Fixed point theorems for contractive mappings in Menger probabilistic metric spaces

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Abstract

Using the theory of countable extension of t-norms we present some new classes of probabilistic contractions in probabilistic metric spaces.

Keywords: fixed point, Menger space, triangular norm, non-Archimedean Menger space, φ-probabilistic contraction, $(h,\, q)$-contraction of $(\varepsilon,\, \lambda)$-type.

1 Introduction

The notion of a probabilistic metric space was introduced by Menger [6] and since then the theory of probabilistic metric spaces has been developed in many directions [5]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to the situations when we do not know exactly the distance between two points, but only probabilities of possible values of this distance.

In 1972, the notion of q- contraction mappings on probabilistic metric spaces was introduced by V.M. Sehgal and A.T. Bharucha-Reid [7].

Definition 1 [7] *Let $(S, \mathcal{F})$ be a probabilistic metric spaces. A mapping $f : S \to S$ is a q-contraction mapping on $(S, \mathcal{F})$ if and only if there is an $q \in (0,\, 1)$ such that*

$$F_{fp_1, fp_2}(s) \geq F_{p_1, p_2}(\frac{s}{q}) \qquad (1)$$

for every $p_1, p_2 \in S$ and $s > 0$.

They proved that every such a mapping on a complete Menger space $(S, \mathcal{F},\, T_M)$ has unique fixed point, where T_M is the t-norm min. Subsequently, H. Sherwood [8] showed that for a very large class of t-norms it is possible to construct complete Menger spaces together with contraction mappings which have no fixed point. In [3] a lot of generalizations of Sehgal and Bharucha-Reid fixed point theorem are given.

In this paper using the theory of countable extension of t-norms [3] we present two new classes of probabilistic contraction in probabilistic metric spaces and proved a fixed point theorems.

2 Preliminaries

Definition 2 *A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (a t-norm) if the following conditions are satisfied:*

$T(a, 1) = a$ *for every* $a \in [0, 1]$ *;*

$T(a, b) = T(b, a)$ *for every* $a, b \in [0, 1]$;

$a \geq b,\ c \geq d \Rightarrow\ \ T(a, c) \geq T(b, d)\ \ (a, b, c, d \in [0, 1])$;

$T(a, T(b, c)) = T(T(a, b), c)\ \ (a, b, c \in [0, 1])$.

Definition 3 *If T is a t-norm, then its dual t-conorm $S : [0,\, 1]^2 \to [0,\, 1]$ is given by*

$$S(x,\, y) = 1 - T(1 - x,\, 1 - y).$$

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The following are the four basic t-norms together with their dual t-conorm:

$$T_M(x,y) = \min(x,y),\ S_M(x,y) = \max(x,y)$$
$$T_P(x,y) = x \cdot y,\ S_P(x,y) = x + y - xy$$
$$T_L(x,y) = \max(x+y-1,0)$$
$$S_L(x,y) = \min(x+y,1)$$
$$T_D(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y)=1, \\ 0 & \text{otherwise} \end{cases}$$
$$S_D(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y)=0, \\ 1 & \text{otherwise} \end{cases}$$

Definition 4 *(i) A t-norm T is said to be strictly monotone if $T(x,y) < T(x,z)$ whenever $x \in (0,1)$ and $y < z$.*
(ii) A t-norm T is called strict if it is continuous and strictly monotone.
(iii) A continuous t-norm T is called Archimedean if $T(x,x) < x$, for all $x \in (0,1)$.

Theorem 1 *A function $T : [0,1]^2 \to [0,1]$ is a continuous Archimedean t-norm if and only if there exists a continuous, strictly decreasing function $\mathbf{t}$:$[0,1] \to [0,+\infty]$ such that for all $x,y \in [0,1]$*

$$T(x,y) = \mathbf{t}^{-1}(\min(\mathbf{t}(x)+\mathbf{t}(y),\mathbf{t}(0))).$$

The function $\mathbf{t}$ is then called an additive generator of T; it is uniquely determined by T up to a positive multiplicative constant.

Let Δ^+ be the set of all distribution functions F such that $F(0) = 0$ (F is a nondecreasing, left continuous mapping from $\mathbb{R}$ into $[0,1]$ such that $\sup_{x\in\mathbb{R}} F(x) = 1$).

