Triangular norms which are meet-morphisms in Atanassov's intuitionistic fuzzy set theory

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Abstract

In this paper we consider a special class of t-norms $\mathcal{T}_{T_1,T_2,t}$ on the lattice $\mathcal{L}^I$, where $\mathcal{L}^I$ is the underlying lattice of both intuitionistic fuzzy set theory (Atanassov, 1983) and interval-valued fuzzy set theory (Sambuc, 1975). We investigate under which conditions these t-norms are meet-morphisms. Using these results, we obtain a characterization for t-norms on $\mathcal{L}^I$ which are both join- and meet-morphisms and which satisfy an additional condition.

Keywords: Triangular norm, join-morphism, meet-morphism, Atanassov's intuitionistic fuzzy set.

1 Introduction

Atanassov's intuitionistic fuzzy set theory [1, 2] is an extension of fuzzy set theory in which to each element of the universe a membership and a non-membership degree is assigned. Unlike in fuzzy set theory, the sum of these two degrees is only required to be less than or equal to 1. Interval-valued fuzzy set theory [13, 16] is another extension of fuzzy set theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. In [9] it is shown that Atanassov's intuitionistic fuzzy set theory is equivalent to interval-valued fuzzy set theory and that both are equivalent to L-fuzzy set theory in the sense of Goguen [12] w.r.t. a special lattice $\mathcal{L}^I$.

Triangular norms on the unit interval are all join- and meet-morphism, since the unit interval is a chain. On the lattice $\mathcal{L}^I$, however, the situation is more complicated, as there exist t-norms which are not a join- or an inf-morphism. There exist several characterizations for t-norms on $\mathcal{L}^I$ which are join-morphisms and which satisfy additional conditions (see e.g. [5, 7, 8]). In this paper we start the research on meet-morphisms. We start from the class of t-norms $\mathcal{T}_{T_1,T_2,t}$ introduced in [8] and we investigate under which conditions the t-norms of this class are meet-morphisms. We also show that there are t-norms in this class for which the t-norms T_1 and T_2 involved in the construction are not equal to each other. Finally, we give a characterization of t-norms on $\mathcal{L}^I$ which are join- and meet-morphisms and which satisfy an additional condition.

2 The lattice $\mathcal{L}^I$

Definition 2.1 *We define* $\mathcal{L}^I = (L^I, \leq_{L^I})$, *where*

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\},$$
$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2),$$
for all $[x_1, x_2], [y_1, y_2]$ *in* L^I.

Similarly as Lemma 2.1 in [9] it can be shown that $\mathcal{L}^I$ is a complete lattice.

Definition 2.2 [13, 16] *An interval-valued*

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fuzzy set on U is a mapping $A : U \to L^I$.

Definition 2.3 [1, 2] *An intuitionistic fuzzy set in the sense of Atanassov on U is a set*

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

where $\mu_A(u) \in [0,1]$ denotes the membership degree and $\nu_A(u) \in [0,1]$ the non-membership degree of u in A and where for all $u \in U$, $\mu_A(u) + \nu_A(u) \leq 1$.

An intuitionistic fuzzy set in the sense of Atanassov A on U can be represented by the $\mathcal{L}^I$-fuzzy set A given by

$$\begin{aligned} A \ : \ U \ &\to \ L^I \ : \\ u \ &\mapsto \ [\mu_A(u), 1 - \nu_A(u)], \end{aligned}$$

In Figure 1 the set L^I is shown. Note that $x = [x_1, x_2] \in L^I$ is identified with the point $(x_1, x_2) \in \mathbb{R}^2$.

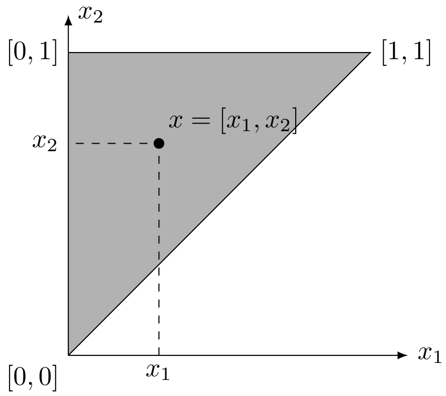

Figure 1: The grey area is L^I.

In the sequel, if $x \in L^I$, then we denote its bounds by x_1 and x_2, i.e. $x = [x_1, x_2]$. The length $x_2 - x_1$ of the interval $x \in L^I$ is called the degree of uncertainty and is denoted by x_π. The smallest and the largest element of $\mathcal{L}^I$ are given by $0_{\mathcal{L}^I} = [0, 0]$ and $1_{\mathcal{L}^I} = [1, 1]$. Note that, for x, y in L^I, $x <_{L^I} y$ is equivalent to $x \leq_{L^I} y$ and $x \neq y$, i.e. either $x_1 < y_1$ and $x_2 \leq y_2$, or $x_1 \leq y_1$ and $x_2 < y_2$. We define the relation $\ll_{L^I}$ by $x \ll_{L^I} y \iff x_1 < y_1$ and $x_2 < y_2$, for x, y in L^I. We define for further usage the set $D = \{[x_1, x_1] \mid x_1 \in [0, 1]\}$.

