Pseudo-linear superposition principle for the Monge-Ampère equation based on generated pseudo-operations with three parameters

Doretta Vivona
Dipartimento di Metodi e Modelli
Matematici per le Scienze Applicate
Sapienza-Università di Roma
Via A. Scarpa 10, 00161 Rome, Italy
vivona@dmmm.uniroma1.it

Ivana Štajner-Papuga
Department of Mathematics
and Informatics
University of Novi Sad
Trg D. Obradovica 4, Novi Sad, Serbia
stajner@im.ns.ac.yu

Abstract

We are considering generated pseudo-operations with three parameters of the following form: $x \oplus y = g^{-1}(\varepsilon_1 g(x) + \varepsilon_2)$, $x \odot y = g^{-1}\left(g^{\delta}(x) g(y)\right)$, where g is a strictly monotone bijection and ε_1, ε_2 and δ arbitrary but fixed positive real numbers. The superposition principle with this type of pseudo-operations in the core for the Monge-Ampère equation is presented.

Keywords: Pseudo-operations with three parameters, Pseudo-superposition principle, Monge-Ampère equation.

1 Introduction

In the field of differential equations one of the highly investigated issues is the issue of obtaining new solutions of differential equations. A nonlinear superposition principle (NLSP), i.e., the principle which insures that if u and v are solutions of some differential equation, then $u * v$ is also a solution of the same differential equation for the certain operation $*$, has proved itself to be a useful tool for constructing new solutions of ordinary and partial differential equations (see [3, 10, 13, 35]). This approach had been extended in the direction of noncommutative operations $*$ ([10, 12, 16]), as well as in the direction of the pseudo-analysis ([19, 25, 26, 28]). Application of the noncommutative operations $*$ had also been investigated in the pseudo-analysis' framework (see [31, 32, 33]). Here, by pseudo-analysis, we consider a generalization of the classical analysis that combines approaches from many different fields and is capable of supplying solutions that were not achieved by the classical tools. Some of important results concerning pseudo-analysis, both theory and application, can be found in [4, 11, 14, 15, 19, 25, 27, 30, 31].

Additionally, important connection between NLSP and Lie symmetry algebras ([5]) for the classical approach had been established in [10].

The focus of this paper is on the well-known Monge-Ampère equation which was introduced in 1815 by Ampère (see [2]) as nonlinear equation of two variables of the second order. Later on, many authors, including Boillat, Donato, Ramgulam, Rogers and Ruggeri, studied the equation of this type in a more general setting ([6, 7, 8, 9, 34]. Subsequently, Oliveri found a connection between the Monge-Ampère equations and their Lie symmetries ([21, 22, 23]). The main aim of this contribution is to present pseudo-analysis' approach to the problem of finding new solutions for the homogeneous Monge-Ampère equation of the form

$$u_{xx}u_{tt} - u_{xt}^2 = 0.$$

More precisely, as a framework for this investigation we are using general pseudo-analysis introduced in [31]. Therefore, the pseudo-linear superposition principle that is being used through this paper is based on the generalized generated pseudo-operations with three parameters, i.e., operations given by a generating function that need not be commutative nor associative. Previously this approach had been applied in [32, 33].

Section 2 contains preliminary notions, such as pseudo-operations, semiring and generalized pseudo-operations. The pseudo-linear superposition principle for the Monge-Ampère equation is considered in the third section. Some concluding remarks are given in the Section 4.

2 Preliminary notions

By pseudo-operations, namely the pseudo-addition and the pseudo-multiplication, we consider operations that are generalizations of the classical addition and multiplication. If $[a,b]$ is a closed subinterval of $[-\infty,+\infty]$ (in some cases semiclosed subintervals) and $\preceq$ a total order on $[a,b]$, *the pseudo-addition* is a function $\oplus : [a,b]\times[a,b] \to [a,b]$ which is commutative, non-decreasing (with respect to $\preceq$), associative and with a zero element $\mathbf{0}$. *The pseudo-multiplication* is a function $\odot : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively non-decreasing ($x \preceq y$ implies $x \odot z \preceq y \odot z,\ z \in [a,b]_+ = \{x : x \in [a,b],\ \mathbf{0} \preceq x\}$), associative and for which there exists a unit element denoted by $\mathbf{1}$.

