Comonotone maxitivity and extended Sugeno integral

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Abstract

Comonotone maxitive idempotent aggregation functions on $[0, 1]$ are characterized and the extended Sugeno integral is introduced. Also the relationship of Sugeno integral and the extended Sugeno integral is clarified. Finally, extended weighted maxima on $[0, 1]$ are discussed.

Keywords: Comonotone maxitivity, Extended Sugeno integral, Sugeno integral, Weighted maximum.

1 Introduction

Riemann, Lebesgue and Choquet integrals are heavily related to the standard arithmetic operations of summation and multiplication on the real line, and thus if we work on a qualitative (discrete) scale they are out of use. To overcome this problem, Sugeno [17] has introduced an integral called now Sugeno integral or fuzzy integral. It can be introduced on any measurable space with respect to any fuzzy measure and on any ordinal scale (i.e., values of measures and functions can be from some complete chain, e.g., from the scale $\{\text{bad}, \ldots, \text{excellent}\}$).

There is a large variety of applications of Sugeno integral in different domains,including diagnostic medicine [15], banking [8], finance [10], image processing [18, 16] and neural networks [11], for instance. Several decision making applications of Sugeno integral are discussed in [6] and in [19]. In this paper authors have focused on the interpretation of Sugeno integral in fuzzy inference systems, when the rules are not independent. This is perhaps the reason, why Sugeno integral was used quite rarely in fuzzy control, and thus a need for some modification of this integral came naturally in the picture, see [12].

A genuine property of Sugeno integral based aggregation is the comonotone maxitivity, see [2, 3, 13]. This property can be defined on any measurable space and any ordinal scale. The aim of this contribution is the investigation of comonotone maxitive idempotent aggregation functions and to study their relationship with Sugeno integral. Though all consideration can be done on any compact ordinal scale, in this contribution we will work on the standard $[0, 1]$ scale only. Similarly, we will work on finite space X only. In the next section, we recall Sugeno integral and its properties. Section 3 is devoted to comonotone maxitive idempotent aggregation functions and their representation as an extended Sugeno integral, which is introduced here. Relationship of the extended Sugeno integral and Sugeno integral is discussed in Section 4, while in Section 5 we introduce the extended weighted maxima operators. Finally, some conclusions are given. Note that full proofs and more details on extended Sugeno integrals will appear in our prepared paper [1].

2 Sugeno integral

Sugeno has introduced the notion of fuzzy integral, now named after him, in his PhD. The-

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sis [17].

Definition 1 Let X be a finite space, m a fuzzy measure on X, $f\colon X \longrightarrow [0,1]$ a function, then the Sugeno integral S_m of f with respect to m is given by

$$S_m(f) = \sup(\min(t, m(f \geq t)) \mid t \in [0,1]) \tag{1}$$

Recall that $m\colon 2^X \longrightarrow [0,1]$ is a fuzzy measure on X if $m(\emptyset) = 0$, $m(X) = 1$ and $m(A) \leq m(B)$ whenever $A \subseteq B$.

Basic properties of the Sugeno integral are summarized in the next proposition, see e.g. [3, 13, 17].

Definition 2 Let $\mathcal{F}$ be the system of all $[0,1]$-valued functions on X. Then a functional $I\colon \mathcal{F} \longrightarrow [0,1]$ is called

(i) idempotent if $I(c \cdot \mathbf{1}_X) = c$ for all $c \in [0,1]$.
(ii) min-homogeneous if $I(f \wedge c \cdot \mathbf{1}_X) = I(f) \wedge c$ for any $f \in \mathcal{F}$ and $c \in [0,1]$.
(iii) max-homogeneous if $I(f \vee c \cdot \mathbf{1}_X) = I(f) \vee c$ for any $f \in \mathcal{F}$ and $c \in [0,1]$.
(iv) horizontal maxitive if $I(f) = I(f \wedge a) \vee I(f \cdot \mathbf{1}_{f>a})$, for all $f \in \mathcal{F}$, $a \in [0,1]$.

Proposition 1 *Let $\mathcal{F}$ be the system of all $[0,1]$-valued functions on X. Then $S_m\colon \mathcal{F} \longrightarrow [0,1]$ is a non-decreasing, idempotent, continuous, comonotone maxitive (and thus max-homogeneous), horizontal maxitive and min-homogeneous functional.*

Recall that two functions $f, g\colon X \longrightarrow [0,1]$ are comonotone if $(f(x)-f(y))(g(x)-g(y)) \geq 0$ for all $x, y \in X$. Equivalently, the comonotonicity of f and g on a finite space $X = \{1, \ldots, n\}$ means that there are two non-decreasing functions $f^*, g^*\colon X \longrightarrow [0,1]$ and a permutation σ on $(1, \ldots, n)$ such that $f^*(i) = f(\sigma(i))$ and $g^*(i) = g(\sigma(i))$ for all $i \in X$. Comonotone maxitivity of S_m means that $S_m(f \vee g) = S_m(f) \vee S_m(g)$ whenever f and g are comonotone. The continuity of Sugeno integral means the standard continuity of real functions of n variables.

