Relevancy Transformation Operators and Aggregation Functions in Fuzzy Inference Process

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Abstract

A generalized fuzzy controller is introduced and its properties are investigated. More attention is devoted to the information boundedness principle and the interaction property. Some techniques of compositions of fuzzy controllers are suggested.

Keywords: relevancy transformation operator, aggregation function, information boundedness principle, fuzzy controller, fuzzy rules, interaction.

1 Introduction

A standard controller performs a mapping

$$\varphi: \boldsymbol{X} \longrightarrow \boldsymbol{Y}$$

where $\boldsymbol{X}$ is the input space and $\boldsymbol{Y}$ is the output space. It can be represented by a crisp relation

$$R : \boldsymbol{X} \times \boldsymbol{Y} \longrightarrow \{0, 1\}$$

such that

$$R(x, y) = \begin{cases} 1 & \text{if } y = \varphi(x) \\ 0 & \text{otherwise} \end{cases}$$

In fuzzy controller [3, 5, 6, 8, 9], the information from the rule base is expressed using fuzzy sets. Instead of points, the fuzzy inference mechanism works with fuzzy subsets of input, output and other spaces. Fuzzy controller maps fuzzy subsets of an input space $\boldsymbol{X}$ onto fuzzy subsets of an output space $\boldsymbol{Y}$. Such control process is given by a control function

$$\Phi : F(\boldsymbol{X}) \longrightarrow F(\boldsymbol{Y})$$

where the symbols $F(\boldsymbol{X})$ and $F(\boldsymbol{Y})$ stand for the sets of all fuzzy subsets of sets $\boldsymbol{X}$ and $\boldsymbol{Y}$ respectively. The sets $\boldsymbol{X}$ and $\boldsymbol{Y}$ are usually supposed to be convex subsets of finite-dimensional real vector spaces. Despite of some imprecision, inputs of fuzzy controllers are often crisp. Then fuzziness is restricted to the computations inside a core of fuzzy controllers, while they communicate with surroundings through crisp values of inputs and outputs. Of course, crisp inputs can be fuzzified or they can be expressed as singletons; characteristic functions of sets with exactly one element. The output can be also fuzzy and a defuzzification is needed. In this paper we will not pay attention to the fuzzification and defuzzification processes. Note that we identify fuzzy sets with their membership functions.

Consider a fuzzy rule base

$$RB = (A_i, C_i)_{i=1}^{n}$$

consisting of a set of if-then rules in the form:

If X is A_1 then Y is C_1

...

If X is A_n then Y is C_n

where $X, A_1, \ldots, A_n \in F(\boldsymbol{X})$ and $Y, C_1, \ldots, C_n \in F(\boldsymbol{Y})$. As $\boldsymbol{X}$ is supposed to be convex subset of finite-dimensional real vector space, we can also deal with more input variables using a conjunction of particular terms or cylindrical extensions [3, 6, 8]. If the output is also multiple, we decompose it to single variables considered independently. Without loss of generality, we restrict attention to multiple outputs and single inputs. The rules in a rule base express the expert's knowledge and can be obtained by various methods (experiences of

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experts, analysis of some models, etc.). Of course, we expect that the rule base is consistent in some sense and fulfills some requirement. We mention some of them [8, 9]:

(i) for all $i \in \{1, 2, \ldots, n\}$ there exists $x \in \boldsymbol{X}$ such that $A_i(x) = 1$ (normality of antecedents),

(ii) $\bigcup_i SuppA_i = \boldsymbol{X}$ (completeness), where $SuppA_i = \{x \in A_i \mid A_i(x) > 0\}$,

(iii) $(A_i = A_j) \Longrightarrow (C_i = C_j)$ (consistency).

Moreover, we usually want the control function Φ to satisfy the following properties:

(iv) $\Phi(A_i) = C_i$, for $i = 1, 2, \ldots, n$ (interaction),

(v) $\Phi(X)$ depends on individual outputs of fired rules only.