The ordered pair $(S,\mathcal{F})$ is said to be a probabilistic metric space if S is a nonempty set and $\mathcal{F} : S \times S \to \Delta^+$ ($\mathcal{F}(p,q)$ written by $F_{p,q}$ for every $(p,q) \in S \times S$) satisfies the following conditions:

1. $F_{u,v}(x) = 1$ for every $x > 0 \Rightarrow u = v$ $(u,v \in S)$.
2. $F_{u,v} = F_{v,u}$ for every $u,v \in S$.
3. $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1 \Rightarrow F_{u,w}(x+y) = 1$ for $u,v,w \in S$ and $x,y \in \mathbb{R}^+$.

A Menger space (see [5]) is an ordered triple $(S,\mathcal{F},T)$, where $(S,\mathcal{F})$ is a probabilistic metric space, T is a triangular norm (abbreviated t -norm) and the following inequality holds

$$F_{u,v}(x+y) \geq T(F_{u,w}(x), F_{w,v}(y))$$

for every $u,v,w \in S$ and every $x > 0, y > 0$.

Definition 5 [*1*] *$(S,\ \mathcal{F},\ T)$ is called a non-Archimedean Menger probabilistic metric space (shortly, a N.A. Menger PM-space) if $(S,\ \mathcal{F},\ T)$ is a Menger PM-space and T satisfies the following condition: for all $x,\ y,\ z \in S$ and $t_1,\ t_2 \geq 0$,*

$$F_{x,z}(max\{t_1,t_2\}) \geq T(F_{x,y}(t_1),\ F_{y,z}(t_2)).$$

Ultrametric spaces belongs to the class of N.A. Menger PM-spaces.

Definition 6 *Let $M \neq \emptyset$ and $d : M \times M \to [0,\ \infty)$ such that the following conditions are satisfied:*

1. $d(x,\ y) = 0 \Leftrightarrow x = y$

2. $d(x,\ y) = d(y,\ x)$ for all $x,\ y \in M$

3. $d(x,\ z) \leq \max\{d(x,\ y),\ d(y,\ z)\}$ for all $x,\ y,\ z \in M$.

Then $(M,\ d)$ is an ultrametric space.

Example 1 Let $(M,\ d)$ be a separable ultrametric space and $(\Omega,\ \mathcal{A},\ P)$ a probability space. Let S be the set of all the equivalence classes of measurable mappings $\hat{X} : \Omega \to M$. If $\hat{X},\ \hat{Y} \in S$ and $x \in \mathbb{R}$ let $F_{\hat{X},\hat{Y}}(x)$ be define in the following way

$$F_{\hat{X},\hat{Y}}(x) = P(\{\omega;\ \omega \in \Omega,\ d(\hat{X}(\omega),\ \hat{Y}(\omega)) < x\}).$$

Then $(S,\ \mathcal{F},\ T_L)$ is a N.A. Menger space.

We shall prove that for every $\hat{X},\ \hat{Y},\ \hat{Z}$ and every $x,\ y \in \mathbb{R}$ the inequality holds

$$F_{\hat{X},\hat{Y}}(\max(x,\ y)) \geq T_L(F_{\hat{X},\hat{Y}}(x),\ F_{\hat{Y},\hat{Z}}(y)).$$

In order to prove the previous inequality we shall prove that

$$F_{\hat{X},\hat{Y}}(\max(x,\,y)) \geq F_{\hat{X},\hat{Y}}(x) + F_{\hat{Y},\hat{Z}}(y) - 1.$$

Let $\mathcal{A} = \{\omega : \omega \in \Omega,\, d(\hat{X}(\omega),\, \hat{Y}(\omega)) < x\}$ and $\mathcal{B} = \{\omega : \omega \in \Omega,\, d(\hat{Y}(\omega),\, \hat{Z}(\omega)) < y\}$.

