On the distributivity of implication operations over t-representable t-norms generated from strict t-norms in Atanassov's intuitionistic fuzzy sets theory

Michał Baczyński
Institute of Mathematics
University of Silesia
ul. Bankowa 14, 40-007 Katowice
Poland
e-mail: michal.baczynski@us.edu.pl

Abstract

Recently, many papers have appeared dealing with the distributivity of fuzzy implications over t-norms, t-conorms and uninorms (see [3, 19, 4, 6, 16, 17, 5]). These equations have a very important role to play in efficient inferencing in approximate reasoning, especially fuzzy control systems (see [7]). In this work we discuss distributivity of functions over some t-representable t-norms in Atanassov's intuitionistic fuzzy sets theory. In particular, some solutions which are implication operations are presented.

Keywords: Atanassov's intuitionistic fuzzy sets, fuzzy implication, t-norm, functional equations.

1 Introduction

Distributivity of fuzzy implications over different fuzzy logic connectives has been studied in the recent past by many authors. This topic was introduced by Combs and Andrews in [7] wherein they exploit the following classical tautology

$$(p \wedge q) \to r \equiv (p \to r) \vee (q \to r)$$

in their inference mechanism towards reduction in the complexity of fuzzy "If-Then" rules. Subsequently, there were many discussions in the journal *IEEE Transaction on Fuzzy Systems*, most of them pointing out the need for a theoretical investigation required for employing such equations in a practice.

It was Trillas and Alsina [19], who were the first to investigate the generalized version of the above law

$$I(T(x,y),z) = S(I(x,z), I(y,z)), \quad (1)$$

where T, S are a t-norm and a t-conorm on $([0,1], \leq)$, respectively, and I is a fuzzy implication on $([0,1], \leq)$. Using similar techniques as above, Balasubramaniam and Rao [6] considered the following dual equations of (1):

$$I(S(x,y),z) = T(I(x,z), I(y,z)), \quad (2)$$

$$I(x, T_1(y,z)) = T_2(I(x,y), I(x,z)), \quad (3)$$

$$I(x, S_1(y,z)) = S_2(I(x,y), I(x,z)), \quad (4)$$

where T, T_1, T_2 and S, S_1, S_2 are t-norms and t-conorms on $([0,1], \leq)$, respectively, and I is an S- or R-implication on $([0,1], \leq)$. Meanwhile, Baczyński in [3, 4] considered the functional equation (3), both independently and along with other equations, and characterized functions I in the case when $T_1 = T_2$ is a strict t-norm. It should be noted that the generalizations of the above equations for uninorms were recently studied by Ruiz and Torrens in [16, 17].

In this paper we will consider the distributivity equations in Atanassov's intuitionistic fuzzy sets theory. We are interested in describing all solutions for t-representable t-norms (t-conorms) generated from continuous and Archimedean t-norms (t-conorms). Due to the page limit, we will concentrate only on the equation (3), when $T_1 = T_2$ is

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a t-representable t-norm generated from the product t-norm and I is any binary function defined on the special lattice $\mathcal{L}^I$.

2 Intuitionistic and interval-valued fuzzy sets theories

Intuitionistic fuzzy sets were introduced by Atanassov as an one possible extension of the fuzzy sets theory in the following way.

Definition 1 ([2]). An intuitionistic fuzzy set A on X is a set

$$A = \{(x, \mu_A(x), \nu_A(x)) \quad : \quad x \in X\},$$

where $\mu_A,\ \nu_A \colon X \to [0,1]$ are called, respectively, the membership function and the non-membership function. Moreover they satisfy the condition

$$\mu_A(x) + \nu_A(x) \leq 1, \qquad\qquad x \in X.$$

An intuitionistic fuzzy set A on X can be represented by the $\mathcal{L}^*$-fuzzy set A in the sense of Goguen given by

$$\begin{aligned} A \colon X &\to L^* \\ x &\mapsto (\mu_A(x), \nu_A(x)), \qquad\qquad x \in X, \end{aligned}$$

where $\mathcal{L}^* = (L^*, \leq_{L^*})$ is the following complete lattice

$$L^* = \{(x_1, x_2) \in [0,1]^2 \quad : \quad x_1 + x_2 \leq 1\}$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1 \wedge x_2 \geq y_2$$

with the units $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$.

