, z$ in $L^I$;
2. $T$ is a meet-morphism if $T(x, \inf(y, z)) = \inf(T(x, y), T(x, z))$, for all $x, y, z$ in $L^I$;
3. $T$ is a sup-morphism if $T(x, \sup Z) = \sup_{z \in Z} T(x, z)$, for all $x \in L^I$ and \( \emptyset \subseteq Z \subseteq L^I \);
4. $T$ is an inf-morphism if $T(x, \inf Z) = \inf_{z \in Z} T(x, z)$, for all $x \in L^I$ and \( \emptyset \subseteq Z \subseteq L^I \);
5. $T$ satisfies the residuation principle if $T(x, y) \leq L^I z \iff y \leq L^I I_T(x, z)$, for all $x, y, z$ in $L^I$, where $I_T(x, z) = \sup \{ y \mid y \in L^I \text{ and } T(x, y) \leq L^I z \}$, for all $x, z$ in $L^I$.

Similarly as for t-norms on the unit interval, a t-norm $T$ on $L^I$ satisfies the residuation principle if and only if $T$ is a sup-morphism [7].

**Theorem 2.7** [8] Let $T : (L^I)^2 \rightarrow L^I$ be a t-norm such that, for all $x \in D$ and $y_2 \in [0, 1]$, $(T(x, [y_2, y_2]))_2 = (T(x, [0, y_2]))_2$. Then $T$ satisfies the residuation principle if and only if there exist two left-continuous t-norms $T_1$ and $T_2$ on $([0, 1], \leq)$ and a real number $t \in [0, 1]$ such that, for all $x, y \in L^I$,

\[
T(x, y) = \begin{cases} 
T_1(x_1, y_1), & \text{max}(T_2(t, T_2(x_2, y_2)), \text{min}(T_2(x_1, y_2), T_2(y_1, x_2)))
\end{cases}
\]

and, for all $x_1, y_1 \in [0, 1]$,

\[
\begin{align*}
T_1(x_1, y_1) &= T_2(x_1, y_1), \\
\text{if } T_2(x_1, y_1) &> T_2(t, T_2(x_1, y_1)), \\
T_1(x_1, y_1) &\leq T_2(x_1, y_1), \text{ else}.
\end{align*}
\]

If for a mapping $f$ on $[0, 1]$ and a mapping $F$ on $L^I$, it holds that $F(D) \subseteq D$, and $F([a, b]) = [f(a), f(b)]$, for all $a \in [0, 1]$, then we say that $F$ is a natural extension of $f$ to $L^I$. E.g. $T_T$, $T_T$, $T_T$, and $T_T$ are all natural extensions of $T$ to $L^I$.

**Example 2.8** Let, for all $x, y \in [0, 1]$,

\[
T_W(x, y) = \max(0, x + y - 1),
\]

\[
T_P(x, y) = xy,
\]

\[
T_D(x, y) = \begin{cases} 
\min(x, y), & \text{if } \max(x, y) = 1, \\
0, & \text{else},
\end{cases}
\]

\[
S_W(x, y) = \min(1, x + y).
\]

Then $T_W$, $T_P$, and $T_D$ are t-norms, and $S_W$ and $S_P$ are t-conorms on $([0, 1], \leq)$. Let now, for all $x, y \in L^I$,

\[
T_W(x, y) = \max(0, x_1 + y_1 - 1),
\]

\[
\max(0, x_1 + y_2 - 1, x_2 + y_1 - 1),
\]

\[
T_P(x, y) = \begin{cases} 
[x_1y_1, \max(x_1y_2, x_2y_1)],
\end{cases}
\]

\[
S_W(x, y) = \min(1, x_1 + y_2, x_2 + y_1), x_2 + y_2.
\]

Then $T_W$ and $T_P$ are t-norms, and $S_W$ is a t-conorm on $L^I$. Furthermore, $T_W$, $T_P$ and $S_W$ are natural extensions of $T_W$, $T_P$, and $S_W$ respectively. The t-norms $T_W$, $T_P$, $T_W$, and $T_P$ satisfy the residuation principle.

We will also need the following result and definition (see [3, 14, 15, 17, 18]).

**Theorem 2.9** Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms and $(T_\alpha)_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0, 1]$. Then the function $T : [0, 1]^2 \rightarrow [0, 1]$ defined by, for all $x, y \in [0, 1]$,

\[
T(x, y) = \begin{cases} 
\alpha_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha \left( \frac{x - a_\alpha}{e_\alpha - a_\alpha}, \frac{y - a_\alpha}{e_\alpha - a_\alpha} \right), \\
\min(x, y), & \text{otherwise},
\end{cases}
\]

is a t-norm on $([0, 1], \leq)$.
Figure 2: The different positions of $x, y \in L^I$, where $T_\alpha([0, 1], [0, 1]) = [0, 1], T_k([0, 1], [0, 1]) = [0, t]$ and $T_\beta([0, 1], [0, 1]) = [0, 0]$. The value of $(T(x, y))_2$ is calculated at the ending points of the arrows.

