Games on distributive lattices and the Shapley interaction transform

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Abstract

The paper proposes a general approach of interaction between players or attributes. It generalizes the notion of interaction defined for players modeled by games, by considering functions defined on distributive lattices. A general definition of the interaction index is provided, as well as the construction of operators establishing transforms between games, their Möbius transforms and their interaction indices.

Keywords: Lattice game, Möbius transform, Interaction index, Equivalent representations.

1 Introduction

Interaction index has been developped in [7] by Grabisch, and can be seen as a generalization of the Shapley value. Roughly speaking, the interaction index quantifies the genuine contribution of a coalition with reference to all its subcoalitions, where a positive (resp. negative) interaction corresponds to a positive (resp. negative) correlation. In game theory, it describes the synergy between players or voters, the interest in forming or not forming certain coalitions. In multicriteria decision, it tells which criteria play a key role (and how), and which criteria are redundant (with which ones) in the decision process.

Games defined over *distributive lattices* are very general objects which enable to capture a large variety of behaviors, since every playable action can be expressed in terms of pure or elementary actions. An alternative use of these games has been proposed by Faigle and Kern [6]: from a partially ordered set of players taking part into the game (the relation of precedence), a game is built over the set of all feasible coalitions. An axiomatization of the Shapley value has been proposed in [9], as well as in [6]. In this paper, we aim at generalizing the interaction index concept for games over distributive lattices, which is based on our Shapley value, and which encompasses the interaction index for classical cooperative games, as axiomatized in [10].

In [4], the authors have worked out a framework in order to underline linear and bijective correspondences between a classical cooperative game, its Möbius transform, and its interaction index. In [13], we generalized this construction for *bi-set functions*, which are functions defined over the set of couples of subsets (A, B) (bi-coalitions) of a basis finite set N, such that $A \cap B = \emptyset$, A representing the coalition of defensive players, and B, the defeaters players. We provided a framework to express any game in TU-form from its interaction index by means of the incidence algebras [5]. The objective in this paper is now to extend this framework to games defined over distributive lattices.

In Section 2, we propose a short introduction to distributive lattices, and provide a general definition of lattice games, with some examples. Section 3 gives definition of the Möbius transform, and brings some mathematical background about derivative of lattice functions. In Section 4, we introduce the interaction index for lattice games. *Group actions* are a useful algebraic tool which enable any bijective linear transformation (isomorphism) to operate over the set of lattice functions. Thanks to them, we set up in Sec-

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tion 5 a commutative diagram in the set of lattice functions, which proves that any lattice function, its Möbius transform and its interaction index characterize the same object. We work out in Section 6 an explicit formula for the Möbius transform of distributive lattice functions, as well as the inversion of a fundamental formula of Section 4 which expresses the interaction index of any game in terms of its Möbius transform. Finally, we provide in Section 7 the inverse interaction operator, and the straight expression of any lattice game from its interaction index.

$\mathbb{N}$ denotes the set of nonnegative integers $\{0, 1, 2, \dots\}$. If no ambiguity occurs, we denote by the lower case letters $s, t, \dots$ the cardinals of sets $S, T, \dots$ and we will often omit braces for singletons.

2 Lattice functions and games

We introduce some basic notions about lattices and distributive lattices. A *lattice* L is any partially ordered set (poset) $(L, \leq)$ in which every pair of elements x, y has a supremum $x \vee y$ and an infimum $x \wedge y$. Note that whenever L is finite[1], L is a *complete lattice*, that is, for any nonempty subset, their supremum and infimum always exist. The greatest element of a lattice (denoted $\top$) and least element $\bot$ always exist. In the sequel, it shall be convenient to lay down the convention $\bigvee \emptyset = \bigwedge \emptyset = \bot$.

A lattice is *distributive* if $\vee, \wedge$ obey distributivity. An element $j \in L$ is *join-irreducible* if it cannot be expressed as a supremum of other elements. Equivalently j is join-irreducible if it covers only one element, where x *covers* y (we also say that y is a *predecessor* of x, and denote $x \succ y$) means that $x > y$ and there is no z such that $x > z > y$. The set of all join-irreducible elements of L is denoted by $\mathscr{J}(L)$.

