Convergence of P-observables

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Abstract

In [2] K. T. Atanassov and B. Riečan studied a new type of probability on the family of IF-events $\mathcal{N} = \{(\mu_A, \nu_A)\ ;\ \mu_A, \nu_A : \Omega \to [0,1], \mu_A + \nu_A \leq 1\}$, where μ_A, ν_A are $\mathcal{S}$-measurable functions, using following pair of connectives

$$\begin{aligned} a \oplus b &= a + b - a \cdot b, \\ a \odot b &= a \cdot b \end{aligned}$$

They called it the P-probability. In this paper we define three types of convergence of P-observables. We show the relation between convergence of P-observables and convergence of corresponding random variables.

Keywords: Convergence in distribution, Convergence in measure m, Convergence m-almost everywhere, P-probability, P-observable.

1 Introduction

Recently the probability on IF-events has been constructed. Let $(\Omega, \mathcal{S}, P)$ be a classical probability space. An IF-event $A = (\mu_A, \nu_A)$ is a couple of $\mathcal{S}$-measurable function with respect to a σ-algebra of subsets of Ω such that $\mu_A(\omega) + \nu_A(\omega) \leq 1$ for each $\omega \in \Omega$ ([1]).

In [4] P. Grzegorzewski and E. Mrówka defined the probability on the family $\mathcal{N} = \{(\mu_A, \nu_A)\ ;\ \mu_A, \nu_A$ are $\mathcal{S}-$measurable and $\mu_A + \nu_A \leq 1\}$ as a mapping $\mathcal{P}$ from the family $\mathcal{N}$ to the set of all compact intervals in $\mathbf{R}$ by the formula

$$\mathcal{P}((\mu_A, \nu_A)) = \left[\int_\Omega \mu_A \ dP, 1 - \int_\Omega \nu_A \ dP \right].$$

This IF-probability was axiomatically characterized by B. Riečan (see[13]).

More general situation was studied in [12], where author introduced the notion of IF-probability on the family $\mathcal{F} = \{(f, g)\ ;\ f, g \in \mathcal{T}, \mathcal{T}$ is Lukasiewicz tribe and $f + g \leq 1\}$ as a mapping $\mathcal{P}$ from the family $\mathcal{F}$ to the family $\mathcal{J}$ of all closed intervals $[a, b]$ such that $0 \leq a \leq b \leq 1$. Variant of Central limit theorem and Weak law of large numbers were proved as an illustration of method applied on these IF-events. It can see in the papers [10], [11].

More general situation was used in [9]. The authors defined the probability on the family $\mathcal{M} = \{(a, b) \in M, a + b \leq u\}$, where M is σ-complete MV-algebra, which can be identified with the unit interval of a unique ℓ-group G with strong unit u, in symbols,

$$M = \Gamma(G, u) = ([0, u], 0, u, \neg, \oplus, \odot)$$

where

$$\begin{aligned} [0, u] &= \{a \in G\ ;\ 0 \leq a \leq u\}, \\ \neg\, a = u - a \quad &, \quad a \oplus b = (a + b) \wedge u, \\ a \odot b &= (a + b - u) \vee 0 \end{aligned}$$

(see [15]). We say that G is the ℓ-group (with strong unit u) corresponding to M.

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By an ℓ-group we shall mean a lattice-ordered Abelian group. For any ℓ-group G, an element $u \in G$ is said to be a strong unit of G, if for all $a \in G$ there is an integer $n \geq 1$ such that $nu \geq a$. Convergence of IF-observables and Strong law of large numbers for IF-events were proved as an illustration of method applied on these IF-events. It can see in the papers [6], [7].

Later M. Krachounov defined an IF-probability theory based on the connectives

$$a \oplus b = \max(a, b),$$
$$a \odot b = \min(a, b)$$

(see [5]). B. Riečan called it M-probability theory and he studied the notion of M-observable and the notion of joint M-observable. He proved the Central limit theorem for this kind of independent IF-observables, too. It can see in a paper [14].