Another extension of the fuzzy sets theory is interval-valued fuzzy sets theory introduced, independently, by Sambuc and Gorzałczany. We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where

$$L^I = \{(x_1, x_2) \in [0,1]^2 : x_1 \leq x_2\}$$
$$(x_1, x_2) \leq_{L^I} (y_1, y_2) \iff x_1 \leq y_1 \wedge x_2 \leq y_2$$

It can be shown that $\mathcal{L}^I = (L^I, \leq_{L^I})$ is a complete lattice with the units $0_{\mathcal{L}^I} = (0,0)$ and $1_{\mathcal{L}^I} = (1,1)$.

Definition 2 ([18, 12]). An interval-valued fuzzy set on X is a mapping $A \colon X \to L^I$.

In fact, an interval-valued fuzzy set can be seen as a $\mathcal{L}^I$-fuzzy set in the sense of Goguen.

Deschrijver and Kerre [8] showed that intuitionistic fuzzy sets theory is equivalent to interval-valued fuzzy sets theory. Therefore we can investigate operations over intuitionistic fuzzy sets by Atanassov in terms of $\mathcal{L}^*$ or $\mathcal{L}^I$. In this article, we will develop our investigations in the terms of $\mathcal{L}^I = (L^I, \leq_{L^I})$, since the main results will be easier to show.

We assume that the reader is familiar with the classical results concerning basic fuzzy logic connectives, but we briefly mention some of the results employed in the rest of the work. By Φ we denote the family of all increasing bijections $\varphi \colon [0,1] \to [0,1]$. We say that functions $f, g \colon [0,1]^n \to [0,1]$, where $n \in \mathbb{N}$, are Φ-conjugate, if there exists $\varphi \in \Phi$ such that $g = f_\varphi$, where

$$f_\varphi(x_1, \ldots, x_n) := \varphi^{-1}\left(f(\varphi(x_1), \ldots, \varphi(x_n))\right),$$

for all $x_1, \ldots, x_n \in [0,1]$.

Definition 3. Let $\mathcal{L} = (L, \leq_L, 0_L, 1_L)$ be a complete lattice. An associative, commutative, increasing operation $T \colon L^2 \to L$ is called a t-norm on $\mathcal{L}$ if 1_L is the neutral element of T.

Definition 4. We say that a t-norm T on $([0,1], \leq)$ is strict, if it is continuous and strictly monotone, i.e., $T(x,y) < T(x,z)$ whenever $x > 0$ and $y < z$.

The following characterization of strict t-norms is well known in the literature.

Theorem 5 ([13], Proposition 5.9). *For a function $T \colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) T is a strict t-norm.

(ii) T is Φ-conjugate with the product t-norm $T_{\mathbf{P}}$, i.e., there exists $\varphi \in \Phi$, which is uniquely determined up to a positive constant exponent, such that

$$T(x,y) = (T_{\mathbf{P}})_\varphi(x,y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y)),$$

for all $x, y \in [0,1]$.

T-norms on $\mathcal{L}^I$ can be defined in many ways. In our article we shall consider the following special class of t-norms.

Definition 6 (see [9]). A t-norm $\mathcal{T}$ on $\mathcal{L}^I$ is called t-representable if there exist t-norms T_1 and T_2 on $([0,1], \leq)$ such that

$$T_1(x,y) \leq T_2(x,y), \qquad x, y \in [0,1]$$

and for all $(x_1,x_2),(y_1,y_2) \in L^I$

$$\mathcal{T}((x_1,x_2),(y_1,y_2)) = (T_1(x_1,y_1), T_2(x_2,y_2)).$$

It should be noted, that not all t-norms on $\mathcal{L}^I$ are t-representable (see [9]).

In the scientific literature one can find several methods for constructing implications in the intuitionistic, as well interval-valued fuzzy sets theory. One possible definition of an implication on $\mathcal{L}^I$ is based on the notation from fuzzy sets theory introduced by Fodor and Roubens in 1994.