The following characterization of functionals on $\mathcal{F}$ representable as the Sugeno integral was shown for the finite X by Marichal [9], and in general by Benvenuti and Mesiar [2].

Proposition 2 *Let $I\colon \mathcal{F} \longrightarrow [0,1]$ be a functional which is continuous, non-decreasing,* min- *and* max-*homogeneous. Then I is the Sugeno integral with respect to the fuzzy measure m_I given by $m_I(A) = I(\mathbf{1}_A)$, i.e., $I = S_{m_I}$.*

Note that min- and max-homogenity of I ensures the idempotency of I, $I(a\mathbf{1}_X) = a$ for any $a \in [0,1]$, and that max-homogenity can be replaced by more specific property of comonotone maxitivity (see [3] or [14]).

3 Comonotone maxitive functionals

Recall that a non-decreasing functional $I\colon \mathcal{F} \longrightarrow [0,1]$ such that $I(\mathbf{1}_\emptyset) = 0$ and $I(\mathbf{1}_X) = 1$ is called an aggregation function on $\mathcal{F}$, see [3].

As stated in Proposition 1, each Sugeno integral S_m on X is an idempotent comonotone maxitive aggregation function on $\mathcal{F}$. However, the class of all aggregation functions $I\colon \mathcal{F} \longrightarrow [0,1]$ which are idempotent and comonotone maxitive is much more wider.

Example 1 Let $X = \{1, 2\}$, i.e., $\mathcal{F} = [0,1]^2$. Define $I\colon [0,1]^2 \longrightarrow [0,1]$ by

$$I(x,y) = \begin{cases} x \wedge y & \text{if } (x,y) \in \left[0, \frac{1}{2}\right[^2, \\ x \vee y & \text{else} \end{cases}$$

(see Figure 1). Then I is an idempotent aggregation function which is comonotone maxitive. However, $I(0,1) = 1$ and $I(0 \wedge \frac{1}{4}, 1 \wedge \frac{1}{4}) = I(0, \frac{1}{4}) = 0 \neq \frac{1}{4} \wedge 1$, i.e., I is not *min*-homogenous and thus it cannot be a Sugeno integral, see Proposition 1.

The next representation of idempotent comonotone maxitive aggregation functions can be derived from our results in [1].

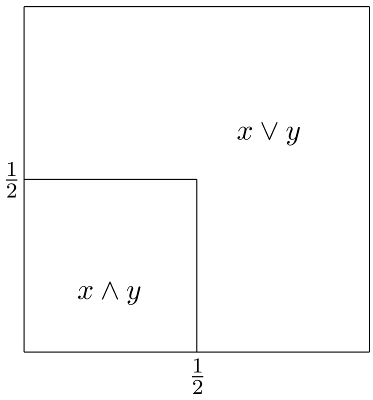

Figure 1: Aggregation function I from Example 1.

Theorem 1 *An idempotent aggregation function $I\colon \mathcal{F} \longrightarrow [0,1]$ is comonotone maxitive if and only if*

$$I(f) = \sup(\min(t, m_t(f \geq t)) \mid t \in [0,1]), \quad (2)$$

where $m_t\colon 2^X \longrightarrow [0,1]$ is for each $t \in [0,1]$ a fuzzy measure on X given by $m_t(A) = I(t \cdot \mathbf{1}_A)$ for $A \neq X$ and $m_t(X) = 1$.

The following definition is inspired by Theorem 1.

Definition 3 Let $M = (m_t)_{t\in[0,1]}$ be a system of fuzzy measures on X. The mapping $S_M\colon \mathcal{F} \longrightarrow [0,1]$ given by

$$S_M(f) = \sup(\min(t, m_t(f \geq t)) \mid t \in [0,1]) \quad (3)$$

is called an extended Sugeno integral.

Observe that for $A \neq X$, $S_M(t \cdot \mathbf{1}_A)$ need not be equal to $m_t(A)$, in general.

Evidently, formula (3) reduces to Sugeno integral formula (1) if $m_t = m$ for all $t \in [0,1]$. Therefore, (3) extends the idea of Sugeno integral.