It is usually requested for the pseudo-addition and pseudo-multiplication to fulfill the following two conditions:

- $\mathbf{0} \odot x = \mathbf{0}$;
- $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$.

Now, the triplet $([a,b],\oplus,\odot)$ is called a *semiring.* There are three basic classes of semirings with continuous (up to some points) pseudo-operations. The first class contains semirings with idempotent pseudo-addition and non idempotent pseudo-multiplication. Semirings with strict pseudo-operations defined by strictly monotone bijection $g : [a,b] \to [0,+\infty]$, i.e., g-semirings, form the second class, and semirings with both idempotent operations belong to the third class. More on this structure as well as on corresponding measures and integrals can be found in [15, 19, 24, 25, 27].

Pseudo-operations $\oplus$ and $\oplus$ are the core of the pseudo-analysis. However, at this point, we are focusing on generalization of the pseudo-analysis, known as *the general pseudo-analysis*, that had been presented in [31] and is based on the generalized pseudo-operations. The complete characterization of the *generalized pseudo-addition and pseudo-multiplication* was given in [31]. Definition of the generalized pseudo-operations follows and, in order to avoid possible confusion, those operations are denoted with $\oplus'$ and $\odot'$.

Definition 1 *Real operations* $\oplus'$ *and* $\odot'$ *are* the generalized pseudo-addition *and* the generalized pseudo-multiplication *from the right (or from the left), if the following hold:*
(i) $\oplus'$ *and* $\odot'$ *are functions from* $C^2(\mathbb{R}^2)$*,*
(ii) the equation $t \oplus' t = z$ *for given* z *is uniquely solvable,*
(iii) $\odot'$ *is right (left) distributive over* $\oplus'$*, i.e.,*

$$(x \oplus' y) \odot' z = (x \odot' z) \oplus' (y \odot' z)$$
$$\big(z \odot' (x \oplus' y) = (z \odot' x) \oplus' (z \odot' y)\big).$$

Now, the basis for the pseudo-superposition principle presented in this paper are generated pseudo-operations with three parameters. Operations in question belong to the special class of generalized generated pseudo-operations firstly introduced in [31] and than extended to the three parameters case in [32].

Definition 2 *Let* $g : [a,b] \to [0,+\infty]$ *be a strictly monotone bijection and let* ε_1, ε_2 *and* δ *be arbitrary but fixed positive real numbers.* The generated pseudo-operations with three parameters *are*

$$u \oplus' v = g^{-1}(\varepsilon_1 g(u) + \varepsilon_2 g(v)) \qquad (1)$$

and

$$u \odot' v = g^{-1}(g^{\delta}(u) g(v)) \qquad (2)$$

Further on by $\oplus'$ and $\odot'$ operations (1) and (2) will be denoted.

It should be stressed that operations given by the previous definition (as well as operations introduced in [31]) need not be commutative nor associative. Therefore, it is necessary to define pseudo-sum of n elements $\alpha_i \in [a,b]$, $i \in \{1,2,\dots n\}$:

$$\bigoplus_{i=1}^{n} \alpha_i = \left(\dots\left(\left(\alpha_1 \oplus' \alpha_2\right) \oplus' \alpha_3\right) \oplus' \dots\right) \oplus' \alpha_n.$$

Remark 3 Operations (1) and (2) are generalization of pseudo-operations from the second class of semirings, i.e., of the strict operations given by the continuous generator $g : [a,b] \to [0,+\infty]$ in the following manner

$$x \oplus y = g^{-1}(g(x) + g(y)) \tag{3}$$

and

$$x \odot y = g^{-1}(g(x)g(y)). \tag{4}$$

This representation of strict pseudo-operations is based on Aczél's representation theorem from [1].

Operations (3) and (4) are the base for g-calculus (see [17, 20, 24, 25, 28]).