Definition 7 (cf. [11], Definition 1.15). Let $\mathcal{L} = (L, \leq_L, 0_L, 1_L)$ be a complete lattice. A function $I \colon L^2 \to L$ is called an implication on $\mathcal{L}$ if it satisfies the following conditions:

$$I \text{ is decreasing in the first variable,}$$
$$I \text{ is increasing in the second variable,}$$
$$I(0_L,0_L) = I(1_L,1_L) = 1_L, \quad I(1_L,0_L) = 0_L.$$

Directly from the above definition we can deduce, that each implication I on $\mathcal{L}$ satisfies the following properties, called left and right boundary condition, respectively:

$$I(0_L, y) = 1_L, \quad y \in L, \tag{5}$$
$$I(x, 1_L) = 1_L\,, \quad x \in L. \tag{6}$$

Therefore, I satisfies also the normality condition $I(0,1) = 1$. Consequently, every implication restricted to the set $\{0_L, 1_L\}^2$ coincides with the classical implication.

When $L = ([0,1], \leq)$, then I is called a fuzzy implication. If $L = \mathcal{L}^*$, then I is called an intuitionistic fuzzy implication, while when $L = \mathcal{L}^I$, then I is called an interval-valued fuzzy implication and will denoted by $\mathcal{I}$. Detailed investigations on different classes of implications on above lattices and their algebraic properties were presented in [15] and [10].

Finally, the first and the second projection mappings pr_1 and pr_2 on $\mathcal{L}^I$ are defined as

$$pr_1(x_1,x_2) = x_1, \qquad pr_2(x_1,x_2) = x_2,$$

for all $(x_1,x_2) \in L^I$.

3 Some new results pertaining to functional equations

Here we show some new results related to the following functional equation:

$$f(x_1 \cdot y_1, x_2 \cdot y_2) = f(x_1,x_2) \cdot f(y_1,y_2). \tag{7}$$

The presented facts, which are important in the proof of the main results, can be seen as the generalizations of the classical facts from the theory of functional equations (see [1]).

Recall, that a function f from one metric space (X, d_X) to another metric space (Y, d_Y) is continuous at the point $x_0 \in X$ if for any positive real number ε, there exists a positive real number δ such that all $x \in X$ satisfying $d_X(x_0, x) < \delta$ will also satisfy $d_Y(f(x_0), f(x)) < \varepsilon$. On $\mathcal{L}^*$ or $\mathcal{L}^I$ we can consider different metrics generated from distances on $\mathbb{R}^2$. From now on, we assume that $\mathcal{L}^I$ is equipped with the classical Euclidean distance. For more discussion about continuity in $\mathcal{L}^*$ and, consequently, in $\mathcal{L}^I$ see [9].

Proposition 8. *For a continuous function $f \colon L^I \to [0,1]$ the following statements are equivalent:*

(i) f satisfies the functional equation (7) *for all $(x_1,x_2),(y_1,y_2) \in L^I$.*

(ii) Either $f = 0$, or $f = 1$, or there exists a unique constant $c \in (0, \infty)$ such that

$$f(a,b) = a^c, \tag{8}$$

or

$$f(a,b) = b^c, \tag{9}$$

or there exist unique constants $c_1, c_2 \in (0,\infty)$ such that

$$f(a,b) = a^{c_1} \cdot b^{c_2}, \tag{10}$$

for all $(a,b) \in L^I$.

Proof. (*ii*) $\implies$ (*i*) It is a direct calculation that all the above functions are continuous and satisfy the functional equation (7).

(*i*) $\implies$ (*ii*) Let a function $f\colon L^I \to [0,1]$ satisfy (7) for all $(x_1, x_2), (y_1, y_2) \in L^I$.