**Definition 2.10** Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms and $(\langle a_\alpha, e_\alpha \rangle)_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0, 1]$. The t-norm $T$ defined by (2) is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, T_\alpha \rangle$, $\alpha \in A$, and we will write

$$T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A}.$$ 

Let $A$ be an arbitrary countable index-set and $T_\alpha$ a t-norm on $L^I$, for all $\alpha \in A$. Define, for all $\alpha \in A$, the following sets and mappings:

- $J_\alpha = \{x \mid x \in L^I \text{ and } a_\alpha \leq_L x \leq_L e_\alpha\}$;
- $J'_\alpha = \{x \mid x \in L^I \text{ and } x_1 > (a_\alpha)_1 \}$
  and $x_2 \leq (e_\alpha)_2$;
- $\Phi_\alpha : J_\alpha \to L^I : x \mapsto \begin{bmatrix} x_1 - (a_\alpha)_1 \\ (e_\alpha)_1 - (a_\alpha)_1 \\ x_2 - (a_\alpha)_2 \\ (e_\alpha)_2 - (a_\alpha)_2 \end{bmatrix}$, for all $x \in J_\alpha$;
- $\Phi_\alpha^{-1} : L^I \to J_\alpha : x \mapsto \begin{bmatrix} (a_\alpha)_1 + x_1 (e_\alpha)_1 - (a_\alpha)_1 \\ (a_\alpha)_2 + x_2 (e_\alpha)_2 - (a_\alpha)_2 \end{bmatrix}$, for all $x \in L^I$;
- $T'_\alpha = \Phi_\alpha^{-1} \circ T_\alpha \circ (\Phi_\alpha \times \Phi_\alpha)$.

Assume that $J'_\alpha \cap J'_\beta = \emptyset$, for any $\alpha, \beta \in A$. Our aim is to construct a t-norm $T$ on $L^I$ satisfying the residuation principle such that $T|_{J'_\alpha \times J'_\alpha} = T'_\alpha$.

Assume that $T_\delta([0, 1], [0, 1]) = [0, t]$, for a certain $k \in A$, where $t \in [0, 1]$. Denote by $A_\delta$ the set $A_\delta = \{\alpha \mid \alpha \in A \text{ and } a_\alpha <_{L^I} a_k\}$ and by $A_> \delta$ the set $A_> \delta = \{\alpha \mid \alpha \in A \text{ and } a_\alpha >_{L^I} a_k\}$. If $t \in [0, 1]$, then $T_\alpha([0, 1], [0, 1]) = [0, 1]$, for all $\alpha \in A_\delta$, and $T_\alpha([0, 1], [0, 1]) = [0, 0]$, for all $\alpha \in A_>$ (see [6, Theorem 4.2]). If $t = 0$ or $t = 1$, then we assume from now on that these equalities hold.

**Theorem 2.11** [6] Let, for all $\alpha \in A$, $T_\alpha : [0, 1]^2 \to [0, 1]$ be the mapping defined by, for all $x_1, y_1$ in $[0, 1]$,

$$T_\alpha(x_1, y_1) = T_\alpha([x_1, x_1], [y_1, y_1])_1,$$

and let $T$ be the ordinal sum of $\langle (a_\alpha)_1, (e_\alpha)_1, T_\alpha \rangle$, $\alpha \in A$. Define the mapping $T : (L^I)^2 \to L^I$ by, for all $x, y \in L^I$,

$$T(x,y)_1 = T(x_1, y_1),$$

$$T(x,y)_2 = \begin{cases} (T_\alpha([\max(x_1, (a_\alpha)_1), \min(x_2, (e_\alpha)_2)]), & \text{if } (x_2 \in [(a_\alpha)_2, (e_\alpha)_2]) \text{ and } y_2 > (a_\alpha)_2 \\
& \text{and } y_1 \leq (e_\alpha)_1 \text{ and } \alpha \in A_\delta \\
or (y_2 \in [(a_\alpha)_2, (e_\alpha)_2]) \text{ and } x_2 > (a_\alpha)_2 \\
& \text{and } x_1 \leq (e_\alpha)_1 \text{ and } \alpha \in A_\delta \\
or (x_1 \in [(a_\alpha)_1, (e_\alpha)_1]) \text{ and } y_2 > (a_\alpha)_2 \\
& \text{and } y_1 \leq (e_\alpha)_1 \text{ and } \alpha \in A_\delta \\
or (y_1 \in [(a_\alpha)_1, (e_\alpha)_1]) \text{ and } x_2 > (a_\alpha)_2 \\
& \text{and } x_1 \leq (e_\alpha)_1 \text{ and } \alpha \in A_\delta \\
or (x_2 > (a_\alpha)_2) \text{ and } x_1 \leq (e_\alpha)_1 \\
& \text{and } y_2 > (a_\alpha)_2 \text{ and } y_1 \leq (e_\alpha)_1 \\
& \text{and } \alpha = k) \text{,}
\end{cases}