Proposition 3 *Let S_M be an extended Sugeno integral. Then it is comonotone maxitive idempotent aggregation function on $\mathcal{F}$.*

Corollary 1 *Each extended Sugeno integral S_M can be represented as an extended Sugeno integral $S_{M'}$, where $M' = (m'_t)_{t\in[0,1]}$ is a non-decreasing system of fuzzy measures.*

Example 2 (i) Let $X = \{1,2\}$ and $M = (m_t)_{t\in[0,1]}$ be given by

$$m_t = \begin{cases} m_* & \text{if } t < \frac{1}{2}, \\ m^* & \text{else,} \end{cases}$$

where m^* is the strongest fuzzy measure on X and m_* is the weakest fuzzy measure on X, i.e.,

$$m^*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{else,} \end{cases}$$

$$m_*(A) = \begin{cases} 1 & \text{if } A = X \\ 0 & \text{else.} \end{cases}$$

Then $S_M = I$, where I is given in Example 1. Observe that S_M is an associative binary function, and that its n-ary extension is just the extended Sugeno integral on $X_n = \{1,\ldots,n\}$ related to the system M as given above (dealing with the strongest and the weakest fuzzy measures on X_n).

(ii) Let $X = \{1,2\}$ and $M = (m_t)_{t\in[0,1]}$ be given by

$$m_t = \begin{cases} m_* & \text{if } \frac{1}{3} < t \leq \frac{2}{3}, \\ m^* & \text{else.} \end{cases}$$

Then $S_M = S_{M'}$ for $M' = (m'_t)_{t\in[0,1]}$, where

$$m'_t(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \frac{1}{3} & \text{if } \emptyset \neq A \neq X, \\ 1 & \text{if } A = X. \end{cases}$$

for $t \in \left[0, \frac{2}{3}\right]$ and $m'_t = m^*$ else. Moreover,

$$S_M(x,y) = \begin{cases} \mathrm{med}_{\frac{1}{3}}(x,y) & \text{if} (x,y) \in \left[0, \frac{2}{3}\right]^2 \\ \max(x,y) & \text{else} \end{cases}$$

is again an associative binary function. Here $\mathrm{med}_{\frac{1}{3}}(x,y) = \mathrm{med}(x,y,\frac{1}{3})$. Again as in the case (i), when working on X_n, the corresponding extended Sugeno integral is the n-ary extension of this binary S_M.

Proposition 4 *Maximum of two extended Sugeno integrals is again an extended Sugeno integral, namely,* $\max(S_{M_1}, S_{M_2}) = S_{M_1 \vee M_2}$, *where for* $M_1 = (m_t^{(1)})_{t\in[0,1]}$ *and* $M_2 = (m_t^{(2)})_{t\in[0,1]}$ *it is* $M_1 \vee M_2 = (m_t^{(1)} \vee m_t^{(2)})_{t\in[0,1]}$.

4 Extended Sugeno integral and the Sugeno integral

In this section we clarify the relationship of the extended Sugeno integral and original Sugeno integral.

Example 3 Let $M = (m_t)_{t\in[0,1]}$, where

$$m_t = \begin{cases} m^* & \text{if } t \leq 0.5, \\ m_* & \text{else.} \end{cases}$$

Then

$$S_M(f) = \begin{cases} \max f & \text{if } \max f \leq 0.5, \\ \min f & \text{if } \min f > 0.5, \\ 0.5 & \text{else.} \end{cases}$$

Observe that if $X = \{1, \ldots, n\}$ then S_M coincide with the nullnorm $\mathrm{med}_{0.5}$, see [4], which is an n-ary extension of an associative binary aggregation function $\mathrm{med}_{0.5}(x, y) = \mathrm{med}(x, y, 0.5)$. Moreover, $S_M = S_m$ is the Sugeno integral with respect to the fuzzy measure m given by

$$m(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A = X, \\ 0.5 & \text{else.} \end{cases}$$

As we have seen in Example 3, even for non-constant system M the corresponding extended Sugeno integral S_M can be represented as a simple Sugeno integral S_m for an appropriate fuzzy measure m. The next Theorem brings a full characterization of all such cases.

Note that the above theorem is already examplified in Example 3.

Theorem 2 *Let* S_M *be an extended Sugeno integral. Then it coincide with the Sugeno integral* S_m, *where* m *is a fuzzy measure on* X *given by*

$$m(A) = S_M(\mathbf{1}_A) = \bigvee_{t\in[0,1]} (t \wedge m_t(A))$$

if and only if for each $A \subset X$, $m_t(A) \leq m(A)$ *whenever* $m(A) \leq t$, *and there is a dense subset* $D \subset \,]0, m(A)[$ *such that* $m_t(A) \geq t$ *if* $t \in D$.

Evidently, if $M = (m_t)_{t\in[0,1]}$ is a non-increasing system of fuzzy measures, then $S_M = S_m$, where $m(A) = S_M(\mathbf{1}_A)$.