In [2] K. T. Atanassov and B. Riečan studied a new type of probability on the family of IF-events

$$\mathcal{N} = \{(\mu_A, \nu_A)\,;\, \mu_A, \nu_A : \Omega \to [0,1], \mu_A + \nu_A \leq 1\},$$

where μ_A, ν_A are $\mathcal{S}$-measurable functions, using following pair of connectives

$$a \oplus b = a + b - a \cdot b,$$
$$a \odot b = a \cdot b$$

They called it the P-probability and they proved the Central limit theorem.

In this paper we define three types of convergence of P-observables. We show the relation between convergence of P-observables and convergence of corresponding random variables. In *Section 2* we introduce the operations on $\mathcal{N}$ and $\mathcal{J}$, where $\mathcal{J}$ is the family of all closed intervals $[a, b]$ such that $0 \leq a \leq b \leq 1$. We introduce the notion of P-probability on $\mathcal{N}$ and the notion of independence of P-observables, too.

2 Basic notions

Now we introduce operations on $\mathcal{N}$. Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$. Then we define

$$A \oplus_P B = (\mu_A + \mu_B - \mu_A \cdot \mu_B, \nu_A \cdot \nu_B),$$
$$A \odot_P B = (\mu_A \cdot \mu_B, \nu_A + \nu_B - \nu_A \cdot \nu_B).$$

If $A_n = (\mu_{A_n}, \nu_{A_n})$, then we write

$$A_n \nearrow A \Longleftrightarrow \mu_{A_n} \nearrow \mu_A,\ \nu_{A_n} \searrow \nu_A.$$

An P-probability $\mathcal{P}$ on $\mathcal{N}$ is a mapping from $\mathcal{N}$ to the family $\mathcal{J}$ of all closed intervals $[a, b]$ such that $0 \leq a \leq b \leq 1$. Here we define

$$[a, b] + [c, d] = [a + c, b + d],$$
$$[a_n, b_n] \nearrow [a, b] \Longleftrightarrow a_n \nearrow a,\ b_n \nearrow b.$$

By an **P-probability** on $\mathcal{N}$ we understand each function $\mathcal{P} : \mathcal{N} \to \mathcal{J}$ satisfying the following properties:

(i) $\mathcal{P}((1, 0)) = [1, 1]$; $\mathcal{P}((0, 1)) = [0, 0]$;

(ii) if $A \odot_P B = (0, 1)$ and $A, B \in \mathcal{N}$, then $\mathcal{P}(A \oplus_P B) = \mathcal{P}(A) + \mathcal{P}(B)$;

(iii) if $A_n \nearrow A$, then $\mathcal{P}(A_n) \nearrow \mathcal{P}(A)$.

By an **P-state** we understand each mapping $m : \mathcal{N} \to [0, 1]$ satisfying the following properties:

(i) $m((1, 0)) = 1, m((0, 1)) = 0$;

(ii) $A \odot_P B = (0, 1) \Longrightarrow m(A \oplus_P B) = m(A) + m(B)$;

(iii) $A_n \nearrow A \Longrightarrow m(A_n) \nearrow m(A)$.

The next important notions are the notion of P-observable and the notion of independence.

By an **P-observable** on $\mathcal{N}$ we understand any mapping $x : \mathcal{B}(\mathbf{R}) \to \mathcal{N}$ satisfying the following conditions:

(i) $x(\mathbf{R}) = (1, 0)$, $x(\emptyset) = (0, 1)$;

(ii) if $A \cap B = \emptyset$, then $x(A) \odot_P x(B) = (0, 1)$ and $x(A \cup B) = x(A) \oplus_P x(B)$;

(iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

By an **joint P-observable** of P-observables $x, y : \mathcal{B}(\mathbf{R}) \to \mathcal{N}$ we understand each mapping $h : \mathcal{B}(\mathbf{R}^2) \to \mathcal{N}$ satisfying the following conditions:

(i) $h(\mathbf{R^2}) = (1,0), h(\emptyset) = (0,1)$;

(ii) if $A \cap B = \emptyset$, then $h(A) \odot_P h(B) = (0,1)$ and $h(A \cup B) = h(A) \oplus_P h(B)$;

(iii) $A_n \nearrow A \Longrightarrow h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C) \cdot y(D)$ for any $C, D \in \mathcal{B}(\mathbf{R})$.