If $\mathcal{C} = \{\omega : \omega \in \Omega,\, d(\hat{X}(\omega),\, \hat{Z}(\omega)) < \max(x,\,y)\}$ then $\mathcal{C} \supset \mathcal{A} \cap \mathcal{B}$ and

$$\begin{aligned} P(\mathcal{C}) &\geq P(\mathcal{A} \cap \mathcal{B}) \\ &= P(\mathcal{A}) + P(\mathcal{B}) - P(\mathcal{A} \cup \mathcal{B}) \\ &\geq \max(P(\mathcal{A}) + P(\mathcal{B}) - 1,\, 0) \\ &= T_L(P(\mathcal{A}),\, P(\mathcal{B})). \end{aligned}$$

Hence

$$\begin{aligned} P(\mathcal{C}) &= F_{\hat{X},\hat{Z}}(\max(x,\,y)) \\ &\geq T_L(F_{\hat{X},\hat{Y}}(x),\, F_{\hat{Y},\hat{Z}}(y)). \end{aligned}$$

The (ϵ, λ)-topology in S is introduced by the family of neighbourhoods $\mathcal{U} = \{U_v(\epsilon,\lambda)\}_{(v,\epsilon,\lambda)\in S\times\mathbb{R}_+\times(0,1)}$, where

$$U_v(\epsilon,\lambda) = \{u;\; F_{u,v}(\epsilon) > 1-\lambda\}.$$

If a t-norm T is such that $\sup_{x<1} T(x,x) = 1$, then S is in the (ϵ,λ) topology a metrizable topological space.

Let $(S,\,\mathcal{F})$ be a probabilistic metric space. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in S is a Cauchy sequence if and only if for every $\varepsilon > 0$ and $\lambda \in (0,\,1)$ there exists $n_0(\varepsilon,\,\lambda) \in \mathbb{N}$ such that for every $n \geq n_0(\varepsilon,\,\lambda)$ and every $p \in \mathbb{N}$

$$F_{x_{n+p},x_n}(\varepsilon) > 1-\lambda.$$

If a probabilistic metric space $(S,\,\mathcal{F})$ is such that every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in S converges in S then $(S,\,\mathcal{F})$ is a complete space.

In [2] the following class of t-norms is introduced, which is useful in the fixed point theory in probabilistic metric spaces.

Let T be a t-norm and $T_n : [0,\,1] \to [0,\,1]$ $(n \in \mathbb{N})$ is defined in the following way:

$$T_1(x) = T(x,\,x),\;\; T_{n+1}(x) = T(T_n(x),\,x)$$

where $x \in [0,\,1]$).

We say that t-norm T is of the H-type if T is continuous and the family $\{T_n(x)\}_{n\in\mathbb{N}}$ is equicontinuous at $x = 1$.

A trivial example of t-norms of H-type is $T = T_M$. A nontrivial example is given in the paper [2].

Each t-norm T can be extended (by associativity) (see [4]) in a unique way to an n-ary operation taking for $(x_1,\ldots,x_n) \in [0,1]^n$ the values

$$\mathbf{T}_{i=1}^{0} x_i = 1,\;\; \mathbf{T}_{i=1}^{n} x_i = T(\mathbf{T}_{i=1}^{n-1} x_i, x_n).$$

We can extend T to a countable infinitary operation taking for any sequence $(x_n)_{n\in\mathbb{N}}$ from $[0,1]$ that [3]:

$$\mathbf{T}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathbf{T}_{i=1}^{n} x_i.$$

Limit of right side exists since the sequence $(\mathbf{T}_{i=1}^{n} x_i)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below.

In the fixed point theory it is of interest to investigate the classes of t-norms T and sequences $(x_n)_{n\in\mathbb{N}}$ from the interval $[0,\,1]$ such that $\lim_{n\to\infty} x_n = 1$, and

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} x_i = \mathbf{T}_{i=1}^{\infty} x_{n+i} = 1. \qquad (2)$$

For some classes of t-norms sufficient conditions for (2) are given in [3].

Example 1. The Dombi family of t-norms $(T_\lambda^D)_{\lambda\in[0,\,\infty]}$ is defined by

$$T_\lambda^D(x,\,y) = \begin{cases} T_D(x,\,y), & \lambda = 0 \\ T_M(x,\,y), & \lambda = \infty \\ \dfrac{1}{1+\left(\left(\frac{1-x}{x}\right)^{\lambda}+\left(\frac{1-y}{y}\right)^{\lambda}\right)^{1/\lambda}}, & \lambda > 0. \end{cases}$$

(ii) The Aczél-Alsina family of t-norms $(T_\lambda^{AA})_{\lambda\in[0,\,\infty]}$ is defined by

$$T_\lambda^{AA}(x,\,y) = \begin{cases} T_D(x,\,y), & \lambda = 0 \\ T_M(x,\,y), & \lambda = \infty \\ e^{-((-\log x)^{\lambda}+(-\log y)^{\lambda})^{1/\lambda}}, & \lambda > 0. \end{cases}$$