Setting $x_1 = y_1 = 0$ in (7) we get

$$f(0, x_2 \cdot y_2) = f(0, x_2) \cdot f(0, y_2).$$

Consider now the following function of one variable $g_0 := f(0, \cdot)$. By our assumptions, since f is continuous, g_0 is the continuous function from $[0,1]$ to $[0,1]$ such that

$$g_0(x_2 \cdot y_2) = g_0(x_2) \cdot g_0(y_2),$$

for all $x_2, y_2 \in [0,1]$. By the well known continuous solutions of the above multiplicative Cauchy functional equation for real numbers on the restricted domain (see [1] or [14], Theorem 13.1.6) we get that either $f(0,b) = 0$ for all $b \in [0,1]$, or $f(0,b) = 1$ for all $b \in [0,1]$, or there exists a unique constant $c \in (0, \infty)$ such that $f(0,b) = b^c$ for all $b \in [0,1]$.

If $f(0,b) = 1$ for all $b \in [0,1]$, then putting $x_1 = 0$ in (7) we obtain

$$f(0, x_2 \cdot y_2) = f(0, x_2) \cdot f(y_1, y_2),$$

hence

$$1 = 1 \cdot f(y_1, y_2),$$

therefore $f(y_1, y_2) = 1$ for all $(y_1, y_2) \in L^I$, so $f = 1$.

If $f(0,b) = b^c$ for all $b \in [0,1]$, then putting $x_1 = 0$ in (7) we obtain

$$f(0, x_2 \cdot y_2) = f(0, x_2) \cdot f(y_1, y_2),$$

hence

$$(x_2 \cdot y_2)^c = x_2^c \cdot f(y_1, y_2).$$

Let $x_2 > 0$, then we obtain, that $f(y_1, y_2) = y_2^c$ for all $(a,b) \in L^I$, so f has the form (9).

From above we can summarize that we have to investigate only the last case, when $f(0,b) = 0$ for all $b \in [0,1]$.

Setting now $x_2 = y_2 = 1$ in (7) we get

$$f(x_1 \cdot y_1, 1) = f(x_1, 1) \cdot f(y_1, 1).$$

Similarly as above we get that either $f(a,1) = 0$ for all $a \in [0,1]$, or $f(a,1) = 1$ for all $a \in [0,1]$, or there exists a unique constant $c \in (0, \infty)$ such that $f(a,1) = a^c$ for all $a \in [0,1]$.

If $f(a,1) = 1$ for all $a \in [0,1]$, then, in particular $f(0,1) = 1$, which is in a contradiction with our assumption that $f(0,b) = 0$ for all $b \in [0,1]$.

If $f(a,1) = 0$ for all $a \in [0,1]$, then putting $x_1 = x_2 = 1$ in (7) we obtain

$$f(y_1, y_2) = f(1,1) \cdot f(y_1, y_2),$$

hence

$$f(y_1, y_2) = 0, \quad (y_1, y_2) \in L^I,$$

so $f = 0$ in this situation.

If $f(a,1) = a^c$ for all $a \in [0,1]$, then putting $x_2 = 1$ in (7) we obtain

$$f(x_1 \cdot y_1, y_2) = f(x_1, 1) \cdot f(y_1, y_2),$$

hence

$$f(x_1 \cdot y_1, y_2) = x_1^c \cdot f(y_1, y_2).$$

Using now some techniques from the theory of functional equations for the Pexider version of the multiplicative Cauchy equation (cf. [14], Theorem 13.3.8) one can show that in this situation either f has the form (8), or (10). $\square$

Example 9. Consider the following function $f\colon L^I \to [0,1]$ given by

$$f(a,b) = \begin{cases} 0, & \text{if } (a,b) = (0,0) \\ 1, & \text{otherwise.} \end{cases}$$

It can be easily checked that it satisfies the functional equation (7), but it is not continuous in the point $(0,0)$. Therefore, the full description of the solutions of (7) is still an open problem.

4 Main results

Using the result from the previous section, we are able to obtain description of some solutions of the equation (3), when both t-norms on $\mathcal{L}^I$ are t-representable and generated from strict t-norms. For the simplicity we will consider the situation, when both t-norms on $\mathcal{L}^I$ are equal and generated from the product t-norm $T_{\mathbf{P}}(x,y) = xy$.