Note that for any non-empty subset A of L^I it holds that

$$\begin{aligned} \sup A = [&\sup\{x_1 \mid x_1 \in [0,1] \text{ and} \\ &(\exists x_2 \in [x_1, 1])([x_1, x_2] \in A)\}, \\ &\sup\{x_2 \mid x_2 \in [0,1] \text{ and} \\ &(\exists x_1 \in [0, x_2])([x_1, x_2] \in A)\}]; \\ \inf A = [&\inf\{x_1 \mid x_1 \in [0,1] \text{ and} \\ &(\exists x_2 \in [x_1, 1])([x_1, x_2] \in A)\}, \\ &\inf\{x_2 \mid x_2 \in [0,1] \text{ and} \\ &(\exists x_1 \in [0, x_2])([x_1, x_2] \in A)\}]. \end{aligned}$$

Theorem 2.4 (Characterization of supremum in $\mathcal{L}^I$) [7] *Let A be an arbitrary non-empty subset of L^I and $a \in L^I$. Then $a = \sup A$ if and only if*

$$\begin{aligned} &(\forall x \in A)(x \leq_{L^I} a) \\ &\quad \textit{and } (\forall \varepsilon_1 > 0)(\exists z \in A)(z_1 > a_1 - \varepsilon_1) \\ &\quad \textit{and } (\forall \varepsilon_2 > 0)(\exists z \in A)(z_2 > a_2 - \varepsilon_2). \end{aligned}$$

Definition 2.5 *A t-norm on $\mathcal{L}^I$ is a commutative, associative, increasing mapping $\mathcal{T} : (L^I)^2 \to L^I$ which satisfies $\mathcal{T}(1_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.*

A t-conorm on $\mathcal{L}^I$ is a commutative, associative, increasing mapping $\mathcal{S} : (L^I)^2 \to L^I$ which satisfies $\mathcal{S}(0_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.

Example 2.6 [8, 10] We give some special classes of t-norms on $\mathcal{L}^I$. Let T, T_1 and T_2 be t-norms on $([0,1], \leq)$ such that $T_1(x_1, y_1) \leq T_2(x_1, y_1)$ for all x_1, y_1 in $[0, 1]$, and let $t \in [0, 1]$. Then we have the following classes:

- t-representable t-norms: $\mathcal{T}_{T_1,T_2}(x, y) = [T_1(x_1, y_1), T_2(x_2, y_2)]$, for all x, y in L^I;
- pseudo-t-representable t-norms: $\mathcal{T}_T(x, y) = [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))]$, for all x, y in L^I;
- $\mathcal{T}_{T,t}(x, y) = [T(x_1, y_1), \max(T(t, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))]$, for all x, y in L^I;
- $\mathcal{T}'_T(x, y) = [\min(T(x_1, y_2), T(x_2, y_1)), T(x_2, y_2)]$, for all x, y in L^I;
- $\mathcal{T}_{T_1,T_2,t}(x, y) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(x_2, y_1))]$, for all

x, y in L^I, where T_1 and T_2 additionally satisfy, for all x_1, y_1 in $[0,1]$,

$$T_2(x_1,y_1) > T_2(t, T_2(x_1,y_1)) \implies T_1(x_1,y_1) = T_2(x_1,y_1). \quad (1)$$

In Theorem 5 of [8] (see Theorem 2.7) it is shown that $\mathcal{T}_{T_1,T_2,t}$ is indeed a t-norm on $\mathcal{L}^I$ if T_1 and T_2 satisfy (1).[1]

Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$. We say that[2]

- $\mathcal{T}$ is a join-morphism if $\mathcal{T}(x, \sup(y,z)) = \sup(\mathcal{T}(x,y), \mathcal{T}(x,z))$, for all x, y, z in L^I;
- $\mathcal{T}$ is a meet-morphism if $\mathcal{T}(x, \inf(y,z)) = \inf(\mathcal{T}(x,y), \mathcal{T}(x,z))$, for all x, y, z in L^I;
- $\mathcal{T}$ is a sup-morphism if $\mathcal{T}(x, \sup Z) = \sup_{z \in Z} \mathcal{T}(x,z)$, for all $x \in L^I$ and $\varnothing \subset Z \subseteq L^I$;
- $\mathcal{T}$ is an inf-morphism if $\mathcal{T}(x, \inf Z) = \inf_{z \in Z} \mathcal{T}(x,z)$, for all $x \in L^I$ and $\varnothing \subset Z \subseteq L^I$;
- $\mathcal{T}$ satisfies the residuation principle if $\mathcal{T}(x,y) \leq_{L^I} z \iff y \leq_{L^I} \mathcal{I}_{\mathcal{T}}(x,z)$, for all x, y, z in L^I, where $\mathcal{I}_{\mathcal{T}}(x,z) = \sup\{y \mid y \in L^I \text{ and } \mathcal{T}(x,y) \leq_{L^I} z\}$, for all x, z in L^I.