(iii) Sugeno-Weber family of t-norms $(T_\lambda^{SW})_{\lambda \in [-1, \infty]}$ is defined by

$$T_\lambda^{SW}(x, y) = \begin{cases} T_D(x, y), & \lambda = -1 \\ T_P(x, y), & \lambda = \infty \\ \max(0, \frac{x+y-1+\lambda xy}{1+\lambda}), & \lambda > -1. \end{cases}$$

In [3] the following results are obtained:

(a) If $(T_\lambda^D)_{\lambda \in (0, \infty)}$ is the Dombi family of t-norms and $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements from $(0, 1]$ such that $\lim_{n\to\infty} x_n = 1$ then we have the following equivalence:

$$\sum_{i=1}^{\infty}(1 - x_i)^\lambda < \infty \Longleftrightarrow \lim_{n\to\infty}(T_\lambda^D)_{i=n}^\infty x_i = 1. \tag{3}$$

(b) Equivalence (3) holds also for the family $(T_\lambda^{AA})_{\lambda \in (0, \infty)}$ i.e.

$$\sum_{i=1}^{\infty}(1 - x_i)^\lambda < \infty \Longleftrightarrow \lim_{n\to\infty}(T_\lambda^{AA})_{i=n}^\infty x_i = 1. \tag{4}$$

(c) If $(T_\lambda^{SW})_{\lambda \in (-1, \infty]}$ is the Sugeno-Weber family of t-norms and $(x_n)_{n \in \mathbb{N}}$ a sequence of elements from $(0, 1]$ such that $\lim_{n\to\infty} x_n = 1$ then we have the following equivalence:

$$\sum_{i=1}^{\infty}(1 - x_i) < \infty \Longleftrightarrow \lim_{n\to\infty}(T_\lambda^{SW})_{i=n}^\infty x_i = 1. \tag{5}$$

Proposition 1 *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of numbers from $[0, 1]$ such that $\lim_{n\to\infty} x_n = 1$ and t-norm T is of H-type. Then*

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^\infty x_i = \lim_{n\to\infty} \mathbf{T}_{i=n}^\infty x_{n+i} = 1.$$

3 Decomposable measures

Let $\mathcal{A}$ be a σ-algebra of subsets of a given set Ω. A classical measure is a set function $m : \mathcal{A} \to [0, \infty]$ such that $m(\emptyset) = 0$ and

$$m(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} m(A_i)$$

for every sequence $(A_i)_{i \in \mathbb{N}}$ of pairwise disjoint set from $\mathcal{A}$.

Definition 7 *Let S be a t-conorm. A $\mathcal{S}$-decomposable measure m is a set function $m : \mathcal{A} \to [0, 1]$ such that $m(\emptyset) = 0$ and*

$$m(A \cup B) = \mathcal{S}(m(A), m(B))$$

for every $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$.

Definition 8 *Let $\mathcal{S}$ be a left continuous t-conorm. A set function $m : \mathcal{A} \to [0, 1]$ is σ-$\mathcal{S}$-decomposable measure if $m(\emptyset) = 0$ and*

$$m(\bigcup_{i=1}^{\infty} A_i) = \mathcal{S}_{i=1}^{\infty} m(A_i)$$

for every sequence $(A_i)_{i \in \mathbb{N}}$ from $\mathcal{A}$ whose elements are pairwise disjoint set.

A measure m is of (NSA)-type if and only if $s \circ m$ is a finite additive measure, where $\mathbf{s}$ is an additive generator of the t-conorm S, which is continuous, non-strict, and Archimedean and with respect to which m is decomposable ($\mathbf{s}(1) = 1$).