An important property is that in a distributive lattice, any element x can be written as an irredundant supremum of join-irreducible elements in a unique way (this is called the *minimal decomposition* of x). We denote by $\eta^*(x)$ the set of join-irreducible elements in the minimal decomposition of x, and we denote by $\eta(x)$ the *normal decomposition* of x, defined as the set of join-irreducible elements smaller or equal to x, i.e., $\eta(x) := \{j \in \mathscr{J}(L) \mid j \leq x\}$. Hence $\eta^*(x) \subseteq \eta(x)$, and

$$x = \bigvee \eta^*(x) = \bigvee \eta(x).$$

For any poset $(P, \leq)$, $Q \subseteq P$ is said to be a *downset* of P if $x \in Q$ and $y \leq x$ imply $y \in Q$. We denote by $\mathscr{O}(P)$ the set of all downsets of P. One can associate to any poset $(P, \leq)$ a distributive lattice which is $\mathscr{O}(P)$ endowed with inclusion. As a consequence of the above results, the mapping η is an isomorphism of L onto $\mathscr{O}(\mathscr{J}(L))$ (Birkhoff's theorem, [1]).

In the whole paper, $N := \{1, \dots, n\}$ is a finite set which can be thought as the set of players or also voters, criteria, states of nature, depending on the application. We consider finite distributive lattices $(L_1, \leq_1), \dots, (L_n, \leq_n)$ and their product $L := L_1 \times \cdots \times L_n$ endowed with the product order $\leq$. Elements x of L can be written in their vector form $(x_1, \dots, x_n)$. L is also a distributive lattice whose join-irreducible elements are of the form $(\bot_1, \dots, \bot_{i-1}, j_i, \bot_{i+1}, \dots, \bot_n)$, for some i and some join-irreducible element j_i of L_i. In the sequel, with some abuse of language, we shall also call j_i this element of L. We denote by $\mathscr{J}(L)$ the set of join-irreducible elements of L (Section 4). A *vertex* of L is any element whose components are either top or bottom. Vertices of L will be denoted by $\top_X$, $X \subseteq N$, whose coordinates are $\top_k$ if $k \in X$, $\bot_k$ else.

Lattice functions are real-valued mappings defined over product lattices of the above form. Lattice functions which vanishes at $\bot$ are called *lattice games* (or games) on $(L, \leq)$. We denote by $\mathbb{R}^L$ the set of lattice functions over L, and by $\mathscr{G}(L)$ the subset of games. Each lattice $(L_i, \leq_i)$ may be different, and represents the (partially) ordered set of actions, choices, levels of participation of player i to the game. A game v is *monotone* if $x \leq y$ implies $v(x) \leq v(y)$ for all $x, y \in L$. Several particular cases of lattice games are of interest.

- $L = \{0, 1\}^n$. This is the classical notion of cooperative game in *pseudo-Boolean functions* form. Indeed, $(L, \leq)$ is isomorphic to the Boolean lattice[2] $(2^N, \subseteq)$ of

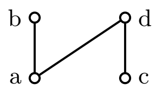
[1] In the context of the paper, all considered lattices are finite.

[2] To avoid heavy notations, we will sometimes de-

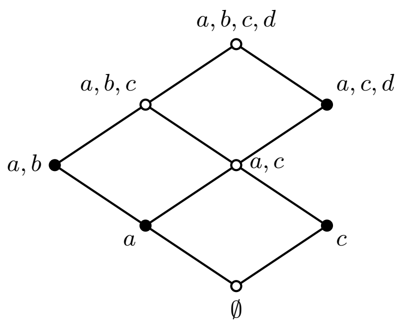
the subsets of N. Monotone games of $\mathscr{G}(2^N)$ are called *capacities* [3].