Similarly as for t-norms on the unit interval, a t-norm $\mathcal{T}$ on $\mathcal{L}^I$ satisfies the residuation principle if and only if $\mathcal{T}$ is a sup-morphism [7].

Theorem 2.7 [8] *Let $\mathcal{T} : (L^I)^2 \to L^I$ be a t-norm such that, for all $x \in D$ and $y_2 \in [0,1]$, $(\mathcal{T}(x, [y_2, y_2]))_2 = (\mathcal{T}(x, [0, y_2]))_2$. Then $\mathcal{T}$ satisfies the residuation principle if and only if there exist two left-continuous t-norms T_1 and T_2 on $([0,1], \leq)$ and a real number $t \in [0,1]$ such that, for all $x, y \in L^I$,*

$$\mathcal{T}(x,y) = [T_1(x_1,y_1), \max(T_2(t, T_2(x_2,y_2)), T_2(x_1,y_2), T_2(y_1,x_2))],$$

and, for all x_1, y_1 in $[0,1]$,

$$\begin{cases} T_1(x_1,y_1) = T_2(x_1,y_1), \\ \qquad \text{if } T_2(x_1,y_1) > T_2(t, T_2(x_1,y_1)), \\ T_1(x_1,y_1) \leq T_2(x_1,y_1), \ \text{else.} \end{cases}$$

If for a mapping f on $[0,1]$ and a mapping F on L^I it holds that $F(D) \subseteq \bar{D}$, and $F([a,a]) = [f(a), f(a)]$, for all $a \in [0,1]$, then we say that F is a natural extension of f to L^I. E.g. $\mathcal{T}_{T,T}$, $\mathcal{T}_T$, $\mathcal{T}_{T,t}$ and $\mathcal{T}'_T$ are all natural extensions of T to L^I.

Example 2.8 Let, for all x, y in $[0,1]$,

$$T_W(x,y) = \max(0, x+y-1),$$
$$T_P(x,y) = xy,$$
$$T_D(x,y) = \begin{cases} \min(x,y), & \text{if } \max(x,y) = 1, \\ 0, & \text{else,} \end{cases}$$
$$S_W(x,y) = \min(1, x+y).$$

Then T_W, T_P and T_D are t-norms, and S_W and S_P are t-conorms on $([0,1], \leq)$. Let now, for all x, y in L^I,

$$\mathcal{T}_W(x,y) = [\max(0, x_1+y_1-1), \max(0, x_1+y_2-1, x_2+y_1-1)],$$
$$\mathcal{T}_P(x,y) = [x_1y_1, \max(x_1y_2, x_2y_1)],$$
$$\mathcal{S}_W(x,y) = [\min(1, x_1+y_2, x_2+y_1), x_2+y_2].$$

Then $\mathcal{T}_W$ and $\mathcal{T}_P$ are t-norms, and $\mathcal{S}_W$ is a t-conorm on $\mathcal{L}^I$. Furthermore, $\mathcal{T}_W$, $\mathcal{T}_P$ and $\mathcal{S}_W$ are natural extensions of T_W, T_P and S_W respectively. The t-norms T_W, T_P, $\mathcal{T}_W$ and $\mathcal{T}_P$ satisfy the residuation principle.

We will also need the following result and definition (see [3, 14, 15, 17, 18]).

Theorem 2.9 *Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the function $T : [0,1]^2 \to [0,1]$ defined by, for all x, y in $[0,1]$,*

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \\ \quad \cdot T_\alpha\left(\dfrac{x - a_\alpha}{e_\alpha - a_\alpha}, \dfrac{y - a_\alpha}{e_\alpha - a_\alpha}\right), \\ \qquad \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y), \ \text{otherwise,} \end{cases} \quad (2)$$

is a t-norm on $([0,1], \leq)$.

[1] Note that the condition in Theorem 5 of [8] that T_1 and T_2 are left-continuous is not used to prove that $\mathcal{T}_{T_1,T_2,t}$ is a t-norm.

[2] Note that for simplicity we call a t-norm a join-morphism if its partial mappings are join-morphisms, and similarly for meet-, sup- and inf-morphisms.

