A Universal Integral Independent of Measurable Spaces and Function Spaces

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Abstract

For $[0,\infty]$-valued (monotone) measures and functions, universal integrals are introduced and investigated. For a fixed pseudo-multiplication $\otimes$ on $[0,\infty]$ the smallest and the greatest universal integrals are given. Finally, a third construction method for obtaining universal integrals is introduced.

Keywords: Universal integral, pseudo-multiplication, Choquet integral, Sugeno integral.

1 Introduction

In [11, 12] we have introduced the framework for integrals with respect to normed monotone measures, acting on measurable functions whose range is a subset of the unit interval. These restrictions are quite natural in several areas where these integrals are applied. However, we want to extend this concept to the case of (nonnegative) real numbers, i.e., we want to integrate nonnegative real-valued functions with respect to arbitrary (nonnegative) monotone set functions. Recall, e.g., the integral in [18] extending the Sugeno integral to $[0,\infty]$, as well as the Choquet integral [4] (see also [25]), again acting on $[0,\infty]$, as prominent examples of integrals not being restricted to $[0,1]$.

The aim of this contribution is to introduce the concept of universal integrals acting on the interval $[0,\infty]$, i.e., integrals which can be defined on an arbitrary measurable space $(X,\mathcal{A})$ based on an arbitrary monotone set function $m\colon \mathcal{A}\to[0,\infty]$ satisfying $m(\emptyset)=0$ and $m(X)>0$ (in the sequel we shall call such an m simply a *measure* on $(X,\mathcal{A})$) and which is applicable to any measurable function $f\colon X\to[0,\infty]$.

2 Universal integrals

For a fixed measurable space $(X,\mathcal{A})$, $\mathcal{F}^{(X,\mathcal{A})}$ denotes the set of all $\mathcal{A}$-measurable functions $f\colon X\to[0,\infty]$. For $a\in\,]0,\infty]$, $\mathcal{M}_a^{(X,\mathcal{A})}$ denotes the set of all monotone set functions $m\colon\mathcal{A}\to[0,\infty]$ such that $m(\emptyset)=0$ and $m(X)=a$, and we put

$$\mathcal{M}^{(X,\mathcal{A})}=\bigcup_{a\in]0,\infty]}\mathcal{M}_a^{(X,\mathcal{A})}.$$

Each nondecreasing function $H\colon\mathcal{F}^{(X,\mathcal{A})}\to[0,\infty]$ with $H(\mathbf{0})=0$ is called an *aggregation function on* $\mathcal{F}^{(X,\mathcal{A})}$ (compare with [2]). Which aggregation functions should be called an integral, this is a classical and still open problem. We give three examples of well-known functions which are used as integrals.

Example 2.1 The *Choquet*, *Sugeno* and *Shilkret* integrals (see [1,17]), respectively, are given, for any $(X,\mathcal{A})$, any $f\in\mathcal{F}^{(X,\mathcal{A})}$ and any $m\in\mathcal{M}^{(X,\mathcal{A})}$, by

$$\mathbf{Ch}(m,f)=\int_0^\infty m(\{f\geq t\})\,dt,$$
$$\mathbf{Su}(m,f)=\sup_{t\in[0,\infty]}\min(t,m(\{f\geq t\})),$$
$$\mathbf{Sh}(m,f)=\sup_{t\in[0,\infty]}t\cdot m(\{f\geq t\}),$$

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where the convention $0 \cdot \infty = 0$ is adopted whenever necessary.

Independently of whatever measurable space $(X, \mathcal{A})$ is actually chosen, all these integrals map $\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}$ into $[0, \infty]$ and, therefore, for any fixed $m \in \mathcal{M}^{(X,\mathcal{A})}$, they are aggregation functions on $\mathcal{F}^{(X,\mathcal{A})}$. Moreover, for any fixed $f \in \mathcal{F}^{(X,\mathcal{A})}$, they are nondecreasing mappings from $\mathcal{M}^{(X,\mathcal{A})}$ into $[0, \infty]$.