The property (iv) says that an antecedent as an input of a fuzzy controller should produce the corresponding consequent as an output. The property (v) means that outputs of non fired rules have no influence on the overall output.

The compositional rule of inference says that the output Y can be obtained by the composition of the input X and fuzzy relation $R \in F(\boldsymbol{X} \times \boldsymbol{Y})$, i.e.,

$$R : \boldsymbol{X} \times \boldsymbol{Y} \longrightarrow [0, 1]$$

and

$$Y = \Phi(X) = X \circ R$$

such that

$$Y(y) = \sup_x \left(T(R(x, y), X(x))\right)$$

for all $y \in \boldsymbol{Y}$, where T is a continuous t-norm. In the case of a residuum-based fuzzy controller, the fuzzy relation R can be obtained from the rule base using t-norm and resituated implication [7, 8]. For Mamdani-Assilian controller we have

$$R(x, y) = \max_i \left(T(A_i(x), C_i(y))\right)$$

and

$$Y(y) = \sup_x \left(T\left(\max_i \left(T(A_i(x), C_i(y))\right), X(x)\right)\right)$$

or

$$Y(y) = \max_i \left(T(r_i(X, A_i), C_i(y))\right)$$

for all $y \in Y$, where the values

$$r_i(X, A_i) = \sup_x \left(T(X(x), A_i(x))\right),$$

i =1, 2,…,n, are the degrees of overlapping of the input X and the antecedents A_i. These values are also called the firing values of the rules. In this case we can divide the process of obtaining of a fuzzy output Y for a given fuzzy input X into 3 steps:

Step 1. Obtain firing values $r_1, r_2, \ldots, r_n$ of individual rules for a given input X.

Step 2. Derive individual outputs ($i = 1,2,\ldots,n$)

$$Y_i(y) = T(r_i, C_i(y))$$

for all $y \in \boldsymbol{Y}$.

3. step: Obtain the global output for all $y \in \boldsymbol{Y}$ by

$$Y(y) = \max_i (Y_i(y))$$

In the next section we shall try to introduce a generalized fuzzy controller with fuzzy output and fuzzy input; GFC for short. We expect that Mamdani-Assilian controllers and some kinds of fuzzy controllers with generalized Boolean implications should be particular examples of it. It is known that some inference models do not admit a description in the described 3 steps; those ones are not included in our generalization. We shall not analyze convenience and utility of introduced examples of our model in practice [3, 4, 6]; they are tightly connected with fuzzification and defuzzification processes. We will preferably investigate some reasonable additional properties of GFC from mathematical point a view.

To generalize Step 2 we will use a relevancy transformation operator (RET operator, for short) which is a generalization both fuzzy conjunction and fuzzy implication. It was originally introduced by R. Yager [11, 12, 13].

Definition 1 Let $e \in [0, 1]$ be a given element. A binary operation $Ret : [0, 1]^2 \longrightarrow [0, 1]$ is called the relevancy transformation (RET) operator with respect to the element e if it satisfies the following axioms:

(r1) $Ret(1,a) = a$ and $Ret(0, a) = e$ for all $a \in [0, 1]$,

(r2) $Ret(r, a_1) \le Ret(r, a_2)$ for all $a_1, a_2 \in [0, 1]$ such that $a_1 < a_2$ and $r \in [0, 1]$,

(r3) if $a \ge e$, then $Ret(r_1, a) \le Ret(r_2, a)$ for all $r_1, r_2 \in [0, 1]$ such that $r_1 < r_2$,

(r4) if $a \le e$, then $Ret(r_1, a) \ge Ret(r_2, a)$ for all $r_1, r_2 \in [0, 1]$ such that $r_1 < r_2$.

Note that the axiom (r1) means that the effective rule output is equal to the consequent of a rule if the rule is fully fired, and the rule output does not distinguish between elements of output space if the rule is not fired. The element e is related to the following aggregation; it should be its neutral element. The last two conditions are called a consistency in the antecedent argument. In [12] one can find more about philosophical background of mentioned properties.