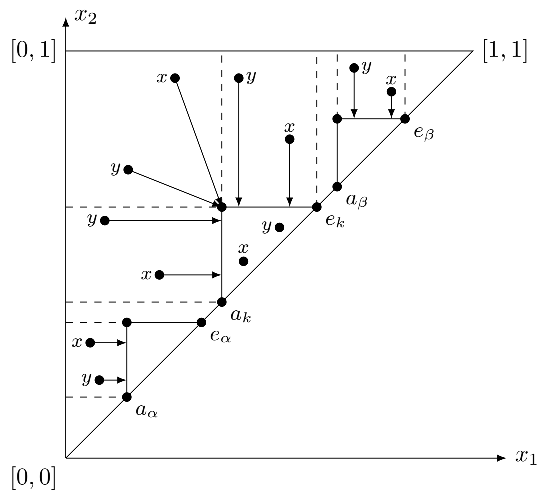

Figure 2: The different positions of $x, y \in L^I$, where $\mathcal{T}_\alpha([0,1],[0,1]) = [0,1]$, $\mathcal{T}_k([0,1],[0,1]) = [0,t]$ and $\mathcal{T}_\beta([0,1],[0,1]) = [0,0]$. The value of $(\mathcal{T}(x,y))_2$ is calculated at the ending points of the arrows.

Definition 2.10 *Let $(T_\alpha)_{\alpha \in A}$ be a family of t-norms and $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. The t-norm T defined by (2) is called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, T_\alpha \rangle$, $\alpha \in A$, and we will write*

$$T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A}.$$

Let A be an arbitrary countable index-set and $\mathcal{T}_\alpha$ a t-norm on $\mathcal{L}^I$, for all $\alpha \in A$. Define, for all $\alpha \in A$, the following sets and mappings:

$$J_\alpha = \{x \mid x \in L^I \text{ and } a_\alpha \leq_{L^I} x \leq_{L^I} e_\alpha\}, \quad \text{where } (a_\alpha, e_\alpha) \in D^2 \text{ and } a_\alpha <_{L^I} e_\alpha;$$

$$J_\alpha^* = \{x \mid x \in L^I \text{ and } x_1 > (a_\alpha)_1 \text{ and } x_2 \leq (e_\alpha)_2\};$$

$$\Phi_\alpha : J_\alpha \to L^I : x \mapsto \left[\frac{x_1 - (a_\alpha)_1}{(e_\alpha)_1 - (a_\alpha)_1}, \frac{x_2 - (a_\alpha)_2}{(e_\alpha)_2 - (a_\alpha)_2}\right], \quad \forall x \in J_\alpha;$$

$$\Phi_\alpha^{-1} : L^I \to J_\alpha : x \mapsto [(a_\alpha)_1 + x_1((e_\alpha)_1 - (a_\alpha)_1), (a_\alpha)_2 + x_2((e_\alpha)_2 - (a_\alpha)_2)], \quad \forall x \in L^I;$$

$$\mathcal{T}'_\alpha = \Phi_\alpha^{-1} \circ \mathcal{T}_\alpha \circ (\Phi_\alpha \times \Phi_\alpha).$$

Assume that $J_\alpha^* \cap J_\beta^* = \varnothing$, for any $\alpha, \beta \in A$. Our aim is to construct a t-norm $\mathcal{T}$ on $\mathcal{L}^I$ satisfying the residuation principle such that $\mathcal{T}\big|_{J_\alpha^* \times J_\alpha^*} = \mathcal{T}'_i$.

Assume that $\mathcal{T}_k([0,1],[0,1]) = [0,t]$, for a certain $k \in A$, where $t \in [0,1]$. Denote by $A_<$ the set $A_< = \{\alpha \mid \alpha \in A \text{ and } a_\alpha <_{L^I} a_k\}$ and by $A_>$ the set $A_> = \{\alpha \mid \alpha \in A \text{ and } a_\alpha >_{L^I} a_k\}$. If $t \in]0,1[$, then $\mathcal{T}_\alpha([0,1],[0,1]) = [0,1]$, for all $\alpha \in A_<$, and $\mathcal{T}_\alpha([0,1],[0,1]) = [0,0]$, for all $\alpha \in A_>$ (see [6, Theorem 4.2]). If $t = 0$ or $t = 1$, then we assume from now on that these equalities hold.

Theorem 2.11 [6] *Let, for all $\alpha \in A$, $T_\alpha : [0,1]^2 \to [0,1]$ be the mapping defined by, for all x_1, y_1 in $[0,1]$,*

$$T_\alpha(x_1, y_1) = (\mathcal{T}_\alpha([x_1, x_1], [y_1, y_1]))_1,$$

and let T be the ordinal sum of $\langle (a_\alpha)_1, (e_\alpha)_1, T_\alpha \rangle$, $\alpha \in A$. Define the mapping $\mathcal{T} : (L^I)^2 \to L^I$ by, for all $x, y \in L^I$,