- We propose the following interpretation for games on L in the general case, i.e., L is any direct product of n distributive lattices. We assume that each player $i \in N$ has at her/his disposal a set of *elementary* or *pure* actions $j_1, \dots, j_{n_i}$. These elementary actions are partially ordered (e.g. in the sense of benefit caused by the action), forming a partially ordered set $(\mathscr{J}_i, \leq_i)$. Then by Birkhoff's theorem (see above), the set $(\mathscr{O}(\mathscr{J}_i), \subseteq)$ of downsets of $\mathscr{J}_i$ is a distributive lattice denoted L_i, whose join-irreducible elements correspond to the elementary actions. The *bottom action* $\perp$ of L_i is the action which amounts to do nothing. Hence, each action in L_i is either a pure action j_k or a combined action $j_k \vee j_{k'} \vee j_{k''} \vee \dots$ consisting of doing all pure actions $j_k, j_{k'}, \dots$ for player i.

For example, let us suppose that for a given player i, elementary actions are a, b, c, d endowed with the order $\leq_i := \{(a,b), (a,d), (c,d)\}$. They form the following poset:



which in turn form the following lattice L_i of possible actions (black circles represent join-irreducible elements of L_i):



3 The Möbius transform and derivatives of lattice functions

We introduce in this section some useful material for lattice functions. The *Möbius transform* initially takes its name from number theory[3], and is a key concept in decision analysis (see e.g. [2]). Let $(P, \leq)$ be any poset. The Möbius transform m^f of a mapping $f : P \to \mathbb{R}$ is the unique solution of the equation

$$f(x) = \sum_{y \leq x} m^f(y), \quad x \in P, \qquad (1)$$

given by

$$m^f(x) := \sum_{y \leq x} \mu(y, x)\, f(y), \quad x \in P, \qquad (2)$$

where μ is an integer-valued function defined on $P \times P$. For instance, whenever P is the Boolean lattice 2^N endowed with inclusion, it is well-known that $\mu(A, B) = (-1)^{|B \setminus A|}$, for all subsets A, B such that $A \subseteq B$.

As it will be seen in the next section, *derivatives* of lattice functions are a very useful tool, and have been generalized (in particular) for distributive lattice functions in [8]. Let $(L, \leq)$ be a distributive lattice and $j \in \mathscr{J}(L)$. The *first-order derivative* of the lattice function f w.r.t. j at element $x \in L$ is given by

$$\Delta_j f(x) := f(x \vee j) - f(x).$$

Using the minimal irredundant decomposition $\eta^*(y) = \{j_1, \dots, j_m\}$ of some $y \in L$, we iteratively define the derivative of f w.r.t. y at $x \in L$ by

$$\Delta_y f(x) := \Delta_{j_m}\big(\dots \Delta_{j_2}\big(\Delta_{j_1} f(x)\big) \dots\big).$$

Note that if for some k, $j_k \leq x$, the derivative is null. Also, this definition does not depend on the order of the j_k's and thus is well defined. In particular, whenever $(L, \leq)$ is the Boolean lattice $(2^N, \subseteq)$, for any nonempty $S \subseteq N$,

$$\Delta_S f(A) := \sum_{T \subseteq S} (-1)^{s-t} f(A \cup T), \quad A \subseteq N.$$

We set $\Delta_\perp f(x) := f(x)$, for any $x \in L$.

4 The interaction index for lattice functions

From now on, L is a direct product of n finite distributive lattices. Let $v \in \mathscr{G}(L)$. We propose a general definition of interaction as presented in the introduction. We begin by defining the *importance index*, introduced in [8], as

note by 2^m any Boolean lattice isomorphic to 2^M, $|M| = m$.

[3]Underlying lattice is in this case the set of all divisors of any positive integer endowed with divisibility relation.

a *power index* of the game defined for elementary actions of every player (that is to say, w.r.t. each join-irreducible element of each lattice L_i). This means that we try to provide an equitable way to share the worth $v(\top)$ between all elementary actions.

For a given elementary action j_i, the importance index is written as a weighted average of the marginal contributions of j_i, taken at vertices of L.

Definition 1 *Let $i \in N$ and j_i any join-irreducible element of L. Let $v \in \mathscr{G}(L)$. The* importance index *w.r.t. j_i of v is defined by*

$$\phi^v(j_i) = \sum_{Y \subseteq N \setminus i} \alpha^1_{|Y|} \, \Delta_{j_i} v(\top_Y),$$

where $\alpha^1_k := \dfrac{k!\,(n-k-1)!}{n!}$, *for all* $k \in \{0, \ldots, n-1\}$.