Example 1 Let $e \in [0, 1]$ be a given element. Define

$$h(r, a) = r\, a + (1-r)\, e$$

$r, a \in [0, 1]$. Then h *is a* RET operator with respect to the element e. It is called Product RET operator

Note that a RET operator is a fuzzy conjunction for $e = 0$ and a fuzzy implication for $e = 1$.

Similarly, we will use an aggregation function as a generalization of the maximum operator (Step 3).

Definition 2 A function

$$Agg : \bigcup_{n \in N} [0,1]^n \to [0,1]$$

is called the aggregation function if

(a) $Agg(x) = x$.

(aa) $Agg(0, \ldots, 0) = 0$, $Agg(1, \ldots, 1) = 1$.

(aaa) $Agg(x_1, x_2, \ldots, x_n) \le Agg(y_1, y_2, \ldots, y_n)$ if $x_1 \le y_1, x_2 \le y_2, \ldots, x_n \le y_n$.

The element $e \in [0, 1]$ is called the neutral element of an aggregation function Agg if

$Agg(x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_n) = Agg(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ for all $x_1, \ldots, x_n \in [0, 1]$.

2 Generalised fuzzy controller

Now consider a generalized fuzzy controller (GFC) with fuzzy inputs $X \in F(\boldsymbol{X})$ and fuzzy outputs $Y \in F(\boldsymbol{Y})$:

$$\Theta = (RB, Fir, Ret, Agg)$$

where

- $RB = (A_i, C_i)_{i=1}^{n}$ is a rule base consisting of a set of simple if-then rules with normal antecedents $A_i \in F(\boldsymbol{X})$ and consequents $C_i \in F(\boldsymbol{Y})$, $i = 1,2, \ldots, n$. We suppose that

$$\bigcup_i suppA_i = \boldsymbol{X},$$

where $Supp\, A_i = \{x \in \boldsymbol{X} \mid A_i(x) > 0\}$.

- $Fir : F(\boldsymbol{X}) \times F(\boldsymbol{X}) \longrightarrow [0,1]$ assigns a firing value r_i to any fuzzy input $X \in F(\boldsymbol{X})$ and the antecedent of the i-th rule ($i = 1,2, \ldots, n$), i. e.,

$$r_i = Fir(X, A_i).$$

Assume that the operation Fir is nondecreasing in both fuzzy arguments, i.e.,

$$Fir(U_1, V_1) \le Fir(U_2, V_2)$$

if $U_1 \le U_2$, $V_1 \le V_2$, $U_1, U_2, V_1, V_2 \in F(\boldsymbol{X})$.

For simplicity, we denote:

$$Fir(X, RB) = (r_1, r_2, \ldots, r_n)$$

Suppose that for all normal $X \in F(\boldsymbol{X})$ it holds:

$$Fir(X, X) = 1,$$

$$Fir(X, RB) = (r_1, r_2, \ldots, r_n) \ne (0,0,\ldots,0)$$

and, moreover, the n-tuple $Fir(A_i, RB)$ contains at most once 1; $i = 1, 2, \ldots, n$. It ensures that for any normal input at least one rule is fired and at most one rule is fully fired for each rule antecedent as an input. It implies that $Fir(A_i, A_j) < 1$ for $i \ne j$.

- Ret: $[0, 1]^2 \longrightarrow [0, 1]$ is a relevancy transformation operator with respect to a given special element $e \in [0, 1]$. The individual output

$$Y_i = Ret\,(r_i, C_i)$$

of each rule is given pointwisely by

$$Y_i(y) = Ret\,(r_i, C_i(y))$$

for all $y \in \boldsymbol{Y}$ and $i = 1,2, \ldots, n$.

- *Agg*: $[0,1]^n \longrightarrow [0, 1]$ is a restriction of an aggregation function with neutral element *e.*

Emphasize that the element *e* is the special element for the operator *Ret* and the neutral element for the aggregation function *Agg* simultaneously.