$$(\mathcal{T}(x,y))_1 = T(x_1, y_1),$$

$$(\mathcal{T}(x,y))_2 = \begin{cases} (\mathcal{T}'_\alpha([\max(x_1,(a_\alpha)_1), \min(x_2,(e_\alpha)_2)], \\ \quad [\max(y_1,(a_\alpha)_1), \min(y_2,(e_\alpha)_2)]))_2, \\ \quad \textit{if } (x_2 \in](a_\alpha)_2,(e_\alpha)_2] \textit{ and } y_2 > (a_\alpha)_2 \\ \quad\quad \textit{and } y_1 \leq (e_\alpha)_1 \textit{ and } \alpha \in A_<) \\ \quad \textit{or } (y_2 \in](a_\alpha)_2,(e_\alpha)_2] \textit{ and } x_2 > (a_\alpha)_2 \\ \quad\quad \textit{and } x_1 \leq (e_\alpha)_1 \textit{ and } \alpha \in A_<) \\ \quad \textit{or } (x_1 \in](a_\alpha)_1,(e_\alpha)_1] \textit{ and } y_2 > (a_\alpha)_2 \\ \quad\quad \textit{and } y_1 \leq (e_\alpha)_1 \textit{ and } \alpha \in A_>) \\ \quad \textit{or } (y_1 \in](a_\alpha)_1,(e_\alpha)_1] \textit{ and } x_2 > (a_\alpha)_2 \\ \quad\quad \textit{and } x_1 \leq (e_\alpha)_1 \textit{ and } \alpha \in A_>) \\ \quad \textit{or } (x_2 > (a_\alpha)_2 \textit{ and } x_1 \leq (e_\alpha)_1 \\ \quad\quad \textit{and } y_2 > (a_\alpha)_2 \textit{ and } y_1 \leq (e_\alpha)_1 \\ \quad\quad \textit{and } \alpha = k), \\ \min(x_2, y_2), \ \textit{if the previous conditions do} \\ \quad \textit{not hold and } (x_2 \leq (a_k)_2 \textit{ or } y_2 \leq (a_k)_2), \\ \min(x_2, y_1), \ \textit{if the previous conditions do} \\ \quad \textit{not hold and } x_1 \leq y_1, \\ \min(y_2, x_1), \ \textit{else.} \end{cases}$$

Then $\mathcal{T}$ is a t-norm on $\mathcal{L}^I$ called the ordinal sum of the summands $\langle a_\alpha, e_\alpha, \mathcal{T}_\alpha \rangle$, $\alpha \in A$, and we write

$$\mathcal{T} = ((\langle a_\alpha, e_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A_<}; \langle a_k, e_k, \mathcal{T}_k \rangle; (\langle a_\alpha, e_\alpha, \mathcal{T}_\alpha \rangle)_{\alpha \in A_>}).$$

In spite of the characterization given in Theorem 2.7, no t-norms of the class $\mathcal{T}_{T_1,T_2,t}$ have yet been found for which $T_1 \neq T_2$. In the following example we show that there do exist *different* t-norms T_1 and T_2 for which the mapping $\mathcal{T}_{T_1,T_2,t}$ defined in Example 2.6 is a t-norm on $\mathcal{L}^I$.

Example 2.12 *Let $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$ be t-norms on $([0,1],\leq)$ such that $\hat{T}_1 \leq \hat{T}_2$. Let furthermore $t \in [0,1]$. Define the t-norms T_1 and T_2 by*

$$T_1 = (\langle 0,t,\hat{T}_1\rangle, \langle t,1,\hat{T}_3\rangle),$$
$$T_2 = (\langle 0,t,\hat{T}_2\rangle, \langle t,1,\hat{T}_3\rangle).$$

Then $T_2(x_1,y_1) > T_2(t,T_2(x_1,y_1)) = \min(t, T_2(x_1,y_1)) \iff T_2(x_1,y_1) > t \implies \min(x_1,y_1) > t$, for all x_1,y_1 in $[0,1]$. It can be easily verified that $T_1 \leq T_2$ and $T_1(x_1,y_1) = T_2(x_1,y_1)$, for all x_1,y_1 in $]t,1]^2$. Clearly, if $\hat{T}_1 \neq \hat{T}_2$, then $T_1 \neq T_2$.

Let $t \in [0,1]$. The mapping $\mathcal{T}_{T_1,T_2,t}$ defined by $\mathcal{T}_{T_1,T_2,t}(x,y) = [T_1(x_1,y_1), \max(T_2(t, T_2(x_2,y_2)), T_2(x_1,y_2), T_2(x_2,y_1))]$, for all x,y in L^I, is a t-norm on $\mathcal{L}^I$ (see Example 2.6).

3 Triangular norms on $\mathcal{L}^I$ which are meet-morphisms

Since $([0,1],\leq)$ is a chain, any t-norm on the unit interval is a join- and a meet-morphism. Furthermore, it is well-known that continuous t-norms on $([0,1],\leq)$ are sup- and inf-morphisms. For t-norms on product lattices, the following result holds.