Let $\mathcal{S}$ be the class of all measurable spaces, and put

$$\mathcal{D} = \bigcup_{(X,\mathcal{A}) \in \mathcal{S}} \mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}.$$

Hence, we require an integral $\mathbf{I}$ to map $\mathcal{D}$ into $[0, \infty]$ and to be nondecreasing in each coordinate. Moreover, each reasonable integral is expected to satisfy the following minimal requirements:

(i) There is a binary function $\otimes\colon [0, \infty]^2 \to [0, \infty]$ with annihilator 0 such that, for all $(m, c \cdot \mathbf{1}_A) \in \mathcal{D}$, we have

$$\mathbf{I}(m, c \cdot \mathbf{1}_A) = c \otimes m(A)$$

(recall the "truth functionality" in propositional logic);

(ii) $\mathbf{I}$ allows to reconstruct the underlying measure: there is a constant $u \in \,]0, \infty]$ such that for all $(m, u \cdot \mathbf{1}_A) \in \mathcal{D}$ we have

$$\mathbf{I}(m, u \cdot \mathbf{1}_A) = m(A);$$

(iii) $\mathbf{I}$ is idempotent in the following sense: there is a constant $v \in \,]0, \infty]$ such that for all measurable spaces $(X, \mathcal{A}) \in \mathcal{S}$, for all constants $c \in [0, \infty]$ and for all measures $m \in \mathcal{M}_v^{(X,\mathcal{A})}$ we have

$$\mathbf{I}(m, c) = c.$$

Proposition 2.2 *A nondecreasing function* $\mathbf{I}\colon \mathcal{D} \to [0, \infty]$ *satisfies* (i)–(iii) *only if the binary operation* $\otimes$ *in* (i) *is a nondecreasing function with neutral element* $e \in \,]0, \infty]$.

Each function $\otimes\colon [0, \infty]^2 \to [0, \infty]$ with the properties mentioned in Proposition 2.2 will be called a *pseudo-multiplication.*

All three integrals mentioned in Example 2.1 fulfill the following equality

$$\mathbf{I}(m, f) = \mathbf{I}(\mu, g)$$

for all $m, \mu \in \mathcal{M}^{(X,\mathcal{A})}$ and $f, g \in \mathcal{F}^{(X,\mathcal{A})}$ satisfying for all $t \in \,]0, \infty]$,

$$m(\{f \geq t\}) = \mu(\{g \geq t\}). \tag{1}$$

Property (1), extended to pairs from possibly different spaces, will be called *integral equivalence*, with the notation $(m, f) \sim (\mu, g)$, and the indistinguishability of integral equivalent pairs will be our last axiom for a reasonable integral.

Example 2.3 Let $\mathbf{I}\colon \mathcal{D} \to [0, \infty]$ be given by

$$\mathbf{I}(m, f) = m(\{f > 0\}) \cdot \sup f.$$

Then $\mathbf{I}$ fulfills (i)–(iii) (here $\otimes$ is the standard product on $[0, \infty]$ with the convention $0 \cdot \infty = 0$), but $(m, f) \sim (\mu, g)$ does not imply $\mathbf{I}(m, f) = \mathbf{I}(\mu, g)$. Take, e.g., $X = \,]0, 1[\,, \mathcal{A} = \mathcal{B}(]0, 1[)$ and $m\colon \mathcal{A} \to [0, \infty]$ given by

$$m(A) = \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(m, \mathbf{1}_\emptyset) \sim (m, \mathrm{id}_X)$, but $\mathbf{I}(m, \mathbf{1}_\emptyset) = 0$ and $\mathbf{I}(m, \mathrm{id}_X) = 1$.

We require universality of the integral, in the sense that they can be defined on any measurable space $(X, \mathcal{A})$. Therefore we will use the name universal integral in what follows.