Here $C \cdot D = (\mu_C, \nu_C) \cdot (\mu_D, \nu_D) = (\mu_C \cdot \mu_D, 1 - (1-\nu_C) \cdot (1-\nu_D))$.

We say that **P-observables** $x_1, \ldots, x_n$ **are independent**, if for each $C_1, ..., C_n \in \mathcal{B}(R)$ holds

$$m(h_n(C_1 \times ... \times C_n)) = m(x_1(C_1)) \cdot \ldots \cdot m(x(C_n)),$$

where $h_n : \mathcal{B}(\mathbf{R}^n) \to \mathcal{N}$ is the joint P-observable of P-observables $x_1, \ldots, x_n$ and $m : \mathcal{N} \to [0,1]$ is the P-state.

3 Convergence on P-probability

In this section we introduce the notion of a function of several P-observables and define three types of convergence for P-observables. We show the relation between convergence of sequence of P-observables and sequence of corresponding random variables, too.

If we have several independent observables and a Borel measurable function, we can define the observable, which is the function of several observables. About this says the following definition.

Definition 3.1 *Let $x_1, \ldots, x_n : \mathcal{B}(\mathbf{R}) \to \mathcal{N}$ be the independent P-observables and $g_n : \mathbf{R}^n \to \mathbf{R}$ be a Borel measurable function. Then the P-observable $y_n = g_n(x_1, \ldots, x_n) : \mathcal{B}(\mathbf{R}) \to \mathcal{N}$ is defined by the equality*

$$y_n = h_n \circ g_n^{-1}$$

where $h_n : \mathcal{B}(\mathbf{R}^n) \to \mathcal{N}$ is the n-dimensional P- observable (joint P-observable of $x_1, \ldots, x_n$).

Example 3.2 *Let $x_1, \ldots, x_n : \mathcal{B}(\mathbf{R}) \to \mathcal{N}$ be independent P-observables and $h_n : \mathcal{B}(\mathbf{R}^n) \to \mathcal{N}$ be their joint P-observable. Then*

1. *the P-observable $y_n = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} x_i - a\Big)$ is defined by the equality*

   $$y_n = h_n \circ g_n^{-1}$$

   where $g_n(u_1, \ldots, u_n) = \frac{\sqrt{n}}{\sigma}\Big(\frac{1}{n}\sum\limits_{i=1}^{n} u_i - a\Big)$;

2. *the P-observable $y_n = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$ is defined by the equality*

   $$y_n = h_n \circ g_n^{-1}$$

   where $g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n} u_i$;

3. *the P-observable $y_n = \frac{1}{n}\sum\limits_{i=1}^{n} (x_i - E(x_i))$ is defined by the equality*

   $$y_n = h_n \circ g_n^{-1}$$

   where
   $g_n(u_1, \ldots, u_n) = \frac{1}{n}\sum\limits_{i=1}^{n} (u_i - E(x_i))$.

Also we need the notion of P-distribution function.

Definition 3.3 *Let $m : \mathcal{N} \to [0,1]$ be an P-state and $x : \mathcal{B}(\mathbf{R}) \to \mathcal{N}$ be an P-observable. Then a mapping $F : \mathbf{R} \to [0,1]$ defined by formula*

$$F(t) = m \circ x((-\infty, t)),$$

for each $t \in \mathbf{R}$, is called a distribution function.