$$

Then $T$ is a t-norm on $L^I$ called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, T_\alpha \rangle$, $\alpha \in A$, and we write

$$T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A_\delta} \cup \langle a_k, e_k, T_k \rangle.$$

$$T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A_\delta} \cup \langle a_k, e_k, T_k \rangle.$$
In spite of the characterization given in Theorem 2.7, no t-norms of the class \( T_{1}, T_{2}, t \) have yet been found for which the mapping \( T_{1}, T_{2}, t \) defined in Example 2.6 is a t-norm on \( L^{I} \).  

Example 2.12 Let \( \hat{T}_{1}, \hat{T}_{2} \) and \( \hat{T}_{3} \) be t-norms on \( ([0, 1], \leq) \) such that \( \hat{T}_{1} \leq \hat{T}_{2} \). Let furthermore \( t \in [0, 1] \). Define the t-norms \( T_{1} \) and \( T_{2} \) by 

\[
T_{1} = ((0, t, \hat{T}_{1}), (t, 1, \hat{T}_{3})), \\
T_{2} = ((0, t, \hat{T}_{2}), (t, 1, \hat{T}_{3})).
\]

Then \( T_{2}(x_{1}, y_{1}) > T_{2}(t, T_{2}(x_{1}, y_{1})) = \min(t, T_{2}(x_{1}, y_{1})) \iff T_{2}(x_{1}, y_{1}) > t \iff \min(x_{1}, y_{1}) > t, \text{ for all } x_{1}, y_{1} \in [0, 1]. \) It can be easily verified that \( T_{1} \leq T_{2} \) and \( T_{1}(x_{1}, y_{1}) = T_{2}(x_{1}, y_{1}), \text{ for all } x_{1}, y_{1} \text{ in } ]t, 1]^{2}. \) Clearly, if \( \hat{T}_{1} \neq \hat{T}_{2}, \text{ then } T_{1} \neq T_{2}. \)  

Let \( t \in [0, 1] \). The mapping \( T_{1}, T_{2}, t, \) defined by \( T_{1}, T_{2}, t, \) \( T_{1}(x_{1}, y_{1}) = T_{2}(x_{1}, y_{1}), \text{ for all } x_{1}, y_{1} \text{ in } L^{I}, \) is a t-norm on \( L^{I} \) (see Example 2.6).

3 Triangular norms on \( L^{I} \) which are meet-morphisms

Since \( ([0, 1], \leq) \) is a chain, any t-norm on the unit interval is a join- and a meet-morphism. Furthermore, it is well-known that continuous t-norms on \( ([0, 1], \leq) \) are sup- and inf-morphisms. For t-norms on product lattices, the following result holds.

**Theorem 3.1** [4] Consider two bounded lattices \( L_{1} = (L_{1}, \leq_{L_{1}}) \) and \( L_{2} = (L_{2}, \leq_{L_{2}}) \) and a t-norm \( T \) on the product lattice \( L_{1} \times L_{2} = (L_{1} \times L_{2}, \leq) \), where \( (x_{1}, x_{2}) \leq (y_{1}, y_{2}) \iff (x_{1} \leq_{L_{1}} y_{1} \text{ and } x_{2} \leq_{L_{2}} y_{2}), \text{ for all } (x_{1}, x_{2}), (y_{1}, y_{2}) \text{ in } L_{1} \times L_{2}. \) The t-norm \( T \) is a join-morphism (resp. meet-morphism) if and only if there exist t-norms \( T_{1} \) on \( L_{1} \) and \( T_{2} \) on \( L_{2} \) which are join-morphisms (resp. meet-morphisms), such that for all \( (x_{1}, x_{2}), (y_{1}, y_{2}) \) in \( L_{1} \times L_{2} \),

\[
T((x_{1}, x_{2}), (y_{1}, y_{2})) = [T_{1}(x_{1}, y_{1}), T_{2}(x_{2}, y_{2})].
\]