Note that if $L = 2^N$, we obtain the definition of the *Shapley value* [14]. In [9], we proposed an axiomatization of the Shapley value for multichoice games, where the obtained formula is also the one given above (all the L_i's are completely ordered).

As an extension of the importance index for every element of L, and every lattice function $f \in \mathbb{R}^L$, we propose a definition for the *interaction transform.* For any $x \in L$, $I^f(x)$ expresses the interaction in the function among all elementary actions j of the minimal decomposition $x = \bigvee_{j \in \eta^*(x)} j$.

An interaction index has been proposed in [8]. However, the formula was only defined for elements of $\mathscr{J}(L)$. We present here I^f as a mapping defined over L. For that, we give the following generalized definition of $\underline{x}$ for any $x \in L$.

Definition 2 *Let $x \in L$. We call* antecessor *of x the unique element of L defined by $\underline{x} := \bigvee \big(\eta(x) \setminus \eta^*(x)\big)$.*

If $x \in \mathscr{J}(L)$, the antecessor of x is obviously its predecessor, in accordance with the notation $\underline{x}$. By the convention of Section 2, the antecessor of $\perp$ is itself. Note also that the definition of $\underline{x} \in L$ is consistent with the structure of direct product of distributive lattices of L. Indeed, we easily check that $\underline{x} = (\underline{x_1}, \ldots, \underline{x_n})$.

The following proposition provides two characterizations and an important property of the antecessor.

Proposition 1 *Let $x \in L$, and $p(x) := \{y \in L \mid y \prec x\}$. Then the following assertions hold.*
(i) $\underline{x} = \bigwedge p(x)$.
(ii) $\underline{x}$ is the unique element s.t. $[\underline{x}, x]$ is Boolean and contains $p(x)$.
(iii) $[\underline{x}, x] \cong 2^{\eta^*(x)}$.

The interaction index $I^v(x)$ is expressed as a weighted average of the derivatives w.r.t. x, taken at vertices of L.

Definition 3 *Let $v \in \mathscr{G}(L)$. Let $x \in L$ and $X := \{i \in N \mid x_i \neq \perp_i\}$. The* interaction index *w.r.t. x of v is defined by*

$$I^v(x) := \sum_{Y \subseteq N \setminus X} \alpha^{|X|}_{|Y|} \, \Delta_x v(\underline{x} \vee \top_Y),$$

where $\alpha^j_k := \frac{k!\,(n-j-k)!}{(n-j+1)!}$, *for all* $j = 0, \ldots, n$ *and* $k = 0, \ldots, n-j$.

This extends Definition 1. Besides, the formula overlaps previous definitions of the interaction introduced and axiomatized in [4, 11] for classical cooperative games, and also in [8] for multichoice games whose all L_i's are identical.

The following result generalizes one given in [8] and express the interaction index in terms of the Möbius transform.

Theorem 2 *Let $v \in \mathscr{G}(L)$ and $x \in L$. Then*

$$I^v(x) = \sum_{z \in [x, \check{x}]} \frac{1}{k(z) - k(x) + 1} \, m^v(z),$$

where $\check{x}_j := \top_j$ if $x_j = \perp_j$, $\check{x}_j := x_j$ else, and $k(y)$ is the number of coordinates of $y \in L$ not equal to $\perp_j$, $j = 1, \ldots, n$.

5 Linear transformations on sets of lattice functions

In [4], the authors laid down a general framework of transformations of set functions by introducing an algebraic structure on set functions and operators (set functions of two variables), which enable the writing of the formulae given in the previous section under a simplified algebraic form. Then in [13], the same has been done for bi-set functions, by introducing *incidence algebras* [5]. Although this tool may be useful in combinatorics of order theory, we do not now proceed in the same way for lattice functions, making the choice to use a more suitable algebraic strucure, namely the group actions.