Definition 3.4 *Let $(y_i)_1^\infty$ be a sequence of P-observables and m be an P-state.*

(i) The sequence is said to be convergent in distribution to a function $F : \mathbf{R} \to [0,1]$ if for each $t \in \mathbf{R}$

$$\lim_{n\to\infty} (m \circ y_n)((-\infty, t)) = F(t).$$

(ii) The sequence is said to be convergent in measure m to 0 if for each $0 < \varepsilon$, $\varepsilon \in \mathbf{R}$

$$\lim_{n\to\infty} (m \circ y_n)((-\varepsilon, \varepsilon)) = 1.$$

(iii) We say that the sequence converges m-almost everywhere to 0, if

$$\lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\Big(\bigwedge_{n=k}^{k+i} y_n\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big) = 1.$$

Theorem 3.5 *Let $(y_i)_1^\infty$ be a sequence of P-observables, $y_n : \mathcal{B}(R) \to \mathcal{N}$, $h_n : \mathcal{B}(\mathbf{R}^n) \to \mathcal{N}$ their joint P-observable. For each $n = 1, 2, ...$ let $g_n : \mathbf{R}^n \to \mathbf{R}$ be a Borel function. Let further the P-observable $y_n : \mathcal{B}(\mathbf{R}) \to \mathcal{N}$ be given by $y_n = h_n \circ g_n^{-1} = g_n(x_1, ..., x_n)$, $n = 1, 2, ...$ Then there exists a probability space $(X, \mathcal{S}, P)$ and a sequence $(\xi_n)_1^\infty$ of random variables, $\xi_n : X \to \mathbf{R}$ such that if $\eta_n = g_n(\xi_1, ..., \xi_n), n = 1, 2, ...,$ then*

(i) the sequence $y_1, y_2, ...$ converges in distribution to a function F if and only if so does the sequence $\eta_1, \eta_2, ...$;

(ii) $y_1, y_2, ...$ converges to 0 in measure m if and only if $\eta_1, \eta_2, ...$ converges to 0 in measure P;

(iii) if $\eta_1, \eta_2, ...$ converges P-almost everywhere to 0, then $y_1, y_2, ...$ converges m-almost everywhere to 0.

Proof. Put $X = \mathbf{R}^N$, $\mathcal{S} = \sigma(\mathcal{C})$, where $\mathcal{C}$ is the family of all cylinders in R^N. Put $m_n = m \circ h_n$. Then $\{m_n; n \in N\}$ form a consistent family of probability measures $m_n : \mathcal{B}(\mathbf{R}^n) \to [0, 1]$, i.e.

$$m_{n+1}(A \times R) = m_n(A), A \in \mathcal{B}(\mathbf{R}^n), n = 1, 2, ...$$

By the Kolmogorov theorem there exists exactly one probability measure $P : \sigma(\mathcal{C}) \to [0, 1]$ such that

$$P \circ \pi_n^{-1} = m_n,\ n = 1, 2, ...$$

where $\pi_n : \mathbf{R}^N \to \mathbf{R}^n$ is the projection. Put

$$\xi_n : \mathbf{R}^N \to \mathbf{R},\ \xi_n((u_i)_{i=1}^\infty) = u_n,\ n = 1, 2, ...$$

Then

$$\begin{aligned} P(\eta_n^{-1}(A)) &= P((g_n(\xi_1, ..., \xi_n))^{-1}(A)) = \\ &= P(\pi_n^{-1}(g_n^{-1}(A))) = \\ &= m(h_n(g_n^{-1}(A))) = m(y_n(A)). \end{aligned}$$

Therefore

$$\begin{aligned} m(y_n(-\infty, t)) &= P(\eta_n^{-1}(-\infty, t)), \\ m(y_n((-\varepsilon, \varepsilon))) &= P(\eta_n^{-1}((-\varepsilon, \varepsilon))), \end{aligned}$$

what implies (i) and (ii). Let now η_n converges to 0 P-almost everywhere. We have

$$\begin{aligned} & P\Big(\bigcap_{n=k}^{k+i} \eta_n^{-1}\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big) = \\ = \ & m\Big(h_{k+i}\Big(\bigcap_{n=k}^{k+i}\Big\{(t_1, ., , , t_{k+i}) : \\ & g_n(t_1, ..., t_n) \in \Big(-\frac{1}{p},\frac{1}{p}\Big)\Big\}\Big)\Big) \leq \\ \leq \ & m\Big(\bigwedge_{n=k}^{k+i} h_{k+i}\Big(\Big\{(t_1, ..., t_{k+i}) : \\ & (t_1, ..., t_n) \in g_n^{-1}\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big\}\Big)\Big) = \\ = \ & m\Big(\bigwedge_{n=k}^{k+i} h_n \circ g_n^{-1}\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big) = \\ = \ & m\Big(\bigwedge_{n=1}^{k+i} y_n\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big). \end{aligned}$$