The overall output (for a given normal fuzzy input *X*) is given by a control function

$$Y = \Phi_{\Theta}(X)$$

such that for all $y \in \boldsymbol{Y}$

$$Y(y) = \underset{i}{Agg}\left(Ret\left(r_i, C_i(y)\right)\right)$$

Note that if the operator *Ret* is a t-norm ($e = 0$), the aggregation function *Agg* is the maximum operator *max* and

$$r_i = Fir(X, A_i) = \sup_x T\left(X(x), A_i(x)\right)$$

for $i = 1, 2, \ldots, n$, we obtain Mamdani – Assilian controller. Recall that the element $e = 0$ is a neutral element of the maximum operator.

3 Properties of GFC

To have a reasonable performance of a generalized fuzzy controller Θ we study conditions under which this controller fulfils some additional requirements.

The information boundedness principle (IBP) for individual rule says that the knowledge obtained as a result of inference process should not have more information than that contained in the consequent of the rule. It means that GFG fulfils IBP if the operator *Ret* has the property:

$$Infor(Ret(r_i, C_i)) \leq Infor(C_i),$$

for all $i \in \{1, 2, \ldots, n\}$, where r_i is a firing value of the *i*-th rule with the consequent C_i and *Infor* is some measure of information. The stronger form of IBP requires

$$Infor(Ret(r_1, C_i)) \leq Infor(Ret(r_2, C_i))$$

whenever $r_1 < r_2$, $r_1, r_2 \in [0, 1]$.

Many information measures have been proposed attached to fuzzy sets (fuzziness, measure of imprecision, Shannon's entropy etc.[7]). If we consider the finite universe $\boldsymbol{X}$, $\boldsymbol{Y}$ with cardinalities

$$|\boldsymbol{X}| = |\boldsymbol{Y}| = m$$

then each fuzzy set *A* is represented by *m*-tuple $(a_1, a_2, \ldots, a_m) \in [0, 1]^m$. Then we can define a special type of information measure [11, 12] which is called a linear specificity.

Definition 3 Let $1 \geq w_2 \geq w_3 \geq \ldots \geq w_m \geq 0$ be given constants for which $w_2 + w_3 + \ldots + w_m = 1$. If $1 \geq a_1 \geq a_2 \geq \ldots \geq a_m \geq 0$, then a mapping *LSp*: $[0, 1]^m \longrightarrow [0, 1]$ given by

$$LSp\left(a_1, a_2, \ldots, a_m\right) = a_1 - \sum_{i=2}^{m} w_i a_i$$

is called the linear specificity measure.

Note that a linear specificity reaches value 1 for singletons only. Now, we are looking for RET operators which fulfill IBP with respect to a linear specificity as an information measure. The next proposition gives a full solution of this problem [11]. Note that a mapping f: $[0, 1]^2 \longrightarrow [0, 1]$ is 2-increasing if

$$f(x_1, y_1) + f(x_2, y_2) \geq f(x_1, y_2) - f(x_2, y_1)$$

whenever $x_1 \leq x_2$ and $y_1 \leq y_2$.

Proposition 1 A relevancy transformation operator *Ret*: $[0, 1]^2 \longrightarrow [0, 1]$ satisfies the stronger form of IBP with respect to a linear specificity if and only if the operator *Ret* is 2-increasing.

Recall that Proposition 1 holds for a wider class of specifities; for the class of shift invariant specifities (see [11]). The notion of 2-increasing binary operation is known in the theory of copulas. A binary operation *Cop*: $[0, 1]^2 \longrightarrow [0, 1]$ is called the copula if 1 is its neutral element, 0 is zero element, and *Cop* is 2-inreasing. If we use some t-norm as a RET operator and we want it to satisfy IBP, we must choose a t-norm which is also a copula.