We call *operator* on L a real-valued function on $L \times L$. A binary operation $\star$ (multiplication or convolution) between operators is introduced as follows:

$$(\Phi \star \Psi)(x, y) := \sum_{t \in L} \Phi(x, t)\, \Psi(t, y).$$

Endowed with $\star$, the set of operators contains the identity element

$$\Delta(x, y) := \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise,} \end{cases} \quad x, y \in L,$$

and also satisfies associativity, which makes it a monoid. When it exists, we denote by Φ^{-1} the inverse of an operator Φ, that is to say the operator satisfying $\Phi \star \Phi^{-1} = \Phi^{-1} \star \Phi = \Delta$. Consequently, the set of all inversible operators is a group. We denote it by $\mathbb{G}$. We denote by ${}^t\Phi$ the *transpose* of the operator Φ, i.e., ${}^t\Phi(x, y) := \Phi(y, x)$ for all $x, y \in L$.

Let $\leqq$ be any partial order on L included in the usual order $\leq$, and $\lneqq$ the associated strict order. We denote by $I(L, \leqq)$ the set of intervals of L w.r.t. the order $\leqq$, i.e., the family of subsets $[x, y]_{\leqq} := \{t \in L \mid x \leqq t \leqq y\}$, with $x \leqq y$. An operator Φ is said to be *unit upper-triangular* (resp. *unit lower-triangular*) relatively to $\leqq$, or shortly $\mathrm{UUT}_{\leqq}$ (resp. $\mathrm{ULT}_{\leqq}$), if it equals 1 on the diagonal of L^2, and vanishes at all pairs (x, y) s.t. $[x, y]_{\leqq} = \emptyset$ (resp. $[y, x]_{\leqq} = \emptyset$):

$$\Phi(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \not\leqq y, \end{cases} \quad x, y \in L.$$

Note that the transpose of any $\mathrm{UUT}_{\leqq}$ operator is $\mathrm{ULT}_{\leqq}$.

Proposition 3 *The subset $G(\leqq)$ of all $UUT_{\leqq}$ operators endowed with $\star$, is a subgroup of $\mathbb{G}$. The inverse Φ^{-1} of $\Phi \in G(\leqq)$ computes recursively through*

$$\Phi^{-1}(x, x) = 1,$$

$$\Phi^{-1}(x, y) = -\sum_{x \leqq t \lneqq y} \Phi^{-1}(x, t)\, \Phi(t, y), \quad x \lneqq y.$$

Applying this result for the *Zeta operator* $Z \in G(\leq)$:

$$Z(x, y) := \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise,} \end{cases} \quad x, y \in L, \tag{3}$$

we obtain the application μ which is the Möbius operator, i.e., $Z^{-1} = \mu$ (see Section 3, (2)).

In order to rewrite formulae (1), (2) and also (7) in a reduced form, we introduce some group actions of $\mathbb{G}$ on the set of lattice functions: for x belonging to L, we define:

$$(\Phi \star f)(x) := \sum_{t \in L} \Phi(x, t)\, f(t), \tag{4}$$

$$(f \star \Phi)(x) := \sum_{t \in L} f(t)\, \Phi(t, x). \tag{5}$$

Now, (1) and (2) respectively rewrites as

$$f = m^f \star Z, \quad \text{and} \quad m^f = f \star Z^{-1}, \quad f \in \mathbb{R}^L.$$

Similarly, if we set down:

$$\Gamma(x, y) := \begin{cases} \frac{1}{k(y) - k(x) + 1}, & \text{if } \forall i \in N, x_i \in \{\perp_i, y_i\}, \\ 0, & \text{otherwise,} \end{cases} \tag{6}$$

we notice that $\Gamma \in G(\leq)$, and we can write from (7) the relation:

$$I^f = \Gamma \star m^f, \quad f \in \mathbb{R}^L, \tag{7}$$

which in turns gives by inversion

$$m^f = \Gamma^{-1} \star I^f, \quad f \in \mathbb{R}^L. \tag{8}$$

It is also possible to do without left group actions. Indeed, we easily show that the left action $\mathbb{G} \times \mathbb{R}^L \to \mathbb{R}^L$ can be converted into the right action $\mathbb{R}^L \times \mathbb{G} \to \mathbb{R}^L$ by $(\Phi, f) \mapsto (f, {}^t\Phi)$. Consequently,

$$I^f = m^f \star {}^t\Gamma, \quad f \in \mathbb{R}^L.$$

Note that ${}^t\Gamma$ and ${}^t\Gamma^{-1}$ are unit lower-triangular.