Remark 4 (i) Pseudo-analysis has been successfully applied on finding weak solutions of the Cauchy problem for the Hamilton-Jacobi equation, i.e., for

$$\frac{\partial u}{\partial t} + H\left(\frac{\partial u}{\partial x}\right) = 0, \quad u(x,0) = u_0(x), \tag{5}$$

where $x \in \mathbb{R}^n$, and the function $H : \mathbb{R}^n \to \mathbb{R}$ is convex (and continuous) (see [15, 19, 26, 27, 28]). For an arbitrary function $u_0(x)$ bounded from below, the weak pseudo-solution of the Cauchy problem (5) is

$$\left(R_t u_0\right)(x) = h \star C_l u_0(x)$$

where $h(x) = t\mathcal{L}^{\oplus}(H)\left(\frac{x}{t}\right)$ for fixed t, $\mathcal{L}^{\oplus}$ is a pseudo-Laplace transform, $C_l k(x) = \sup\{\psi(x) | \psi \in \mathcal{C}(\mathbb{R}^n), \psi \leq k\}$ and $\star$ is a pseudo-convolution, all based on the semiring $((-\infty,+\infty], \min, +)$ (see [26, 27, 29, 30]).

(ii) Application of the g-calculus in the field of nonlinear PDE can be illustrated through the Burgers equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 - \frac{c}{2}\frac{\partial^2 u}{\partial x^2} = 0, \tag{6}$$

$u(x,0) = u_0(x)$, where $x \in \mathbb{R}$, $t > 0$ and c is given positive constant ([18, 27, 28]). If for the generating function is taken function $g(u) = e^{-u/c}$, corresponding pseudo-operations are

$$u \oplus v = -c \ln\left(e^{-u/c} + e^{-v/c}\right) \tag{7}$$

and

$$u \odot v = u + v, \tag{8}$$

and the pseudo-integral for this particular choice of pseudo-operations is of the form

$$\int^{\oplus} f(x)dx = -c \ln\left(\int e^{-f(x)/c}\, dx\right).$$

Now, a solution of the given initial problem is

$$u(x,t) = \frac{c}{2}\ln(2\pi ct) \odot \int^{\oplus} \frac{(x-s)^2}{2t} \odot u_0(s)\, ds.$$

The pseudo-linear superposition principle based on the operations (7) and (8) holds (see [31]). Additionally, pseudo-linear superposition principle based on operations (1) and (2) has been investigated in [32].

3 Pseudo-linear superposition principle based on $\oplus'$ and $\odot'$

Let us consider the homogeneous Monge-Ampère equation

$$u_{xx}u_{tt} = u_{xt}^2, \tag{9}$$

where $u = u(x,t)$ is a function of two real variables with values in $[a,b]$,

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2} \quad \text{and} \quad u_{xt} = \frac{\partial^2 u}{\partial x \partial t}.$$

It is possible to observe two directions for this investigation. The first one considers the family of all solutions of the equation (9), denoted with $\mathcal{F}$. For this direction a restriction on the generator g has to be imposed. On the other hand, the second direction is centered on the subclass of $\mathcal{F}$ of the following form

$$\mathcal{F}_1 = \{u \mid u = \varphi(x+t)\},$$

where φ is some real function with values in $[a, b]$ from the class C^2. In this case, restrictions for the generator g will be avoided.

For this whole section let us suppose that generator g is, additionally, a function from the class $C^2([a, b])$.

3.1 Pseudo-linear superposition principle for the class $\mathcal{F}$

Theorem 5 *Let $u = u(x, t)$ be a solution of the equation (9) and let $g : [a, b] \to [0, +\infty]$ be a generating function for (2) such that*

$$g^{\delta}(\alpha) \cdot \frac{g''(\alpha \odot' u)}{(g'(\alpha \odot' u))^2} = \frac{g''(u)}{(g'(u))^2} \quad (10)$$

holds for a real parameter $\alpha \in [a, b]$. Then, $\alpha \odot' u$ is a solution of (9).

Proof. Let us assume that u is a solution of (9), i.e., $u_{xx}u_{tt} = u_{xt}^2$. Now, after calculating partial derivatives, we have

$$(\alpha \odot' u)_{xx} (\alpha \odot' u)_{tt} - \left((\alpha \odot' u)_{xt}\right)^2 =$$

$$\frac{g^{2\delta}(\alpha) g'(u)}{(g'(\alpha \odot' u))^2} \left(u_{xx}u_t^2 + u_{tt}u_x^2 - 2u_x u_t u_{xt}\right) G_u^{\alpha}, \quad (11)$$

where

$$G_u^{\alpha} = \left(g''(u) - \frac{g^{\delta}(\alpha) (g'(u))^2 g''(\alpha \odot' u)}{(g'(\alpha \odot' u))^2}\right).$$

Expression (11) is equal to 0, i.e., $\alpha \odot' u$ is a solution of the equation (9) for $\alpha \in [a, b]$, if (10) holds for the generating function g. □

For the class $\mathcal{F}$, i.e., for the class of all solutions of (9), pseudo-superposition principle based on $\oplus'$ is restricted to the following corollary.