Therefore

$$1 \leq \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} m\Big(\bigwedge_{n=k}^{k+i} y_n\Big(\Big(-\frac{1}{p},\frac{1}{p}\Big)\Big)\Big) \leq 1,$$

hence $(y_n)_{n=1}^\infty$ converges to 0 m-almost everywhere.

4 Conclusion

The paper is concerned in the P-probability theory. We showed that there exist relation between convergence of P-observables and convergence their corresponding random variables.

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References

[1] K. T. Atanassov (1999). Intuitionistic Fuzzy sets : Theory and Applications. In *Physica Verlag*, New York.

[2] K. T. Atanassov - B. Riečan (2007). On two new types of probability on IF-events. (Submitted to Notes on IFS).

[3] K. Čunderlíková - Lendelová - B. Riečan. The probability theory on B-structures. (Accepted in FSS).

[4] P. Grzegorzewski - E. Mrówka (2002). Probability of intuitionistic fuzzy events. In *Soft Metods in Probability, Statistics and Data Analysis (P. Grzegorzewski et al. eds.)*, Physica Verlag, New York, pages 105-115.

[5] M. Krachounov (2006). Intuitionistic probability and intuitionistic fuzzy sets. In *First Intern. Workshop on IFS (El-Darzi et al. eds.)*, Univ. of Westminster, London , pages 714 - 717.

[6] K. Lendelová (2005). Convergence of IF-observables. In *Issues in the Representation and Processing of Uncertain and Imprecise Information - Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized nets, and Related Topics*, pages 232-240.

[7] K. Lendelová (2006). Strong law of large numbers for IF-events. In *Proceedings of the Eleventh International Conference IPMU 2006*, July, 2-7, 2006, Paris, France, pages 2363-2366.

[8] K. Lendelová (2006). Conditional IF-probability. In *Advances in Soft Computing: Soft Methods for Integrated Uncertainty Modelling*, pages 275-283.

[9] K. Lendelová, - J. Petrovičová, (2005). Representation of IF-probability on MV-algebras. In *Soft Computing - A Fusion of Foundation, Methodologies and Applications*, Springer-Verlag.

[10] K. Lendelová - B. Riečan (2004). Weak law of large numbers for IF-events. In *Current Issues in Data and Knowledge Engineering (Bernard De Baets et al. eds.)*, EXIT, Warszawa, pagrs 309-314.

[11] J. Petrovičová - B. Riečan (2005). On the central limit theorem on IFS-events. In *Mathware & Soft Computing*, 1.

[12] B. Riečan (2004). Representation of Probabilities on IFS Events. In *Soft Methodology and Random Information Systems (López-Diáz et al. eds.)*, Springer, Berlin Heidelberg New York, pages 243-248.

[13] B. Riečan (2003). A descriptive definition of the probability on intuitionistic fuzzy sets. In *EUSFLAT '2003 (M. Wagenecht, R. Hampet eds.)*, Zittau-Goerlitz Univ. Appl. Sci., pages 263-266.

[14] B.Riečan (2007). Probability theory on IF-events. In *A volume in honor of Daniele Mundici's 60th birthday. Lecture Notes in Computer Science*, Springer, Berlin.

[15] B. Riečan - D. Mundici (2002). Probability on MV-algebras. In *Handbook of Measure Theory (E. Pap ed.)*, Elsevier, Amsterdam, pages 869-909.

[16] B. Riečan - T. Neubrunn (1997), Integral, Measure, and Ordering. In *Kluwer, Dordrecht and Ister Science*, Bratislava.

[17] L. A. Zadeh (1968). Probability measures of fuzzy events. In *J. Math. Anal. Appl.*, 23, pages 421-427.