As a conclusion of these results, any lattice function may be seen as the interaction index or the Möbius transform of some lattice function. This actually generalizes a result *(equivalent representations)* of [7] by the result below (see Figure 1).

Theorem 4 *Operators Z and Γ generate a commutative diagram in $\mathbb{R}^L$.*

We call *interaction operator*, the operator $\mathbb{I} := Z^{-1} \star {}^t\Gamma$. Hence, the interaction index of $f \in \mathbb{R}^L$ is given by $I^f = f \star \mathbb{I}$. Note that $\mathbb{I}$ is neither UUT nor ULT.

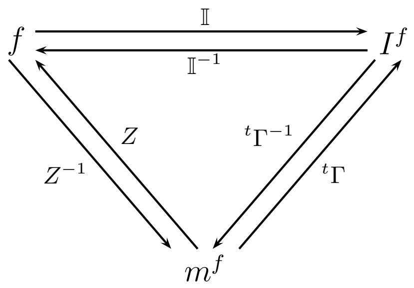


Figure 1: Lattice functions and their representations (operators act on the right)

6 The Möbius and Bernoulli operators

We now aim at giving an explicit formula for the Möbius operator and the Bernoulli operator[4] Γ^{-1}. Let $\sim$ be an equivalence relation on the set $I(L, \leqq)$. We denote by $\overline{[x,y]_{\leqq}}$ the class of any interval $[x,y]_{\leqq}$ by the relation $\sim$. We consider the following property for operators of $G(\leqq)$ relatively to this relation:

Φ is constant on each equivalence class of $\sim$, *i.e.*, $\forall [x,y], [x',y'] \in I(L, \leqq)$, if $\overline{[x,y]_{\leqq}} = \overline{[x',y']_{\leqq}}$, then $\Phi(x,y) = \Phi(x',y')$. (9)

The relation $\sim$ is said to be *compatible*, if the set of operators satisfying (9) is stable under multiplication.

We now consider the particular equivalence relation $\cong$ (order isomorphism) on $I(L, \leqq)$. Then it is a compatible equivalence relation (see [5]). One can notice that relatively to $\cong$ and the usual order, Z satisfies (9). However, it is not the case of Γ in the general case; for instance, if $L := L_1 = \{0,1,2\}$, $\frac{1}{2} = \Gamma(0,1) \neq \Gamma(1,2) = 0$, although $[0,1] \cong [1,2]$.

We denote by $\tilde{G}(\leqq)$ the subset of $G(\leqq)$ of operators satisfying property (9) relatively to the compatible relation $\cong$. It is possible to reduce the algebra structure of operators when dealing with the elements of $\tilde{G}(\leqq)$: to any $\Phi \in \tilde{G}(\leqq)$, we associate the following function φ defined on $\tilde{I}(L, \leqq)$, quotient set of $I(L, \leqq)$ by $\cong$:

$$\varphi(\overline{[x,y]_{\leqq}}) := \Phi(x,y), \quad [x,y]_{\leqq} \in I(L, \leqq). \tag{10}$$

The identity operator Δ clearly belongs to $\tilde{G}(\leqq)$, and has for associated function

$$\delta(\overline{[x,y]_{\leqq}}) := \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\tilde{g}(\leqq) := \{\varphi : \tilde{I}(L, \leqq) \to \mathbb{R} \mid \forall x \in L, \varphi(\overline{\{x\}}) = 1\}$. Clearly, (10) being reversible, we see that any real-valued mapping φ on $\tilde{I}(L, \leqq)$ such that $\varphi(\overline{\{x\}}) = 1$, $x \in L$, determines uniquely an operator of $\tilde{G}(\leqq)$. For $\varphi, \psi \in \tilde{g}(\leqq)$, we define

$$\varphi \star \psi(\overline{[x,y]_{\leqq}}) := \Phi \star \Psi(x,y), \quad [x,y]_{\leqq} \in I(L, \leqq),$$

where Φ and Ψ are the operators of $\tilde{G}(\leqq)$ respectively induced by φ and ψ.