Proposition 2 [3] *Let $(\Omega, \mathcal{A}, m)$ be a measure space, where m is a continuous $\mathcal{S}$-decomposable measure of (NSA)-type with monotone increasing generator $\mathbf{s}$. Then $(S, \mathcal{F}, T)$ is a Menger space, where $\mathcal{F}$ and t-norm T are given in the following way ($\mathcal{F}(\hat{X}, \hat{Y}) = F_{\hat{X}, \hat{Y}}$):*

$$F_{\hat{X}, \hat{Y}}(u) = m\{\omega : \omega \in \Omega, d(X(\omega), Y(\omega)) < u\}$$
$$= m\{d(X, Y) < u\}$$

(for every $\hat{X}, \hat{Y} \in S$, $u \in \mathbb{R}$),

$T(x, y) = \mathbf{s}^{-1}(\max(0, \mathbf{s}(x) + \mathbf{s}(y) - 1))$ *for every $x, y \in [0, 1]$.*

4 The fixed point theorems

Definition 9 [3] *Let $(S, \mathcal{F})$ be a probabilistic metric space. A mapping $f : S \to S$ is said to be a q-contraction of (ε, λ)-type, where $q \in (0, 1)$, if for every $p_1, p_2 \in S$, every $\varepsilon > 0$ and every $\lambda \in (0, 1)$ the following implication holds:*

$$F_{p_1,p_2}(\varepsilon) > 1 - \lambda \Rightarrow F_{fp_1,fp_2}(q\varepsilon) > 1 - q\lambda.$$

Definition 10 *Let $(S, \mathcal{F})$ be a probabilistic metric space. A mapping $f : S \to S$ is said to be a (h, q)-contraction of (ε, λ)-type, where $h : (0, 1) \to (0, 1)$, $q \in (0, 1)$, if for every $p_1, p_2 \in S$, every $\varepsilon > 0$ and every $\lambda \in (0, 1)$ the following implication holds:*

$$F_{p_1,p_2}(\varepsilon) > h(\lambda) \tag{6}$$

$$\Rightarrow F_{fp_1,fp_2}(q\varepsilon) > h(q\lambda).$$

Every q-contraction of (ε, λ)-type is a (h, q)-contraction of (ε, λ)-type if $h(\lambda) = 1 - \lambda$.

Theorem 2 *Let $(S, \mathcal{F}, T)$ be a complete Menger space, $h : (0, 1] \to (0, 1]$, $\lim\limits_{t\to 0^+} h(t) = 1$ and $\lim\limits_{t\to 1} h(t) = 0$, h is nonincreasing function and $f : S \to S$ is a (h, q)-contraction of (ε, λ)-type. If t-norm T satisfies the following condition:*

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} h(q^i) = 1 \tag{7}$$

then there exists unique fixed point x of the mapping f and $x = \lim\limits_{n\to\infty} f^n p$ for every $p \in S$.

Proof: Let $p \in S$ and $\delta > 0$ be such that $F_{p,fp}(\delta) > 0$. Since $F_{p,fp} \in \Delta^+$ such a δ exists. Let $\lambda_1 \in (0, 1)$ be such that $F_{p,fp}(\delta) > h(\lambda_1)$. From (6) we have that

$$F_{fp,f^2p}(q\delta) > h(q\lambda_1)$$

and generally, for every $n \in \mathbb{N}$

$$F_{f^np,f^{n+1}p}(q^n\delta) > h(q^n\lambda_1). \tag{8}$$

We prove that $(f^np)_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., that for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists $n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that

$$F_{f^np,f^{n+m}p} > 1 - \lambda \text{ for every } n \geq n_0(\varepsilon, \lambda)$$

and every $m \in \mathbb{N}$.

Let $\varepsilon > 0$ and $\lambda \in (0, 1)$ be given. Since $q^n\delta < 1$ the series $\sum\limits_{n=1}^{\infty} q^n\delta$ converges and so there exists $n_0 = n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that $\sum\limits_{n=n_0}^{\infty} q^n\delta < \varepsilon$. Then for every $n \geq n_0$

$$\begin{aligned}
F_{f^np,f^{n+m}p}(\varepsilon) &\geq F_{f^np,f^{n+m}p}\Big(\sum_{n=n_0}^{\infty} q^n\delta\Big) \\
&\geq F_{f^np,f^{n+m}p}\Big(\sum_{i=n}^{n+m-1} q^i\delta\Big) \\
&\geq \underbrace{T(T(\dots T}_{(m-1)-times}(F_{f^np,f^{n+1}p}(q^n\delta), \\
&\quad F_{f^{n+1}p,f^{n+2}p}(q^{n+1}p), \dots \\
&\quad \dots F_{f^{n+m-1}p,f^{n+m}p}(q^{n+m-1}\delta)).
\end{aligned}$$