The requirement of interaction in GFC says that an antecedent as an input of GFC produces the corresponding consequent as an output. This requirement is generally violated in many fuzzy controllers. For Mamdani-Assilian controller it leads to a complicated solution of a system of fuzzy relational equations [1, 10]. To simplify this problem, a fuzzy controller with conditionally fired rules (CFR) was introduced by Navara, Moser and Petrik [8, 9]. It can be obtained by some modification of Mamdani-Assilian controller and it satisfies the interaction property. The CFR controller allows satisfying the interaction property without a change of the rule base but it modifies firing values of individual rules.

We try to find conditions under which our GFC fulfills interaction property:

$$\Phi_\Theta(A_i) = C_i, \quad i = 1,2,...,n.$$

The next proposition gives some sufficient conditions. Remind that

$$c = max\{Fir(A_j, A_k), k \neq j, \ k, j \in \{1,2,...,n\}\} < 1$$

Proposition 2 Consider a generalized fuzzy controller $\Theta = (RB, Fir, Ret, Agg)$ such that for all $r, b \in [0, 1]$, $0 \le r \le c$ holds:

$$Ret(r, b) = e,$$

where *Ret* is a relevancy transformation operator with respect to the given element *e* and

$$c = max\{Fir(A_j, A_k), k \neq j, \ k, j \in \{1,2,...,n\}\}$$

Then

$$\Phi_\Theta(A_i) = C_i.$$

for all $i \in \{1, 2, \ldots, n\}$.

Proof. Because of $Fir(A_j, A_k) < 1$ for $i \neq j$ we have $c < 1$. Put $X = A_i$, $i \in \{1, 2, \ldots, n\}$ as an input of GFC,. Then

$$Fir(X, RB) = (r_1, r_2, ..., r_{i-1}, 1, r_{i+1}, ..., r_n)$$

and

$$r_1, r_2, ..., r_{i-1}, r_{i+1}, ..., r_n \le c < 1$$

Thus, for all $y \in \boldsymbol{Y}$ we have

$$Ret(r_i, C_i(y)) = Ret(1, C_i(y)) = C_i(y)$$

and

$$Ret(r_j, C_j(y)) = e$$

if $i \neq j$. The neutrality of the element *e* for aggregation function *Agg* implies that for all $y \in \boldsymbol{Y}$ we have

$$Y(y) = \underset{i}{Agg}(Ret(r_i, C_i(y))) =$$
$$Agg(e,..,e, C_i(y), e,..., e) = C_i(y)$$

which proves our claim.

Proposition 2 offers a way how to construct GFC with interaction property. For given rule base *RB* and given mapping *Fir* we find out a constant

$$c = max\{Fir(A_j, A_k), k \neq j, k, j \in \{1,2,...,n\}\}.$$

We expect that $c < 1$. If not, we must change the rule base or the operation *Fir*. Let $c < 1$. Then we choose a RET operator with respect to a given element *e*, satisfying

$$Ret(r, b) = e$$

for all $r, b \in [0, 1]$, $0 \le r \le c$ and some appropriate aggregation function with *e* as a neutral element *e*; e. g., an uninorm [13] with neutral element *e*.

The obtained GFC has property (v), as well.

The next example gives a possibility of the construction of a RET operator with respect to a special element $e \in [0, 1]$ having the required property.

Example 2 Let $c, e \in [0, 1]$ be given elements. Put $h : [0,1]^n \longrightarrow [0, 1]$ by

$$h(r,b) = \begin{cases} e & \text{if } r \le c \\ b + (b-e)(r-1)/(1-c) & elsewhere \end{cases}$$

Then *h* is a RET operator with respect to the element *e* and fulfills

$$h(r, b) = e$$

for all $r, b \in [0, 1]$, $0 \le r \le c$

Moreover *h* is 2-increasing and so it fulfills IBP.

Conclusion

We have introduced a generalized fuzzy controller which is a generalization of Mamdani-Assilian controllers and some kinds of fuzzy controllers with generalized Boolean implications. We have investigated some of its properties. Finally, some techniques of possible compositions of fuzzy controllers were suggested.

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