Proposition 5 *$(\tilde{G}(\leqq), \star)$ and $(\tilde{g}(\leqq), \star)$ are isomorphic groups. δ is the identity element of $(\tilde{g}(\leqq), \star)$.*

We now address the particular order relation $\leqq$ that enables the writing of operation $\star$ in $\tilde{g}(\leqq)$ in terms of binomial coefficients, which makes brighter the terminology "convolution". From the description of (6) of Γ, we define the following binary relation in L:

$$x \trianglelefteq y \quad \text{iff} \quad \forall i \in N, x_i = \bot_i \text{ or } x_i = y_i.$$

One can easily check that $\trianglelefteq$ in an order relation. Besides, for all x, y s.t. $x \trianglelefteq y$, we naturally define the element $y - x$ of L by

$$(y-x)_i := \begin{cases} y_i, & \text{if } x_i = \bot_i, \\ \bot_i, & \text{if } x_i = y_i, \end{cases} \quad i \in N.$$

Note that if $x \trianglelefteq y$, $k(y-x) = k(y) - k(x)$.

Let $w(\mathscr{J}(L))$ be the *width* of $\mathscr{J}(L)$, that is to say the cardinal of a maximal antichain of $\mathscr{J}(L)$, that is also the sum of the cardinals of maximal antichains of the $\mathscr{J}(L_i)$'s. As a result, the greatest intervals of L isomorphic to a Boolean lattice, are isomorphic to $2^{w(\mathscr{J}(L))}$. Note that $n \leq w(\mathscr{J}(L)) \leq |\mathscr{J}(L)|$.

Considering the elements of $\tilde{I}(L, \leq)$, we denote by $\overline{m}$ the class of all Boolean intervals isomorphic to 2^m, $m = 0, \ldots, w(\mathscr{J}(L))$.

[4] This name is justified at the end of the section.

In the same way, $\overline{\overline{m}}$ denotes the element of $\tilde{I}(L,\trianglelefteq)$ representing all intervals $[x,y]_{\trianglelefteq}$ s.t. $k(y-x)=m$, $m=0,\dots,n$. Clearly, all these classes are nonempty. In particular, $\overline{0}$ and $\overline{\overline{0}}$ are the unique elements of $\tilde{g}(\leq)$ and $\tilde{g}(\trianglelefteq)$ containing singletons of L: $\overline{0}=\overline{\overline{0}}=\{\{x\}\mid x\in L\}$. Consequently, the identity element of $\tilde{g}(\leq)$ (resp. $\tilde{g}(\trianglelefteq)$) simply writes as the function which is 1 onto $\overline{0}$ (resp. $\overline{\overline{0}}$), and 0 elsewhere. One can note that in the general case, $\tilde{I}(L,\trianglelefteq)=\{\overline{\overline{0}},\dots,\overline{\overline{n}}\}$, but $\tilde{I}(L,\leq)\supsetneq\{\overline{0},\dots,\overline{w(\mathscr{J}(L))}\}$ (there are some classes having not a "Boolean type").

By (3) and (6), the associated functions $\zeta\in\tilde{g}(\leq)$ of Z and $\gamma\in\tilde{g}(\trianglelefteq)$ of Γ respectively write

$$\zeta(\alpha)=1,\quad \alpha\in\tilde{I}(L,\leq),$$
$$\text{and}\quad \gamma(\overline{\overline{m}})=\frac{1}{m+1},\quad m=0,\dots,n.$$

Theorem 6 *For all $\varphi,\psi\in\tilde{g}(\leq)$, and any $m\in\{0,\dots,w(\mathscr{J}(L))\}$,*

$$\varphi\star\psi(\overline{m})=\sum_{j=0}^{m}\binom{m}{j}\varphi(\overline{j})\,\psi(\overline{m-j}).$$

Besides, the inverse of φ computes recursively through

$$\varphi^{-1}(\overline{0})=1,$$
$$\varphi^{-1}(\overline{m})=-\sum_{j=0}^{m-1}\binom{m}{j}\varphi^{-1}(\overline{j})\,\varphi(\overline{m-j}).$$

The same formulae hold for $\varphi\star\psi(\overline{\overline{m}})$ and $\varphi^{-1}(\overline{\overline{m}})$, $\varphi,\psi\in\tilde{g}(\trianglelefteq)$ and $m\in\{0,\dots,n\}$.