Example 4 In this example we show an increasing $M = (m_t)_{t\in[0,1]}$ such that S_M is Sugeno integral. Let $X = \{1, 2\}$ and $m_t(1) = m_t(2) = t$. For $x < y$ $S_M((x, y)) = (\bigvee_0^x (t \wedge 1)) \vee (\bigvee_x^y t \wedge t) = y$. After minor processing we see that $S_M = S_{m^*} = \max$.

5 Extended weighted maximum

For maxitive fuzzy measure $m \colon 2^X \longrightarrow [0, 1]$, $m(A) = \max(w_i \mid i \in A)$, the corresponding Sugeno integral $S_m \colon \mathcal{F} \longrightarrow [0, 1]$ yields the weighted maximum operator, i.e.,

$$S_m(f) = \max(\min(w_i, f(i)) \mid i \in X) \qquad (4)$$

Let $M = (m_t)_{t\in[0,1]}$ be a system of maxitive fuzzy measures which is constant on intervals $\mathrm{Int}_1, \ldots, \mathrm{Int}_k$. Evidently, S_M restricted to the functions with range contained in Int_1 is a weighted maximum operator, similarly for $\mathrm{Int}_2, \ldots, \mathrm{Int}_k$ (these are domains where S_M acts as a weighted maximum operator; obviously, on different domains different weighted maxima may act). Then S_M is an aggregation function extending all partial weighted maxima acting on $\mathrm{Int}_1, \ldots, \mathrm{Int}_k$ into $[0, 1]$ domain.

Proposition 5 *Let* $X = \{1, \ldots, n\}$ *and let* $M = (m_t)_{t\in[0,1]}$ *be a system of maxitive fuzzy measures on* X. *Then*

$$S_M(f) = \max(w_i(f(i)) \mid i \in X),$$

where $w_i(x) = \sup(\min(t, m_t(i)) \mid t \in [0, x])$. *Moreover, each* $w_i \colon [0, 1] \longrightarrow [0, 1]$ *is non-decreasing and* $\max(w_i \mid i \in X) = \mathrm{id}$.

Example 5 (i) Let $X = \{1,2\}$ and $M = (m_t)_{t\in[0,1]}$, where for $t \in \left[0, \frac{1}{2}\right] = \mathrm{Int}_1$, $m_t(A) = \mathbf{1}_A(1)$, and for $t \in \left]\frac{1}{2}, 1\right] = \mathrm{Int}_2$, $m_t(A) = \mathbf{1}_A(2)$. Then all m_t are maxitive fuzzy measures, $S_M(x,y) = x$ if $(x,y) \in \left[0, \frac{1}{2}\right]^2$ and $S_M(x,y) = y$ if $(x,y) \in \left]\frac{1}{2}, 1\right]^2$. In general $S_M(x,y) = w_1(x) \vee w_2(y)$, where $w_1, w_2 \colon [0,1] \longrightarrow [0,1]$ are given by $w_1(t) = \min(t, \frac{1}{2})$ and

$$w_2(t) = \begin{cases} 0 & \text{if } t \le \frac{1}{2}, \\ t & \text{else.} \end{cases}$$

Observe that $w_1 \vee w_2 = \mathrm{id}$.

(ii) On $X = \{1,2\}$, let $M = (m_t)_{t\in[0,1]}$ be given by $m_t(\{1\}) = 1 - t$ and $m_t(\{2\}) = m_t(X) = 1$. Then all m_t are maxitive fuzzy measures and $S_M(x,y) = w_1(x) \vee w_2(y)$, where $w_1(t) = \min(t, \frac{1}{2})$ and $w_2(t) = t$, $t \in [0,1]$.

6 Conclusion

Examining the comonotone maxitive idempotent aggregation functions, we have introduced the extended Sugeno integral.

We have examined some properties of this interesting integral and its relationship to the ordinary Sugeno integral. Extended Sugeno integral can be introduced on arbitrary (compact) ordinal scale, e.g., on a finite scale. We expect possible applications of this new concept in several areas where Sugeno integral was applied, and where the behavior of an aggregation only on small values differs from the behavior of aggregation only on high values.

Observe that recently Giove et al. [5] have introduced generalized Choquet integral which is based on a special system $M = (m_t)_{t\in[0,1]}$ of fuzzy measures. Note that though Sugeno and Choquet integrals coincide whenever the underlying fuzzy measure attains only trivial values 0 and 1, this is not more true for the extended Sugeno integral and the generalized Choquet integral.

Note also that a concept based on ultrafilters which is closely related to the extended Sugeno integral can be found in a recent paper of Havranová and Kalina [7].

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