Theorem 10. *For the t-representable t-norm $\mathcal{T}$ on $\mathcal{L}^I$ generated from the product t-norm $T_{\mathbf{P}}$ and a function $\mathcal{I}\colon (L^I)^2 \to L^I$ which is continuous with respect to the second variable, the following statements are equivalent:*

(i) The pair of functions $\mathcal{T},\mathcal{I}$ satisfies the functional equation

$$\mathcal{I}(x,\mathcal{T}(y,z)) = \mathcal{T}(\mathcal{I}(x,y),\mathcal{I}(x,z)), \quad (11)$$

for all $x,y,z \in L^I$.

(ii) For every fixed $x = (x_1,x_2) \in L^I$ the vertical section $\mathcal{I}((x_1,x_2),\cdot)$ has one of the following forms

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (0,0),$$

or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (0,1),$$

or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (1,1),$$

or there exist unique constants $c_x,d_x,e_x,f_x \in (0,\infty)$ such that

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (0,y_1^{e_x}),$$

or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (0,y_2^{f_x}),$$

or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (0,y_1^{e_x}\cdot y_2^{f_x}),$$

or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (y_1^{c_x},1),$$

or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (y_2^{d_x},1),$$

or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (y_1^{c_x}\cdot y_2^{d_x},1),$$

or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (y_1^{c_x},y_1^{e_x}),$$

with $c_x \geq e_x$, or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (y_1^{c_x},y_2^{f_x}),$$

with $c_x \geq f_x$, or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (y_1^{c_x},y_1^{e_x}\cdot y_2^{f_x}),$$

with $c_x - e_x \geq f_x$, or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (y_1^{c_x}\cdot y_2^{d_x},y_1^{e_x}),$$

with $c_x \geq e_x$, or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (y_2^{d_x},y_2^{f_x}),$$

with $d_x \geq f_x$, or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (y_1^{c_x}\cdot y_2^{d_x},y_2^{f_x}),$$

with $d_x \geq f_x$, or

$$\mathcal{I}((x_1,x_2)(y_1,y_2)) = (y_1^{c_x}\cdot y_2^{d_x},y_1^{e_x}\cdot y_2^{f_x}),$$

with $c_x - e_x \geq f_x - d_x$, for $(y_1,y_2) \in L^I$.

Proof. $(ii) \Longrightarrow (i)$ The proof in this direction can be checked by a direct substitution.

$(i) \Longrightarrow (ii)$ Let us assume that a t-representable t-norm $\mathcal{T}$ and a function $\mathcal{I}$ are the solutions of the functional equation (11) satisfying the required properties. At this situation our equation has the following form

$$\mathcal{I}((x_1,x_2),(y_1\cdot z_1,y_2\cdot z_2)) = (pr_1(\mathcal{I}((x_1,x_2),(y_1,y_2)))\cdot pr_1(\mathcal{I}((x_1,x_2),(z_1,z_2))), pr_2(\mathcal{I}((x_1,x_2),(y_1,y_2)))\cdot pr_2(\mathcal{I}((x_1,x_2),(z_1,z_2))))$$

for all $(x_1,x_2),(y_1,y_2),(z_1,z_2) \in L^I$. As a consequence we obtain the following two equations

$$pr_1(\mathcal{I}((x_1,x_2),(y_1\cdot z_1,y_2\cdot z_2))) = pr_1(\mathcal{I}((x_1,x_2),(y_1,y_2)))\cdot pr_1(\mathcal{I}((x_1,x_2),(z_1,z_2))),$$

and

$$pr_2(\mathcal{I}((x_1,x_2),(y_1\cdot z_1,y_2\cdot z_2))) = pr_2(\mathcal{I}((x_1,x_2),(y_1,y_2)))\cdot pr_2(\mathcal{I}((x_1,x_2),(z_1,z_2)))$$

which are satisfied for all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in L^I$.