Note that the above result is not general and does not apply for any $\tilde{g}(\leqq)$. Actually, $\tilde{G}(\leq)$ and $\tilde{G}(\trianglelefteq)$ are very particular subgroups of $G(\leq)$, which refer to particular algebras, namely of *binomial type* in the framework of incidence algebras.

Let $(B_m)_{m\in\mathbb{N}}$ be the *sequence of Bernoulli numbers*, computing recursively through

$$B_0=1,$$
$$B_m=-\frac{1}{m+1}\sum_{j=0}^{m-1}\binom{m+1}{j}B_j,\quad m\in\mathbb{N}\setminus\{0\}.$$

$(B_m)_m$ starts with $1,-1/2,1/6,0,-1/30,0,1/42\dots$, and it is well-known that $B_m=0$ for $m\geq 3$ odd. From Theorem 6, we derive the explicit expressions of ζ^{-1} and γ^{-1}. Thus by the bijection (10), we derive the following results.

$$Z^{-1}(x,y):=\begin{cases}(-1)^m, & \text{if } [x,y]\cong 2^m,\\ 0, & \text{otherwise,}\end{cases}$$

$$\text{and}\quad \Gamma^{-1}(x,y):=\begin{cases}B_{k(y-x)}, & \text{if } x\trianglelefteq y,\\ 0, & \text{otherwise,}\end{cases}$$

$x,y\in L$.

7 The interaction operator and its inverse

By means of the expression of the Bernoulli operator and Eq. (8), for any lattice function f, we get

$$m^f(x)=\sum_{y\trianglerighteq x}B_{k(y-x)}\,I^f(y).$$

For any $p\in\mathbb{N}$, and $m=0,\dots,p$, we define

$$b_m^p:=\sum_{j=0}^{m}\binom{m}{j}B_{p-j},$$

These numbers have been introduced in [4] to express a lattice function v from its interaction index I^v. It is easy to compute them from the sequence of Bernoulli: $b_0^p=B_p$, $p\in\mathbb{N}$, and by applying the recursion of the Pascal's triangle:

$$b_{m+1}^{p+1}=b_m^{p+1}+b_m^p,\quad 0\leq m\leq p.$$

The coefficients also satisfy the following symmetry:

$$b_m^p=(-1)^p\,b_{p-m}^p,\quad 0\leq m\leq p.$$

The values of b_m^p, $p\leq 6$, are

We finally give an explicit formula for the inverse interaction operator $\mathbb{I}^{-1}=Z\star{}^t\Gamma^{-1}$ (cf. Section 5).

Theorem 7 *For all $x,y\in L$,*

$$\mathbb{I}^{-1}(x,y)=b_{k(x_y)}^{k(x)},$$

where $(x_y)_i:=\begin{cases}x_i, & \textit{if } x_i\leq y_i\\ \perp_i, & \textit{otherwise}\end{cases}$, $i\in N$.

Consequently, for any lattice function f,

$$f(x)=\sum_{z\in L}b_{k(z_x)}^{k(z)}\,I^f(z),\quad x\in L.$$

		m				
		0	1	2	3	4
p	0	1				
	1	$-\frac{1}{2}$	$\frac{1}{2}$			
	2	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$		
	3	0	$\frac{1}{6}$	$-\frac{1}{6}$	0	
	4	$-\frac{1}{30}$	$-\frac{1}{30}$	$\frac{2}{15}$	$-\frac{1}{30}$	$-\frac{1}{30}$

8 Concluding remarks

We provided in this paper a complete framework in order to determine the whole expression of any game defined on a distributive game from its interaction transform. As a final result, we obtain a quite simple expression for computing this transformation, which denotes the existence of a simple polynomial algorithm for realising this task.

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