Let $n_1 = n_1(\lambda) \in \mathbb{N}$ be such that

$$\mathbf{T}_{i=n_1}^{\infty} h(q^i) > 1 - \lambda.$$

Since (7) holds, such a number n_1 exists. Now, for every $n \geq \max(n_0, n_1)$ and every $m \in \mathbb{N}$

$$\begin{aligned}
F_{f^np,f^{n+m}p}(\varepsilon) &\geq \mathbf{T}_{i=n}^{n+m-1} h(q^i\lambda_1) \\
&\geq \mathbf{T}_{i=n}^{n+m-1} h(q^i) \\
&\geq \mathbf{T}_{i=n}^{\infty} h(q^i) \\
&> 1 - \lambda.
\end{aligned}$$

The mapping f is uniformly continuous. Indeed, let $\mu > 0$ and $\zeta \in (0, 1)$ be given. Since $\lim\limits_{t\to 0^+} h(t) = 1$ there exists $\eta \in (0, 1)$ such that $h(\eta) > 1 - \zeta$. Then for $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that $\varepsilon = \frac{\mu}{q}$, $\lambda = \frac{\eta}{q}$ we have the implication

$F_{p_1,p_2}(\varepsilon) > h(\lambda) \Rightarrow$

$F_{fp_1,fp_2}(q\varepsilon) = F_{fp_1,fp_2}(\mu) > h(q\lambda) = h(\eta) > 1 - \zeta.$

So

$$(p_1, p_2) \in N(\frac{\mu}{q}, 1-h(\frac{\eta}{q})) \Rightarrow (fp_1, fp_2) \in N(\mu, \zeta)$$

where $N(\varepsilon, \lambda) = \{(u, v) : u, v \in S, F_{u,v}(\varepsilon) > 1 - \lambda\}$.

The relation $x = \lim_{n\to\infty} f^n p$ implies that

$$fx = f(\lim_{n\to\infty} f^n p) = \lim_{n\to\infty} f^n p = x.$$

It remains to prove the uniqueness of the fixed point x. Suppose that $y = fy$, $y \neq x$. Then there exists $\varepsilon > 0$ and $\lambda \in (0, 1)$ such that $F_{x,y}(\varepsilon) > h(\lambda)$. Then we have $F_{fx,fy}(q\varepsilon) > h(q\lambda)$ and similarly

$$F_{x,y}(q^n\varepsilon) = F_{f^nx,f^ny}(q^n\varepsilon) > h(q^n\lambda)$$

for every $n \in \mathbb{N}$.

Let $u > 0$ and $\eta \in (0, 1)$ be given. If $n_0 \in \mathbb{N}$ is such that $q^{n_0}\varepsilon < u$ and $h(q^{n_0}\lambda) > 1 - \eta$ then

$$\begin{aligned} F_{x,y}(u) &\geq F_{x,y}(q^{n_0}\varepsilon) > h(q^{n_0}\varepsilon) \\ &> 1 - \eta. \end{aligned}$$

Therefore, $F_{x,y}(u) = 1$, for every $u > 0$, which contradicts to $x \neq y$.

Corollary 1 *Let $(S, \mathcal{F}, T)$ be a complete Menger space and $f : S \to S$ is a q-contraction of (ε, λ)-type. If t-norm T satisfies the following condition:*

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty}(1 - q^i) = 1 \qquad (9)$$

then there exists unique fixed point for the mapping f and $x = \lim_{n\to\infty} f^n p$ for every $p \in S$.

Proof: Let $h(x) = 1 - x$. Then all the conditions of previous theorem are satisfied.

Corollary 2 *Let $(S, \mathcal{F}, T)$ be a complete Menger space, $h : (0, 1] \to (0, 1]$, $\lim_{t\to 0^+} h(t) = 1$, $\lim_{t\to 1} h(t) = 0$, h is nonincreasing function and $f : S \to S$ is a (h, q)-contraction of (ε, λ)-type. If t-norm T is a t-norm of H-type then there exists unique fixed point for the mapping f and $x = \lim_{n\to\infty} f^n p$ for every $p \in S$.*

Proof: By Proposition 1 all the conditions of the Theorem 2 are satisfied.