Fix arbitrarily $(x_1, x_2) \in L^I$. Define a function $\mathcal{I}_{x_1,x_2} \colon L^I \to L^I$ by the formula

$$\mathcal{I}_{x_1,x_2}(y_1, y_2) = \mathcal{I}((x_1, x_2), (y_1, y_2)),$$

for all $(y_1, y_2) \in L^I$. This function is continuous. By the substitutions, $g_{x_1,x_2} = pr_1 \circ \mathcal{I}_{x_1,x_2}$, and $h_{x_1,x_2} = pr_2 \circ \mathcal{I}_{x_1,x_2}$, we obtain the following two functional equations

$$g_{x_1,x_2}(y_1 \cdot z_1, y_2 \cdot z_2) = g_{x_1,x_2}(y_1, y_2) \cdot g_{x_1,x_2}(z_1, z_2),$$

and

$$h_{x_1,x_2}(y_1 \cdot z_1, y_2 \cdot z_2) = h_{x_1,x_2}(y_1, y_2) \cdot h_{x_1,x_2}(z_1, z_2),$$

which are satisfied for all $(y_1, y_2), (z_1, z_2) \in L^I$. Let us observe now, that both equations are just the other versions of the functional equation (7). From Proposition 8 we obtain all possible continuous solutions for g_{x_1,x_2} and h_{x_1,x_2}. Since in this proposition we have 5 possible solutions, we should have 25 different solutions of (11). But observe, that some of these solutions are not good, since the range of $\mathcal{I}$ is L^I. For example the solution when the vertical section $(1, 0)$ is not good, since 1 is not less than or equal to 0. Considering all possible pairs and above assumption we obtain exactly 16 different solutions. For example we show the full solution when

$$g_{x_1,x_2}(a, b) = a^{c_x} \cdot b^{d_x}$$

and

$$h_{x_1,x_2}(a, b) = a^{e_x} \cdot b^{f_x},$$

with some positive real constants c_x, d_x, e_x, f_x, which depend on the fixed $x = (x_1, x_2)$. Then we get

$$pr_1 \circ \mathcal{I}_{x_1,x_2}(a, b) = a^{c_x} \cdot b^{d_x}$$

and

$$pr_2 \circ \mathcal{I}_{x_1,x_2}(a, b) = a^{e_x} \cdot b^{f_x}.$$

Therefore

$$\mathcal{I}((x_1, x_2)(y_1, y_2)) = (y_1^{c_x} \cdot y_2^{d_x}, y_1^{e_x} \cdot y_2^{f_x}).$$

Finally, the range of $\mathcal{I}$ is L^I, so

$$y_1^{c_x} \cdot y_2^{d_x} \le y_1^{e_x} \cdot y_2^{f_x}$$

for all $(y_1, y_2) \in L^I$, which implies, that

$$y_1^{c_x - e_x} \le y_1^{e_x - f_x},$$

thus $c_x - e_x \ge f_x - d_x$. □

We would like to notice, that not all obtained vertical solutions in the above theorem can be used for obtaining an implication on $\mathcal{L}^I$ in the sense of Definition 7. By (6) one can easily see that the following vertical sections are not possible: $(0, 0)$, $(0, 1)$, $(0, y_1^{e_x})$, $(0, y_2^{f_x})$ and $(0, y_1^{e_x} \cdot y_2^{f_x})$. This mean, that only 11 different vertical sections can be considered to obtaining an implication operation. The full description of solutions of (11), which are implications on $\mathcal{L}^I$, even with continuous sections, is still unknown, but in the full version of this article we expect to present such result.

Example 11. Let us consider the following function

$$\mathcal{I}((x_1, x_2)(y_1, y_2)) = \begin{cases} 1_{\mathcal{L}^I}, & \text{if } x_1 = y_1 = 0 \\ (y_1^{x_2}, y_2^{x_1}), & \text{otherwise.} \end{cases}$$

defined for all $((x_1, x_2), (y_1, y_2)) \in L^I$. One can easily check, that this function is an implication on $\mathcal{L}^I$. Indeed

$$\begin{aligned} \mathcal{I}(0_{\mathcal{L}^I}, 0_{\mathcal{L}^I}) &= \mathcal{I}((0, 0), (0, 0)) = 1_{\mathcal{L}^I}, \\ \mathcal{I}(1_{\mathcal{L}^I}, 1_{\mathcal{L}^I}) &= \mathcal{I}((1, 1), (1, 1)) = (1^1, 1^1) \\ &= (1, 1) = 1_{\mathcal{L}^I}, \\ \mathcal{I}(1_{\mathcal{L}^I}, 0_{\mathcal{L}^I}) &= \mathcal{I}((1, 1), (0, 0)) = (0^1, 0^1) \\ &= (0, 0) = 0_{\mathcal{L}^I}. \end{aligned}$$