Definition 2.4 A function $\mathbf{I}\colon \mathcal{D} \to [0, \infty]$ is called a *universal integral* if the following axioms hold:

(I1) For any measurable space $(X, \mathcal{A})$, $\mathbf{I}$ restricted to $\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}$ is nondecreasing in each coordinate;

(I2) there exists a pseudo-multiplication $\otimes\colon [0, \infty]^2 \to [0, \infty]$ with neutral element $e \in \,]0, \infty]$ such that for all $(m, c \cdot \mathbf{1}_A) \in \mathcal{D}$

$$\mathbf{I}(m, c \cdot \mathbf{1}_A) = c \otimes m(A);$$

(I3) for all integral equivalent pairs $(m, f), (\mu, g) \in \mathcal{D}$ we have

$$\mathbf{I}(m, f) = \mathbf{I}(\mu, g).$$

Due to axiom (I3), for each universal integral $\mathbf{I}$ and for each pair $(m, f) \in \mathcal{D}$, the value $\mathbf{I}(m, f)$ depends only on the function $h^{(m,f)}\colon\,]0, \infty] \to [0, \infty]$ given by

$$h^{(m,f)}(x) = m(\{f \geq x\}).$$

Note that, for each $(m, f) \in \mathcal{D}$, the function $h^{(m,f)}$ is nonincreasing and thus Borel measurable.

Denote by $\mathcal{H}$ the subset of all nonincreasing functions from $\mathcal{F}^{(]0,\infty],\mathcal{B}(]0,\infty])}$.

Proposition 2.5 *A function $\mathbf{I}\colon \mathcal{D} \to [0, \infty]$ is a universal integral related to some pseudo-multiplication $\otimes$ if and only if there is a function $J\colon \mathcal{H} \to [0, \infty]$ satisfying the following conditions:*

(J1) *J is nondecreasing;*

(J2) *$J(d \cdot \mathbf{1}_{]0,c]}) = c \otimes d$ for all $c, d \in [0, \infty]$;*

(J3) *$\mathbf{I}(m, f) = J(h^{(m,f)})$ for all $(m, f) \in \mathcal{D}$.*

An approach to universal integrals similar to Proposition 2.5 can be traced back to [23], compare also with [10].

Example 2.6 *Let $\mathbf{I}\colon \mathcal{D} \to [0, \infty]$ be given by*

$$\mathbf{I}(m, f) = \frac{\displaystyle\int_0^1 \frac{m(\{f \geq \frac{t}{1-t}\})}{1+m(\{f \geq \frac{t}{1-t}\})}\, dt}{\displaystyle\int_0^1 \frac{1}{1+m(\{f \geq \frac{t}{1-t}\})}\, dt}.$$

Then $\mathbf{I}$ is a universal integral. Moreover, we have

$$\mathbf{I}(m, c \cdot \mathbf{1}_A) = \frac{c \cdot m(A)}{1 + c + m(A)},$$

i.e., $\mathbf{I}$ is based on the pseudo-multiplication $\otimes\colon [0, \infty]^2 \to [0, \infty]$ given by

$$a \otimes b = \frac{a \cdot b}{1 + a + b}$$

which has ∞ as its neutral element. The function $J\colon \mathcal{H} \to [0, \infty]$ is then given by

$$J(h) = \frac{\displaystyle\int_0^1 \frac{h(\frac{t}{1-t})}{1+h(\frac{t}{1-t})}\, dt}{\displaystyle\int_0^1 \frac{1}{1+h(\frac{t}{1-t})}\, dt}.$$

3 Extremal universal integrals

Following the ideas of inner and outer measures in Caratheodory's approach [8], the following result is not difficult to check.