Corollary 3 *Let $(S, \mathcal{F}, T_\lambda^D)$ for some $\lambda > 0$ be a complete Menger space, $h : (0, 1] \to (0, 1]$, $\lim_{t\to 0^+} h(t) = 1$, $\lim_{t\to 1} h(t) = 0$, h is non-increasing function and $f : S \to S$ is a (h, q)-contraction of (ε, λ)-type. If $\sum_{i=1}^{\infty}(1 - h(q^i))^\lambda < \infty$ then there exists unique fixed point for the mapping f and $x = \lim_{n\to\infty} f^n p$ for every $p \in S$.*

Proof. From equivalence (3) we have

$$\sum_{i=1}^{\infty}(1 - h(q^i))^\lambda < \infty \Longleftrightarrow \lim_{n\to\infty} (T_\lambda^D)_{i=n}^{\infty} h(q^i) = 1.$$

Corollary 4 *Let $(S, \mathcal{F}, T_\lambda^{AA})$ for some $\lambda > 0$ be a complete Menger space, $h : (0, 1] \to (0, 1]$, $\lim_{t\to 0^+} h(t) = 1$, $\lim_{t\to 1} h(t) = 0$, h is non-increasing function and $f : S \to S$ is a (h, q)-contraction of (ε, λ)-type. If $\sum_{i=1}^{\infty}(1 - h(q^i))^\lambda < \infty$ then there exists unique fixed point for the mapping f and $x = \lim_{n\to\infty} f^n p$ for every $p \in S$.*

Proof. From equivalence (4) we have

$$\sum_{i=1}^{\infty}(1 - h(q^i))^\lambda < \infty \Longleftrightarrow$$

$$\lim_{n\to\infty} (T_\lambda^{AA})_{i=n}^{\infty} h(q^i) = 1.$$

Corollary 5 *Let $(S, \mathcal{F}, T_\lambda^{SW})$ for some $\lambda > -1$ be a complete Menger space, $h : (0, 1] \to (0, 1]$, $\lim_{t\to 0^+} h(t) = 1$, $\lim_{t\to 1} h(t) = 0$, h is non-increasing function and $f : S \to S$ is a (h, q)-contraction of (ε, λ)-type. If $\sum_{i=1}^{\infty}(1 - h(q^i)) <$*

∞ *then there exists unique fixed point for the mapping* f *and* $x = \lim_{n\to\infty} f^n p$ *for every* $p \in S$.

Proof. From equivalence (5) we have

$$\sum_{i=1}^{\infty}(1 - h(q^i)) < \infty \Longleftrightarrow$$

$$\lim_{n\to\infty} (T_\lambda^{SW})_{i=n}^{\infty} h(q^i) = 1.$$

Corollary 6 *Let* $(\Omega, \mathcal{A}, m)$ *be a measure space, where* m *is a continuous* $\mathcal{S}$ *decomposable measure of (NSA)-type,* $\mathbf{s}$ *is a monotone increasing additive generator of* $\mathcal{S}$, (M, d) *a complete separable metric space and* $f : \Omega \times S \to M$ *a random operator such that for some* $q \in (0, 1)$ *and every measurable mappings* $X, Y : \Omega \to M$

$$(\forall u > 0)(\forall \lambda \in (0,1))(m\{\omega; \omega \in \Omega, d(X(\omega), Y(\Omega)) < u\}) > h(\lambda) \Rightarrow m(\{\omega; \omega \in \Omega, d((\widehat{f}X), (\widehat{f}Y)) < qu\}) > h(q\lambda).$$

If $h : (0, 1] \to (0, 1]$, *such that* $\lim_{t\to 0^+} h(t) = 1$, $\lim_{t\to 1} h(t) = 0$ *and t-norm* T *defined by*

$T(x, y) = \mathbf{s}^{-1}(\max(0,\mathbf{s}(x)+\mathbf{s}(y) - 1))$, $x, y \in [0, 1]$,

satisfies condition

$$\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} h(q^i) = 1 \qquad (10)$$

then there exists a random fixed point of the operator f.

Definition 11 *A function* $\varphi : \mathbb{R} \to \mathbb{R}$ *is said to satisfy the condition (A) if it satisfies the following conditions:*

(i) $\varphi(t) = 0$ *if and only if* $t = 0$

(ii) $\lim_{t\to\infty} \varphi(t) = \infty$

(iii) φ *is left continuous in* $(0, \infty)$

(iv) φ *is continuous at* 0.