Corollary 6 *Let (10) holds for the generator g. If $u = u(x, t)$ is a solution of the equation (9), then, for all $n \in \mathbb{N}$, $\underbrace{u \oplus' u \oplus' \ldots \oplus' u}_{n}$ is a solution of (9).*

Proof. Since for all $n \in \mathbb{N}$ we have

$$\underbrace{u \oplus' u \oplus' \ldots \oplus' u}_{n} = g^{-1}\left(E(n)^{1/\delta}\right) \odot' u,$$

where

$$E(n) = \varepsilon_1^{n-1} + \varepsilon_2 \sum_{i=0}^{n-2} \varepsilon_1^i,$$

this claim follows directly from Theorem 5. □

Remark 7 For $g : [a, b] \to [0, +\infty]$ being a strictly increasing function from the class $C^2([a, b])$, by reducing (10) to the classical Cauchy functional equation, the following conclusion can be obtained: g is a solution of the equation (10) if and only if it is of the form

$$g(z) = e^{Az+B}, \text{ while } A > 0 \text{ and } B \in \mathbb{R},$$

or

$$g(z) = (Az + B)^p, \text{ while } A, p > 0 \text{ and } B \in \mathbb{R}.$$

Similar conclusion can be obtained for a strictly decreasing generator.

3.2 Pseudo-linear superposition principle for the subclass $\mathcal{F}_1$

If we consider the subclass $\mathcal{F}_1$, complete pseudo-linear superposition principle is obtained, i.e., no restrictions for generator g are requested.

Theorem 8 *Let u be a solutions of (9) from $\mathcal{F}_1$, let $\odot'$ be a pseudo-multiplication of the form (2) given by some generator g and let $\alpha \in [a, b]$. Then, $\alpha \odot' u \in \mathcal{F}_1$.*

Proof. Since u is a solutions from $\mathcal{F}_1$, we have $u_x = u_t$ and $u_{xx} = u_{xt} = u_{tt}$, and the claim follows directly from (11). □

Theorem 9 *Let u and v be solutions of (9) from $\mathcal{F}_1$, let $\oplus'$ and $\odot'$ be pseudo-operations (1) and (2) given by some generator g and let $\alpha \in [a, b]$. Then*

a) $u \oplus' v \in \mathcal{F}_1$,

b) $u \odot' \alpha \in \mathcal{F}_1$.

Proof. a) Let u and v be a solutions of the equation (9) from $\mathcal{F}_1$ and let $u \oplus' v = g^{-1}(\varepsilon_1 g(v) + \varepsilon_2 g(u))$. Since u and v are not just solutions, but solutions from $\mathcal{F}_1$, which insures that $u(x,t) = \varphi_1(x+t)$ and $u(x,t) = \varphi_2(x+t)$ for some real functions φ_1 and φ_2 from the class C^2 with values in $[a,b]$, $u \oplus' v$ can also be represented by a real function from the class C^2 with values in $[a,b]$ that contains φ_1, φ_2 and g. Therefore, $u \oplus' v \in \mathcal{F}_1$.

b) Let u be a solution from $\mathcal{F}_1$, α some real parameter from $[a,b]$ and $u \odot' \alpha = g^{-1}\left(g^{\delta}(u)g(\alpha)\right)$. As in the previous case, it is easily obtained that $u \odot' \alpha$ can be represented by a real function from the class C^2 with values in $[a,b]$, which insures $u \oplus' v \in \mathcal{F}_1$. Now, this function consists of φ and g, where for φ holds $u(x,t) = \varphi(x+t)$.