Theorem 3.1 [4] *Consider two bounded lattices $\mathcal{L}_1 = (L_1, \leq_{L_1})$ and $\mathcal{L}_2 = (L_2, \leq_{L^2})$ and a t-norm $\mathcal{T}$ on the product lattice $\mathcal{L}_1 \times \mathcal{L}_2 = (L_1 \times L_2, \leq)$, where $(x_1,x_2) \leq (y_1,y_2) \iff (x_1 \leq_{L_1} y_1$ and $x_2 \leq_{L_2} y_2)$, for all (x_1,x_2), (y_1,y_2) in $L_1 \times L_2$. The t-norm $\mathcal{T}$ is a join-morphism (resp. meet-morphism) if and only if there exist t-norms T_1 on $\mathcal{L}_1$ and T_2 on $\mathcal{L}_2$ which are join-morphisms (resp. meet-morphisms), such that for all (x_1,x_2), (y_1,y_2) in $L_1 \times L_2$,*

$$\mathcal{T}((x_1,x_2),(y_1,y_2)) = [T_1(x_1,y_1), T_2(x_2,y_2)].$$

On $\mathcal{L}^I$, the situation is more complicated. Not all t-norms on $\mathcal{L}^I$ are join- and meet-morphisms. Consider the t-norm $\mathcal{T}'_{T_P}$ given by $\mathcal{T}'_{T_P}(x,y) = [\min(x_1y_2,\ x_2y_1), \max(x_2y_2)]$, for all x,y in L^I. Then we have $\mathcal{T}'_{T_P}([0.2, 0.5], \sup([0.5,0.5], [0,1])) = \mathcal{T}'_{T_P}([0.2,0.5], [0.5,1]) = [0.2,0.5] \neq [0.1,0.5] = \sup([0.1, 0.25], [0,0.5]) = \sup(\mathcal{T}'_{T_P}([0.2,0.5], [0.5,0.5]), \mathcal{T}'_{T_P}([0.2,0.5], [0,1]))$. So $\mathcal{T}'_{T_P}$ is not a join-morphism. Similarly the t-norm $\mathcal{T}_{T_P}$ is not a meet-morphism.

Gehrke *et al.* [11] used the following definition for a t-norm on $\mathcal{L}^I$: a commutative, associative binary operation $\mathcal{T}$ on $\mathcal{L}^I$ is a t-norm if for all x,y,z in L^I,

(G.1) $\mathcal{T}(D,D) \subseteq D$,

(G.2) $\mathcal{T}(x,\sup(y,z)) = \sup(\mathcal{T}(x,y),\mathcal{T}(x,z))$,

(G.3) $\mathcal{T}(x,\inf(y,z)) = \inf(\mathcal{T}(x,y),\mathcal{T}(x,z))$,

(G.4) $\mathcal{T}(1_{\mathcal{L}^I},x) = x$,

(G.5) $\mathcal{T}([0,1],x) = [0,x_2]$.

They showed that such a t-norm is increasing, so their t-norms are a special case of the t-norms on $\mathcal{L}^I$ as defined in Definition 2.5.

Clearly, commutative, associative binary operations on $\mathcal{L}^I$ satisfying (G.1)–(G.5) are t-norms on $\mathcal{L}^I$ which are join- and meet-morphisms. The two additional conditions (G.1) and (G.5) ensure that these t-norms are t-representable, as is shown in the next theorem.

Theorem 3.2 [11] *For every commutative, associative binary operation $\mathcal{T}$ on $\mathcal{L}^I$ satisfying (G.1)–(G.5) there exists a t-norm T on $([0,1],\leq)$ such that, for all x,y in L^I,*

$$\mathcal{T}(x,y) = [T(x_1,y_1), T(x_2,y_2)].$$

We can extend this result as follows.

Theorem 3.3 *For any t-norm $\mathcal{T}$ on $\mathcal{L}^I$ satisfying (G.2) and (G.5) there exist t-norms T_1 and T_2 on $([0,1],\leq)$ such that, for all x,y in L^I,*

$$\mathcal{T}(x,y) = [T_1(x_1,y_1), T_2(x_2,y_2)].$$

Clearly, (G.5) is a rather restrictive condition. We will show that if this condition is not imposed, then the class of t-norms on $\mathcal{L}^I$ satisfying the other conditions is much larger.

For continuous t-norms on $\mathcal{L}^I$ we have the following relationship between sup- and join-morphism, and inf- and meet-morphisms.

Theorem 3.4 *Let $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}^I$. Then*

(i) $\mathcal{T}$ is a sup-morphism if and only if $\mathcal{T}$ is a join-morphism;

(ii) $\mathcal{T}$ is an inf-morphism if and only if $\mathcal{T}$ is a meet-morphism.

We extend Theorem 2.7 to t-norms on $\mathcal{L}^I$ which are join-morphisms.