Definition 12 *Let* $(S, \mathcal{F}, T)$ *be a Menger space. A mapping* $f : S \to S$ *is said to be a* φ*-probabilistic contraction if*

$$F_{fx,fy}(\varphi(t)) \geq F_{x,y}(\varphi(\frac{t}{c})) \qquad (11)$$

where $0 < c < 1$, $x, y \in S$ *and* $t > 0$ *and function* φ *satisfies the condition (A).*

Theorem 3 *Let* $(S, \mathcal{F}, T)$ *be a complete N.A. Menger PM-space with continuous t-norm* T. *Let* $f : S \to S$ *be continuous and* φ*-probabilistic contraction. If there exists* $x_0 \in S$ *and* $x_1 = fx_0$ *such that t-norm* T *satisfies condition* $\lim_{n\to\infty} \mathbf{T}_{i=n}^{\infty} F_{x_0,x_1}\varphi(\frac{r}{c^i}) = 1$, *for every* $r > 0$, $0 < c < 1$ *then there exists unique fixed point* z *of the mapping* f.

Proof: In view of the condition (i) and (iv) in Definition 11, for all $s > 0$ we can find a positive number r such that $s > \varphi(r)$. Let $x_0 \in S$ and $x_n = fx_{n-1}$, $n = 1, 2, \ldots$

Then

$$\begin{aligned}
F_{x_n,x_{n+1}}(s) &> F_{x_n,x_{n+1}}(\varphi(r)) \\
&= F_{fx_{n-1},fx_n}(\varphi(r)) \\
&\geq F_{x_{n-1},x_n}(\varphi(\frac{r}{c})) \\
&= F_{fx_{n-2},fx_{n-1}}(\varphi(\frac{r}{c})) \\
&\geq F_{x_{n-2},x_{n-1}}(\varphi(\frac{r}{c^2})) \\
&\ldots \\
&\geq F_{x_0,x_1}(\varphi(\frac{r}{c^n}))
\end{aligned}$$

It remains to be proved that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned}
F_{x_{n+1},x_{n+m+1}}(s) &> F_{x_{n+1},x_{n+m+1}}(\varphi(r)) \\
&= F_{fx_n,fx_{n+m}}(\varphi(r)) \\
&\geq \underbrace{T(T(\ldots T}_{m-times}(F_{fx_n,fx_{n+1}}(\varphi(r)), \\
&\qquad \ldots, F_{fx_{n+m},fx_{n+m+1}}(\varphi(r)))
\end{aligned}$$

$$\begin{aligned}
&\geq \underbrace{T(T(\ldots T}_{m-times}(F_{x_0,x_1}(\varphi(\frac{r}{c^n}),\\
&\quad \ldots, F_{x_0,x_1}(\varphi(\frac{r}{c^{n+m}})\\
&= \mathbf{T}_{i=n}^{n+m} F_{x_0,x_1}\varphi(\frac{r}{c^i})\\
&\geq \mathbf{T}_{i=n}^{\infty} F_{x_0,x_1}\varphi(\frac{r}{c^i})
\end{aligned}$$

So, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and since the space S is complete there exists $z \in S$ such that

$$\lim_{n\to\infty} x_n = z = \lim_{n\to\infty} f x_{n-1} = fz.$$

Next we show that the fixed point is unique. Suppose that there exists w such that $fw = w$ and $w \neq z$. From the property of φ it follows that for a given $\varepsilon > 0$ we can find $\varepsilon_1 > 0$ such that $\varepsilon > \varphi(\varepsilon_1) > 0$. Then

$$\begin{aligned}
F_{w,z}(\varepsilon) &= F_{fw,fz}(\varepsilon)\\
&\geq F_{fw,fz}(\varphi(\varepsilon_1))\\
&\geq F_{w,z}(\varphi(\frac{\varepsilon_1}{c}))\\
&= F_{fw,fz}(\varphi(\frac{\varepsilon_1}{c}))\\
&\geq F_{w,z}(\varphi(\frac{\varepsilon_1}{c^2}))\\
&\quad \ldots\\
&\geq F_{w,z}(\varphi(\frac{\varepsilon_1}{c^n}))
\end{aligned}$$

Letting $n \to \infty$ in the above inequality and we obtain $z = w$. This completes the proof.

Using Proposition 2 and Theorem 3 a random fixed point theorem can be proved similarly as in Corollary 6.

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