Proposition 3.1 *Let $\otimes\colon [0, \infty]^2 \to [0, \infty]$ be a pseudo-multiplication on $[0, \infty]$. Then the smallest universal integral $\mathbf{I}_\otimes$ and the greatest universal integral $\mathbf{I}^\otimes$ based on $\otimes$ are given by*

$$\mathbf{I}_\otimes(m, f) = \sup\{t \otimes m(\{f \geq t\}) \mid t \in [0, \infty]\},$$
$$\mathbf{I}^\otimes(m, f) = (\mathrm{essup}_m f) \otimes (\sup_{t>0} m(\{f \geq t\})),$$

where

$$\mathrm{essup}_m f = \sup\{t \in [0, \infty] \mid m(\{f \geq t\}) > 0\}.$$

Clearly, we have $\mathbf{Su} = \mathbf{I}_{\mathrm{Min}}$ and $\mathbf{Sh} = \mathbf{I}_{\mathrm{Prod}}$, where $\mathrm{Min}(a, b) = \min(a, b)$ and $\mathrm{Prod}(a, b) = a \cdot b$.

There is neither a smallest nor a greatest pseudo-multiplication $\otimes$ on $[0, \infty]$. However, if we fix the neutral element $e \in\,]0, \infty]$, then the smallest pseudo-multiplication $\otimes_e$ and the greatest pseudo-multiplication $\otimes^e$ with neutral element e are given by

$$a \otimes_e b = \begin{cases} 0 & \text{if } (a, b) \in [0, e[^2, \\ \max(a, b) & \text{if } (a, b) \in [e, \infty]^2, \\ \min(a, b) & \text{otherwise,} \end{cases}$$

$$a \otimes^e b = \begin{cases} \min(a, b) & \text{if } (a, b) \in [0, e]^2, \\ \infty & \text{if } (a, b) \in\,]e, \infty]^2, \\ \max(a, b) & \text{otherwise.} \end{cases}$$

Proposition 3.2 *Denote by $\mathcal{K}$ the set of all universal integrals $\mathbf{I}$ such that*

(i) *for each $m \in \mathcal{M}_e^{(X,\mathcal{A})}$ and each $c \in [0, \infty]$ we have $\mathbf{I}(m, c) = c$,*

(ii) *for each* $m \in \mathcal{M}^{(X,\mathcal{A})}$ *and each* $A \in \mathcal{A}$ *we have* $\mathbf{I}(m, e \cdot \mathbf{1}_A) = m(A)$.

Then $\mathbf{I}_{\otimes_e}$ *and* $\mathbf{I}^{\otimes^e}$ *are the smallest and greatest element of* $\mathcal{K}$*, respectively, their explicit formulas being given by*

$$\mathbf{I}_{\otimes_e}(m,f) = \max(m(\{f \geq e\}), \mathrm{essinf}_m f),$$

where

$$\mathrm{essinf}_m f = \sup\{t \in [0,\infty] \mid m(\{f \geq t\}) \geq e\},$$

and

$$\mathbf{I}^{\otimes^e}(m,f) = \begin{cases} \min(\mathrm{essup}_m f, m(\{f>0\})) \\ \textit{if } \max(\mathrm{essup}_m f, m(\{f>0\})) \leq e, \\ \infty \\ \textit{if } \min(\mathrm{essup}_m f, m(\{f>0\})) > e, \\ \mathrm{essup}_m f \\ \textit{if } m(\{f>0\}) < e \textit{ and } \mathrm{essup}_m f \geq e, \\ m(\{f>0\}) \\ \textit{otherwise.} \end{cases}$$

4 A construction of universal integrals

Proposition 3.1 gives two construction methods for universal integrals based on a given pseudo-multiplication $\otimes$ on $[0,\infty]$. Based on [1], we introduce another construction method.