On \( L^{I} \), the situation is more complicated. Not all t-norms on \( L^{I} \) are join- and meet-morphisms. Consider the t-norm \( T_{1}^{I} \) defined by \( T_{1}^{I}(x_{1}, y_{1}) = \min(x_{1}y_{2}, x_{2}y_{1}), \text{ max}(x_{2}y_{2}) \), for all \( x_{1}, x_{2}, y_{1}, y_{2} \in L^{I} \). Then we have \( T_{1}^{I}((0.2, 0.5), (0.2, 0.5)) = 0.4 \neq 0.2, 0.2 \neq 0.2, 0.5 \neq 0.1, 0.5 = \sup(0.1, 0.25), [0, 0.5] = \sup(T_{1}^{I}(0.2, 0.5), [0, 0.5], T_{1}^{I}((0.2, 0.5), [0, 1])). \) So \( T_{1}^{I} \) is not a join-morphism. Similarly the t-norm \( T_{2}^{I} \) is not a meet-morphism.

Gehrke et al. [11] used the following definition for a t-norm on \( L^{I} \): a commutative, associative binary operation \( T \) on \( L^{I} \) is a t-norm if for all \( x, y, z \in L^{I}, \)

\[
(\text{G.1}) \ T(D, D) \leq D, \\
(\text{G.2}) \ T(x, s_{p}(y, z)) = \sup(T(x, y), T(x, z)), \\
(\text{G.3}) \ T(x, \inf_{p}(y, z)) = \inf(T(x, y), T(x, z)), \\
(\text{G.4}) \ T(1_{\mathbb{L}^{I}}, x) = x, \\
(\text{G.5}) \ T([0, 1], x) = [0, x].
\]

They showed that such a t-norm is increasing, so their t-norms are a special case of the t-norms on \( L^{I} \) as defined in Definition 2.5.

Clearly, commutative, associative binary operations on \( L^{I} \) satisfying (G.1)–(G.5) are t-norms on \( L^{I} \) which are join- and meet-morphisms. The two additional conditions (G.1) and (G.5) ensure that these t-norms are t-representable, as is shown in the next theorem.

**Theorem 3.2** [11] For every commutative, associative binary operation \( T \) on \( L^{I} \) satisfying (G.1)–(G.5) there exists a t-norm \( T \) on \( [0, 1], \leq \) such that, for all \( x, y \in L^{I}, \)

\[
T(x, y) = [T(x_{1}, y_{1}), T(x_{2}, y_{2})].
\]

We can extend this result as follows.

**Theorem 3.3** For any t-norm \( T \) on \( L^{I} \) satisfying (G.2) and (G.5) there exist t-norms \( T_{1} \) and \( T_{2} \) on \( [0, 1], \leq \) such that, for all \( x, y \in L^{I}, \)

\[
T(x, y) = [T_{1}(x_{1}, y_{1}), T_{2}(x_{2}, y_{2})].
\]
Clearly, (G.5) is a rather restrictive condition. We will show that if this condition is not imposed, then the class of t-norms on $L^I$ satisfying the other conditions is much larger.

For continuous t-norms on $L^I$ we have the following relationship between sup- and join-morphism, and inf- and meet-morphisms.

**Theorem 3.4** Let $T$ be a continuous t-norm on $L^I$. Then

(i) $T$ is a sup-morphism if and only if $T$ is a join-morphism;

(ii) $T$ is an inf-morphism if and only if $T$ is a meet-morphism.

We extend Theorem 2.7 to t-norms on $L^I$ which are join-morphisms.

**Theorem 3.5** Let $T : (L^I)^2 \to L^I$ be a t-norm such that, for all $x \in D$ and $y_1, y_2 \in [0, 1]$, $(T(x, [y_2, y_2]))_2 = (T(x, [0, y_2]))_2$. Then $T$ is a join-morphism if and only if there exist two t-norms $T_1$ and $T_2$ on $([0, 1], \leq)$ and a real number $t \in [0, 1]$ such that

$$T(x, y) = \{ T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(y_1, x_2)), $$

and, for all $x_1, y_1 \in [0, 1]$,

$$T_1(x_1, y_1) = T_2(x_1, y_1),$$

if $T_2(x_1, y_1) > T_2(t, T_2(x_1, y_1))$,

$T_1(x_1, y_1) \leq T_2(x_1, y_1)$, else.

Now we characterize the t-norms on $L^I$ belonging to the class $T_{T_1, T_2, t}$ which are meet-morphisms. First we need some lemmas.

**Lemma 3.8** Assume that $T_{T_1, T_2, t}$ is a meet-morphism. Then the t-norm $T_2$ in the representation of $T_2$ given in Corollary 3.7 is equal to the minimum.