Moreover, if we fix arbitrarily $(y_1, y_2) \in L^I$, then for $(x_1, x_2) \le (x_1', x_2')$ we get $x_1 \le x_1'$ and $x_2 \le x_2'$. Thus $y_1^{x_2} \ge y_1^{x_2'}$ and $y_2^{x_1} \ge y_2^{x_1'}$. Therefore $(y_1^{x_2}, y_2^{x_1}) \ge (y_1^{x_2'}, y_2^{x_1'})$, so $\mathcal{I}$ is decreasing in the first variable. In a similar way one can show that $\mathcal{I}$ is increasing in the second variable. Finally observe that it satisfies the functional equation (11) with the t-representable t-norm $\mathcal{T}$ on $\mathcal{L}^I$ generated from the product t-norm $T_{\mathbf{P}}$:

$$\mathcal{T}((x_1, x_2)(y_1, y_2)) = (x_1 y_1, x_2 y_2).$$

Indeed, for all $x, y, z \in L^I$ we get

$$
\begin{aligned}
&\mathcal{T}(\mathcal{I}(x,y),\mathcal{I}(x,z)) \\
&= \mathcal{T}(\mathcal{I}((x_1,x_2),(y_1,y_2)),\mathcal{I}((x_1,x_2),(z_1,z_2))) \\
&= \begin{cases} \mathcal{T}(1_{\mathcal{L}^I},1_{\mathcal{L}^I}), & \text{if } x_1=y_1=z_1=0 \\ \mathcal{T}(1_{\mathcal{L}^I},(z_1^{x_2},1)), & \text{if } x_1=y_1=0 \\ \mathcal{T}((y_1^{x_2},1),1_{\mathcal{L}^I}), & \text{if } x_1=z_1=0 \\ \mathcal{T}((y_1^{x_2},y_2^{x_1}),(z_1^{x_2},z_2^{x_1})), & \text{otherwise} \end{cases} \\
&= \begin{cases} 1_{\mathcal{L}^I}, & \text{if } x_1=y_1=z_1=0 \\ (z_1^{x_2},1), & \text{if } x_1=y_1=0 \\ (y_1^{x_2},1), & \text{if } x_1=z_1=0 \\ (y_1^{x_2}\cdot z_1^{x_2},y_2^{x_1}\cdot z_2^{x_1}), & \text{otherwise} \end{cases} \\
&= \begin{cases} 1_{\mathcal{L}^I}, & \text{if } x_1=0 \wedge (y_1=0 \vee z_1=0) \\ ((y_1\cdot z_1)^{x_2},(y_2\cdot z_2)^{x_1}), & \text{otherwise} \end{cases} \\
&= \mathcal{I}(x,\mathcal{T}(y,z)) = \mathcal{I}((x_1,x_2),(y_1z_1,y_2z_2)).
\end{aligned}
$$

The above implication can be seen as the interval-valued generalization of the classical Yager implication (see [20]):

$$
I_{\mathbf{YG}}(x,y) = \begin{cases} 1, & \text{if } x=0 \text{ and } y=0 \\ y^x, & \text{otherwise} \end{cases}
$$

for $x, y \in [0,1]$, which satisfies the distributive equation (3) with the product t-norm $T_{\mathbf{P}}$ (cf. [3], Corollary 2).

5 Summary

In this paper we have examined one possible distributive equation defined on lattice $\mathcal{L}^I$. More precisely, we have obtained some solutions of the equation (11), when $\mathcal{T}$ is a t-representable t-norm generated from the product t-norm. It should be noted, that obtained facts can be easily transformed to the solutions in the lattice $\mathcal{L}^*$. Also, we would like to underline that the situation when $\mathcal{T}$ is t-representable and generated from continuous, Archimedean t-norms can be examined by using similar techniques as above. In our future works we will concentrate on the other possible distributive equations on $\mathcal{L}^I$ for t-representable operations.

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