Since assumption $u \in \mathcal{F}_1$ insures $u_x = u_t$ and $u_{xx} = u_{xt} = u_{tt}$, the fact that $u \odot' \alpha$ is a solution also follows from

$$\left(u \odot' \alpha\right)_{xx}\left(u \odot' \alpha\right)_{tt} - \left(\left(u \odot' \alpha\right)_{xt}\right)^2 = D_u^{\alpha}\left(u_x^2 u_{tt} + u_t^2 u_{xx} - 2u_x u_t u_{xt}\right),$$

where

$$D_u^{\alpha} = \frac{g^2(\alpha)\delta^2 g^{2\delta-4}(u)g'(u)}{\left(g'\left(u \odot' \alpha\right)\right)^2} \cdot \left\{ g(u)(g'(u))^2(\delta-1) + g''(u)g^2(u) - \frac{g(\alpha)\delta g^{\delta+1}(u)(g'(u))^2 g''\left(u \odot' \alpha\right)}{\left(g'\left(u \odot' \alpha\right)\right)^2} \right\}.$$

□

Corollary 10 *If u and v are solutions of the equation (9) from the class $\mathcal{F}_1$ and α_1 and α_2 are parameters from $[a,b]$, then pseudo-linear combinations*

$$\alpha_1 \odot' u \oplus' \alpha_2 \odot' v \quad \text{and} \quad u \odot' \alpha_1 \oplus' v \odot' \alpha_2$$

are solutions of the equation (9) that belong to the class $\mathcal{F}_1$.

Proof. Follows directly from the previous theorem. □

4 Conclusion

Presented approach was based on the generalized generated pseudo-operations with three parameters given by some generating function. For the family of all solutions of the equation in question, we obtained partial pseudo-linear superposition principle for some specific generators. However, for solutions from the subfamily $\mathcal{F}_1$, the pseudo-linear superposition principle holds regardless to the choice of the generating function.

Acknowledgement. The support of grants GMFM of MIUR (for the first author) and projects MNZŽSS 144012 and "Mathematical Models for Decision Making under Uncertain Conditions and Their Applications" of the Academy of Sciences and Arts of Vojvodina supported by Provincial Secretary for Science and Technological Development (for the second author) is kindly announced.

References

[1] J. Aczel (1966). Lectures on Functional Equations and their Applications, Academic Press, New York.

[2] A. M. Ampère (1815). Considérations générales sur les intégrales des équations aux différences partielles.In *Journal de l'École polytechnique*, volume (17), pages 549-611.

[3] W. F. Ames (1965). Ad hoc Exact techniques for Nonlinear Partial Differential Equations. In *Nonlinear Partial Differential Equations in Engineering*, Academic Press, New York.

[4] P. Benvenuti, R. Mesiar, D. Vivona (2002). Monotone Set Functions-Based Integrals. In *Handbook of Measure Theory* (Ed. E. Pap), Elsevier, North-Holland, volume II, pages 1329-1379.

[5] G. W. Bluman, S. Kumei (1989). Symmetries and differential equations, Springer-Verlag, New York.

[6] G. Boillat (1968). Le champ scalaire de Monge-Ampère. In *Det Kgl. Norske Vid.Selsk. Forth.*, volume 41, pages 78-81.

[7] G. Boillat (1991). Sur l'équation générale de Monge-Ampère à plusieur variables. In *C.R.Acad.Sci.Paris Sér.I.Math.*, volume 313, pages 805-808.

[8] G. Boillat (192). Sur l'équation générale de Monge-Ampère d'ordre supérieur. In *C.R.Acad.Sci.Paris Sér.I.Math.*, volume 315, pages 1211-1214.

[9] A. Donato, U. Ramgulam, C. Rogers (1992). The 3 + 1-dimensional Monge-Ampère equation in discontinuity were theory: application of a reciprocal transformation. In *Meccanica*, volume 27, pages 257-262.

[10] J. M. Goard, P. Broadbridge (1997). Nonlinear Superposition Principles Obtained Lie Symmetry Methods. In *J. Math. Anal. Appl.*, volume 214, pages 633-657.

[11] M. Grabisch, H. T. Nguyen, E. A. Walker (1995). Fundamentals of Uncertainty Calculi with Applications to Fuzzy Inference, Kluwer Academic Publishers, Dordrecht-Boston-London.