**Corollary 3.9** Assume that $T_{T_1, T_2, t}$ is a meet-morphism. Then there exists a t-norm $\hat{T}_1$ on $([0, 1], \leq)$ such that

$$T_2 = (\langle 0, t, \hat{T}_1 \rangle, \langle t, 1, \min \rangle).$$

**Lemma 3.10** Assume that there exists a t-norm $\hat{T}_1$ on $([0, 1], \leq)$ such that $T_2 = (\langle 0, t, \hat{T}_1 \rangle, \langle t, 1, \min \rangle)$, then $T_{T_1, T_2, t}$ is a meet-morphism.

Now we obtain the main theorem.

**Theorem 3.11** For any t-norms $T_1$ and $T_2$ on $([0, 1], \leq)$ and $t \in [0, 1]$, $T_{T_1, T_2, t}$ is a meet-morphism if and only if there exist a t-norm $\hat{T}_1$ on $([0, 1], \leq)$ such that $T_2 = (\langle 0, t, \hat{T}_1 \rangle, \langle t, 1, \min \rangle)$.

If we assume that $T_1 = T_2$, then we do not only obtain that $T_1$ is the ordinal sum of two t-norms on $([0, 1], \leq)$, but we can also write the t-norm $T_{T_1, T_1, t}$ as $T_{T_1, t}$ as an ordinal sum of two t-norms on $L^I$. This is shown in the next theorem.

**Theorem 3.12** For any t-norm $T$ on $([0, 1], \leq)$ and $t \in [0, 1]$, $T_{T, t}$ is a meet-morphism if and only if there exists a t-norm $\hat{T}_1$ on $([0, 1], \leq)$ such that

$$T_{T, t} = (\emptyset : (0, t), t, [t, 1], \min \langle t, 1, L^I, T_{\min} \rangle),$$

where, for all $x, y \in L^I$,

$$T_{T_1, T_1, t}(x, y) = \{ \hat{T}_1(x_1, y_1), \hat{T}_1(x_2, y_2) \},$$

$$T_{\min}(x, y) = \min\{ x_1, y_1 \}, \max\{ x_1, y_2 \}, \min\{ x_2, y_1 \}).$$

By combining Theorems 2.7 and 3.11, we obtain the following result.

**Theorem 3.13** Let $T : (L^I)^2 \to L^I$ be a t-norm such that, for all $x \in D$ and $y_2 \in [0, 1]$, $(T(x, [y_2, y_2]))_2 = (T(x, [0, y_2]))_2$. Then $T$ is a join-morphism and a meet-morphism if and only if there exist two t-norms $T_1$ and $T_2$ on $([0, 1], \leq)$ and a real number $t \in [0, 1]$ such that

$$T_2 = (\langle 0, t, \hat{T}_1 \rangle, \langle t, 1, \hat{T}_2 \rangle).$$
(i) $T_1(x_1, y_1) \leq T_2(x_1, y_1)$, for all $x_1, y_1$ in $[0, 1]$,

(ii) $T_1(x_1, y_1) = T_2(x_1, y_1)$, for all $x_1, y_1$ in $[0, 1]$ such that $T_2(x_1, y_1) > t$,

(iii) there exists a t-norm $\hat{T}_1$ on $([0, 1], \leq)$ such that $T_2 = (\langle 0, t, \hat{T}_1 \rangle, \langle t, 1, \min \rangle)$,

(iv) for all $x, y$ in $L^I$,

$T(x, y) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(x_2, y_1))]$.

4 Conclusion

In this paper we have investigated the class $T_{T_1, T_2, t}$ of t-norms on $L^I$ which were introduced in [8]. We have found examples of t-norms in this class for which $T_1 \neq T_2$. We have found that a t-norm $T_{T_1, T_2, t}$ is a meet-morphism if and only if $T_2$ can be represented as the ordinal sum of any t-norm on $([0, 1], \leq)$ and the minimum. If we restrict ourselves to the case when $T_1 = T_2$, then $T_{T_1, T_2, t}$ can itself be written as the ordinal sum of a t-representable t-norm on $L^I$ and the pseudo-t-representable extension of the minimum on $([0, 1], \leq)$ to $L^I$. We have found a characterization of t-norms on $L^I$ which are join- and meet-morphisms and which satisfy an additional condition. For continuous t-norms $T$ on $L^I$ we have found that $T$ is a join-morphism if and only if $T$ is a sup-morphism (or, equivalently, $T$ satisfies the residuation principle); a similar relationship was found between inf- and meet-morphisms.

References