Theorem 3.5 *Let $\mathcal{T} : (L^I)^2 \to L^I$ be a t-norm such that, for all $x \in D$ and $y_2 \in [0,1]$, $(\mathcal{T}(x,[y_2,y_2]))_2 = (\mathcal{T}(x,[0,y_2]))_2$. Then $\mathcal{T}$ is a join-morphism if and only if there exist two t-norms T_1 and T_2 on $([0,1],\leq)$ and a real number $t \in [0,1]$ such that, for all $x, y \in L^I$,*

$$\mathcal{T}(x,y) = [T_1(x_1,y_1), \max(T_2(t,T_2(x_2,y_2)), T_2(x_1,y_2), T_2(y_1,x_2))],$$

and, for all x_1, y_1 in $[0,1]$,

$$\begin{cases} T_1(x_1,y_1) = T_2(x_1,y_1), \\ \qquad \text{if } T_2(x_1,y_1) > T_2(t,T_2(x_1,y_1)), \\ T_1(x_1,y_1) \leq T_2(x_1,y_1), \ \text{else.} \end{cases}$$

Now we characterize the t-norms on $\mathcal{L}^I$ belonging to the class $\mathcal{T}_{T_1,T_2,t}$ which are meet-morphisms. First we need some lemmas.

Lemma 3.6 *Assume that $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism. Then $T_2(t,y_1) = \min(t,y_1)$, for all $y_1 \in [0,1]$.*

Corollary 3.7 *Assume that $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism. Then there exists two t-norms $\hat{T}_1$ and $\hat{T}_2$ on $([0,1],\leq)$ such that*

$$T_2 = (\langle 0,t,\hat{T}_1\rangle, \langle t,1,\hat{T}_2\rangle).$$

Lemma 3.8 *Assume that $\mathcal{T}_{T_1,T_2,t}$ is a meet morphism. Then the t-norm $\hat{T}_2$ in the representation of T_2 given in Corollary 3.7 is equal to the minimum.*

Corollary 3.9 *Assume that $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism. Then there exists a t-norm $\hat{T}_1$ on $([0,1],\leq)$ such that*

$$T_2 = (\langle 0,t,\hat{T}_1\rangle, \langle t,1,\min\rangle).$$

Lemma 3.10 *Assume that there exists a t-norm $\hat{T}_1$ on $([0,1],\leq)$ such that $T_2 = (\langle 0,t,\hat{T}_1\rangle, \langle t,1,\min\rangle)$, then $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism.*

Now we obtain the main theorem.

Theorem 3.11 *For any t-norms T_1 and T_2 on $([0,1],\leq)$ and $t \in [0,1]$, $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism if and only if there exists a t-norm $\hat{T}_1$ on $([0,1],\leq)$ such that $T_2 = (\langle 0,t,\hat{T}_1\rangle, \langle t,1,\min\rangle)$.*

If we assume that $T_1 = T_2$, then we do not only obtain that T_1 is the ordinal sum of two t-norms on $([0,1],\leq)$, but we can also write the t-norm $\mathcal{T}_{T_1,T_1,t} = \mathcal{T}_{T_1,t}$ as an ordinal sum of two t-norms on $\mathcal{L}^I$. This is shown in the next theorem.

Theorem 3.12 *For any t-norm T on $([0,1],\leq)$ and $t \in [0,1]$, $\mathcal{T}_{T,t}$ is a meet-morphism if and only if there exists a t-norm $\hat{T}_1$ on $([0,1],\leq)$ such that*

$$\mathcal{T}_{T,t} = (\varnothing; \langle 0_{\mathcal{L}^I}, [t,t], \mathcal{T}_{\hat{T}_1,\hat{T}_1}\rangle; \langle [t,t], 1_{\mathcal{L}^I}, \mathcal{T}_{\min}\rangle),$$

where, for all x, y in L^I,

$$\mathcal{T}_{\hat{T}_1,\hat{T}_1}(x,y) = [\hat{T}_1(x_1,y_1), \hat{T}_1(x_2,y_2)],$$
$$\mathcal{T}_{\min}(x,y) = [\min(x_1,y_1), \max(\min(x_1,y_2), \min(x_2,y_1))].$$

By combining Theorems 2.7 and 3.11, we obtain the following result.

Theorem 3.13 *Let $\mathcal{T} : (L^I)^2 \to L^I$ be a t-norm such that, for all $x \in D$ and $y_2 \in [0,1]$, $(\mathcal{T}(x,[y_2,y_2]))_2 = (\mathcal{T}(x,[0,y_2]))_2$. Then $\mathcal{T}$ is a join-morphism and a meet-morphism if and only if there exist two t-norms T_1 and T_2 on $([0,1],\leq)$ and a real number $t \in [0,1]$ such that* *(i)* $T_1(x_1, y_1) \leq T_2(x_1, y_1)$, *for all* x_1, y_1 *in* $[0, 1]$,