[12] A. Inselberg (1970). Noncommutative Superpositions for Nonlinear Operators. In *J. Math. Anal. Appl.*, volume 29, pages 294-298.

[13] S. E. Jones, W. F. Ames (1967). Nonlinear Superposition. In *J. Math. Anal. Appl.*, volume 17, pages 484-487.

[14] E. P. Klement, R. Mesiar, E. Pap (2000). Triangular Norms, Kluwer Academic Publishers, Dordrecht.

[15] V. N. Kolokoltsov, V. P. Maslov (1997). Idempotent Analysis and Its Applications, Kluwer Academic Publishers, Dordrecht.

[16] S. A. Levin (1970). Principles of nonlinear superposition. In *J. Math. Anal. Appl.* volume 30, pages 197-205.

[17] A. Marková-Stupňanová (1996). A note on g-derivative and g-integral. In *Tatra Mt. Math. Publ.* volume 8, pages 71-76.

[18] V. P. Maslov (1987). Méthods Opératorielle, Mir, Moscou.

[19] V. P. Maslov, S. N. Samborskij (1992). Idempotent Analysis. In: *Advances in Soviet Mathematics* (V.P. Maslov and S.N. Samborskij, Ed.), American Mathematical Society, Providence, volume 13.

[20] R. Mesiar (1996). Pseudo-linear integrals and derivatives based on a generator g. In *Tatra Mt. Math. Publ.*, volume 8, pages 67-70.

[21] F. Oliveri (1998). Linearizable second order Monge-Ampère equation. In *J. Math. Anal. Appl.*, volume 218, pages 329-345.

[22] F. Oliveri (2004). Lie group analisys of differntial equations: direct and inverse problems. In *Note di Matematica*, volume 23, pages 195-216.

[23] F. Oliveri (2006). Sur une Propriété Remarquable des Équations de Monge-Ampère. In *Rend. del Circ. Mat. di Palermo*, Serie II, Suppl.78, pages 243-258..

[24] E. Pap (1993). g-calculus. In *Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat.*, volume 23 (1), pages 145-156.

[25] E.Pap (1995). Null-Additive Set Functions, Kluwer Academic Publishers, Dordrecht- Boston-London.

[26] E. Pap (1997). Solving nonlinear equations by non-additive measures. In *Nonlinear Analysis*, volume 30, pages 31-40.

[27] E. Pap (2002). Pseudo-additive measures and their applications. In *Handbook of Measure Theory* (Ed. E. Pap), Volume II, Elsevier, North-Holland, pages 1403-1465.

[28] E. Pap (2005). Applications of the generated pseudo-analysis to nonlinear partial differential equations. In *Contemporary Mathematics*, volume 377, pages 239-259.

[29] E. Pap, N. Ralević (1998). Pseudo-Laplace Transform. In *Nonlinear Analysis*, volume 33, pages 533-550.

[30] E. Pap, I. Štajner (1999). Generalized pseudo-convolution in the theory of probabilistic metric spaces, information, fuzzy numbers, optimization, system theory. In *Fuzzy Sets and Systems*, volume 102, pages 393-415.

[31] E. Pap, D. Vivona (2000). Non-commutative and non-associative pseudo-analysis and its applications on nonlinear partial differential equations. In *J. Math. Anal. Appl.*, volume 246/2, pages 390-408.

[32] E. Pap, D. Vivona, I. Štajner-Papuga (2002). On a pair of generated pseudo-operations with three parameters and its application on the Burgers type equations. In *Atti del Seminario Matematico e Fisico dellUniversita di Modena*, volume L, pages 451-464.

[33] N. M. Ralević, Lj. M. Nedović, T. Grbić (2005). The pseudo-linear superposition principle for nonlinear partial differential equations and representation of their solution by the pseudo-integral. In *Fuzzy Sets and Systems*, volume 155, pages 89-101.

[34] T. Ruggeri (1973). Su una naturale estensione a tre variabili dell'equazione di Monge-Ampère. in *Rend.Accad.Naz.Lincei*, volume 55, pages 445-449.

[35] M. N. Sprijker (1970). Superposition in linear and nonlinear ordinary differential equations. In *J. Math. Anal. Appl.*, volume 30, pages 206-222.