For a given pseudo-multiplication $\otimes$ on $[0,\infty]$, we suppose the existence of a pseudo-addition $\oplus\colon [0,\infty]^2 \to [0,\infty]$ which is continuous, associative, nondecreasing and has 0 as neutral element (then the commutativity of $\oplus$ follows, see [9]), and which is left-distributive with respect to $\otimes$, i.e., for all $a,b,c \in [0,\infty]$ we have

$$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c).$$

The pair $(\oplus, \otimes)$ is then called an *integral operation pair*. For all $(X,\mathcal{A})$ and $f \in \mathcal{F}^{(X,\mathcal{A})}$ with a finite range $\mathrm{Ran} f = \{a_1, \ldots, a_n\}$ such that $a_1 < \cdots < a_n$ we have

$$f = \bigoplus_{i=1}^{n} b_i \cdot \mathbf{1}_{A_i}, \tag{2}$$

where $A_i = \{f \geq a_i\}$, $a_0 = 0$, and

$$b_i = \inf\{c \in [0,\infty] \mid a_{i-1} \oplus c = a_i\}.$$

We denote the set of all functions in $\mathcal{F}^{(X,\mathcal{A})}$ with finite range by $\mathcal{F}_{\mathrm{fin}}^{(X,\mathcal{A})}$. For any $\otimes$-based universal integral $\mathbf{I}$ we have

$$\mathbf{I}(m, b_i \cdot \mathbf{1}_{A_i}) = b_i \otimes m(A_i).$$

We define the function $\mathbf{I}_{\oplus,\otimes}$ acting on $\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}_{\mathrm{fin}}^{(X,\mathcal{A})}$ (the elements of $\mathcal{F}_{\mathrm{fin}}^{(X,\mathcal{A})}$ are written in the form (2)) by

$$\mathbf{I}_{\oplus,\otimes}(m,f) = \bigoplus_{i=1}^{n} b_i \otimes m(A_i),$$

and its extension to $\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}$ by

$$\mathbf{I}_{\oplus,\otimes}(m,g) = \sup\{\mathbf{I}_{\oplus,\otimes}(m,f) \mid f \in \mathcal{F}_{\mathrm{fin}}^{(X,\mathcal{A})}, f \leq g\}.$$

Proposition 4.1 *For each integral operation pair* $(\oplus,\otimes)$ *the function* $\mathbf{I}_{\oplus,\otimes}$ *is a universal integral.*

Note that Choquet-like integrals studied in [13] are a special case of universal integrals of the type $\mathbf{I}_{\oplus,\otimes}$, with $(\oplus,\otimes)$ being an appropriate integral operation pair.

Example 4.2

(i) For each pseudo-multiplication $\otimes$ on $[0,\infty]$, the pair $(\vee,\otimes)$, where $\vee = \sup$, is an integral operation pair and we have $\mathbf{I}_{\vee,\otimes} = \mathbf{I}_{\otimes}$.

(ii) The Choquet integral is related to the pair $(+, \mathrm{Prod})$, i.e., $\mathbf{Ch} = \mathbf{I}_{+,\mathrm{Prod}}$.

(iii) For $p \in \,]0,\infty[$, define the pseudo-addition $+_p\colon [0,\infty]^2 \to [0,\infty]$ by $a +_p b = (a^p + b^p)^{1/p}$. The pair $(+_p, \mathrm{Prod})$ is an integral operation pair, and we have

$$\mathbf{I}_{+_p,\mathrm{Prod}}(m,f) = (\mathbf{Ch}(m^p, f^p))^{1/p}.$$

Moreover, we get $\lim_{p\to\infty} \mathbf{I}_{+_p,\mathrm{Prod}} = \mathbf{Sh}$, i.e., $\lim_{p\to\infty} \mathbf{I}_{+_p,\mathrm{Prod}} = \mathbf{I}_{\mathrm{Prod}}$.

Similarly, $\lim_{p\to 0^+} \mathbf{I}_{+_p,\mathrm{Prod}} = \mathbf{I}^{\mathrm{Prod}}$. Note that $\mathbf{I}^{\mathrm{Prod}}$ cannot be constructed as described in Proposition 4.1.

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