(ii) $T_1(x_1, y_1) = T_2(x_1, y_1)$, *for all* x_1, y_1 *in* $[0, 1]$ *such that* $T_2(x_1, y_1) > t$,

(iii) there exists a t-norm $\hat{T}_1$ *on* $([0, 1], \leq)$ *such that* $T_2 = (\langle 0, t, \hat{T}_1 \rangle, \langle t, 1, \min \rangle)$,

(iv) for all x, y *in* L^I,

$$\mathcal{T}(x, y) = [T_1(x_1, y_1), \max(T_2(t, T_2(x_2, y_2)), T_2(x_1, y_2), T_2(x_2, y_1))].$$

4 Conclusion

In this paper we have investigated the class $\mathcal{T}_{T_1,T_2,t}$ of t-norms on $\mathcal{L}^I$ which were introduced in [8]. We have found examples of t-norms in this class for which $T_1 \neq T_2$. We have found that a t-norm $\mathcal{T}_{T_1,T_2,t}$ is a meet-morphism if and only if T_2 can be represented as the ordinal sum of any t-norm on $([0, 1], \leq)$ and the minimum. If we restrict ourselves to the case when $T_1 = T_2$, then $\mathcal{T}_{T_1,T_1,t}$ can itself be written as the ordinal sum of a t-representable t-norm on $\mathcal{L}^I$ and the pseudo-t-representable extension of the minimum on $([0, 1], \leq)$ to $\mathcal{L}^I$. We have found a characterization of t-norms on $\mathcal{L}^I$ which are join- and meet-morphisms and which satisfy an additional condition. For continuous t-norms $\mathcal{T}$ on $\mathcal{L}^I$ we have found that $\mathcal{T}$ is a join-morphism if and only if $\mathcal{T}$ is a sup-morphism (or, equivalently, $\mathcal{T}$ satisfies the residuation principle); a similar relationship was found between inf- and meet-morphisms.

References

[1] K. T. Atanassov, Intuitionistic fuzzy sets, 1983, VII ITKR's Session, Sofia (deposed in Central Sci.-Technical Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian).

[2] K. T. Atanassov, *Intuitionistic fuzzy sets*, Physica-Verlag, Heidelberg, New York, 1999.

[3] A. H. Clifford, Naturally totally ordered commutative semigroups, *Amer. J. Math.*, **76** (1954) 631–646.

[4] B. De Baets and R. Mesiar, Triangular norms on product lattices, *Fuzzy Sets and Systems*, **104**(1) (1999) 61–75.

[5] G. Deschrijver, A representation of t-norms in interval-valued L-fuzzy set theory, *Fuzzy Sets and Systems*, in press.

[6] G. Deschrijver, Ordinal sums in interval-valued fuzzy set theory, *New Mathematics and Natural Computation*, **1**(2) (2005) 243–259.

[7] G. Deschrijver, C. Cornelis and E. E. Kerre, On the representation of intuitionistic fuzzy t-norms and t-conorms, *IEEE Transactions on Fuzzy Systems*, **12**(1) (2004) 45–61.

[8] G. Deschrijver and E. E. Kerre, Classes of intuitionistic fuzzy t-norms satisfying the residuation principle, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, **11**(6) (2003) 691–709.

[9] G. Deschrijver and E. E. Kerre, On the relationship between some extensions of fuzzy set theory, *Fuzzy Sets and Systems*, **133**(2) (2003) 227–235.

[10] G. Deschrijver and E. E. Kerre, Implicators based on binary aggregation operators in interval-valued fuzzy set theory, *Fuzzy Sets and Systems*, **153**(2) (2005) 229–248.

[11] M. Gehrke, C. L. Walker and E. A. Walker, Some comments on interval valued fuzzy sets, *International Journal of Intelligent Systems*, **11**(10) (1996) 751–759.

[12] J. A. Goguen, L-fuzzy sets, *Journal of Mathematical Analysis and Applications*, **18**(1) (1967) 145–174.

[13] M. B. Gorzałczany, A method of inference in approximate reasoning based on interval-valued fuzzy sets, *Fuzzy Sets and Systems*, **21**(1) (1987) 1–17.

[14] S. Jenei, A note on the ordinal sum theorem and its consequence for the construction of triangular norms, *Fuzzy Sets and Systems*, **126**(2) (2002) 199–205.

[15] C.-H. Ling, Representation of associative functions, *Publ. Math. Debrecen*, **12** (1965) 189–212.

[16] R. Sambuc, *Fonctions Φ-floues. Application à l'aide au diagnostic en pathologie thyroidienne*, Ph.D. thesis, Université de Marseille, France, 1975.

[17] S. Saminger, On ordinal sums of triangular norms on bounded lattices, *Fuzzy Sets and Systems*, **157**(10) (2006) 1403–1416.

[18] B. Schweizer and A. Sklar, Associative functions and abstract semigroups, *Publ. Math. Debrecen*, **10** (1963) 69–81.