Unipolar parametric evaluation of aggregation functions

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Abstract

The aim of this contribution is to present the concept of unipolar parametric evaluation of aggregation functions, and unify the ideas of global and local unipolar parametric evaluations. We show that several well–known parameters introduced in special classes of aggregation functions are included in this framework, and we also propose some new parameters for aggregation functions.

Keywords: Aggregation function, Unipolar measure of similarity, Global parametric evaluation, Local parametric evaluation

1 Introduction

Aggregation functions used for aggregating input data depend on the solved problem and the expectations of evaluators. In order to choose an appropriate aggregation function, we should be able to measure the degree of the required properties of aggregation functions (or their defect), that is, to assign to aggregation functions some values - *parameters* - characterizing the properties of aggregation functions. The process of assigning parameters to aggregation functions we call *parametric evaluation of aggregation functions.*

Although in special classes of aggregation functions, for example, in the class of OWA operators, root–power operators, triangular norms, copulas, etc., certain parameters expressing the degree of an investigated property were already introduced, a systematic approach to the parametric evaluation of aggregation functions is missing.

If a class $\mathcal{A}$ of aggregation functions contains an extremal element E (with respect to the ordering of aggregation functions) possessing a property $\mathcal{P}$, then this extremal element can be chosen as the prototype, and all other members of the class can be compared to it. A *unipolar measure of similarity* is a function $\mu : \mathcal{A} \to [0,1]$, such that $\mu(E) = 1$ (0 is attained by the other extremal element of $\mathcal{A}$ if such an element exists) and with the property
$|E - A| \leq |E - B| \Rightarrow \mu(A) \geq \mu(B)$
for all $A, B \in \mathcal{A}$. The function μ measures the degree of the property $\mathcal{P}$ for all members of the class and expresses similarity between aggregation functions of the considered class and the prototype with respect to $\mathcal{P}$. An example of a unipolar measure of similarity is, e.g., the measure of the degree of orness [16] with prototype Max, or, in the dual case, the measure of the degree of andness with prototype Min.

In classes of aggregation functions with the greatest and smallest elements $\overline{A}$ and $\underline{A}$, respectively, which possess a property $\mathcal{P}$ in two dual forms (e.g., increasing and decreasing functional dependence) and where exists an element O representing total absence of that property (e.g., independence), we can define a function measuring the difference of aggregation functions from the element O - a central element of the class. The range of the

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function is the interval $[-1,1]$; for the central element O its value is zero, the values -1 and 1 are usually attained for the extremal elements of the class only. The functions of this type are called *bipolar measures of dissimilarity*. A non–trivial example of a bipolar measure of dissimilarity (with respect to the product copula Π as the central element) is, e.g., the Spearman rho, a well–known measure of association introduced in statistics [13]. Another bipolar parametric measure of dissimilarity (again with central element Π) in the class of copulas is Blomqvist's beta [13]. A more detailed discussion of bipolar parametric evaluation can be found in [10].

2 Preliminaries

As mentioned above, in special classes of aggregation functions certain parameters expressing the degree of some property $\mathcal{P}$ have already been introduced. For example, for OWA operators Yager [16] defined the measure of the degree of orness/andness, expressing the possibility of an OWA operator to stand as an operator for disjunction/conjunction. In [17] Yager et al. studied possible generalizations of "and" operator for conjunction in fuzzy logics and discussed a special parametric evaluation of triangular norms [7].

Another distinguished class of aggregation functions is the class of root–power operators $(A_p)_{p\in[-\infty,\infty]}$, see, e.g., [6]. For a fixed $n \in \mathbb{N}$, and $p \in]-\infty, 0[\cup]0,\infty[$, the n–ary root–power operator $A_p : [0,1]^n \to [0,1]$ is defined by

$$A_p(x_1,\ldots,x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^p\right)^{\frac{1}{p}},$$

and the limit operators are: A_0 - the geometric mean G, $A_{-\infty} = Min$ and $A_\infty = Max$. Evidently, A_1 is the arithmetic mean M.

For measuring the degree of disjunctive/conjunctive behavior of root–power operators Dujmovic [3] proposed the concept of mean local orness/andness. E.g., the mean local orness of A_p was defined by

$$\int\limits_{[0,1]^n} \frac{A_p(\mathbf{x}) - Min(\mathbf{x})}{Max(\mathbf{x}) - Min(\mathbf{x})}\, d\mathbf{x}. \qquad (1)$$

This value was later again studied as the orness average value in the class of averaging (mean) operators by Fernández Salido and Murakami in [14], see also [5] and [12].

In [4] Dujmovic characterized root–power operators by the *mean value*, defined for an n–ary operator A_p by

$$m(A_p) = \int\limits_{[0,1]^n} A_p(x_1,\ldots,x_n)\, dx_1 \ldots dx_n. \qquad (2)$$

The family $(A_p)_{p\in[-\infty,\infty]}$ is increasing and continuous with respect to the parameter p. It holds $m(A_{-\infty}) = \frac{1}{n+1}$, $m(A_\infty) = \frac{n}{n+1}$, and for any fixed $n \in \mathbb{N}$ and $\alpha \in \left[\frac{1}{n+1}, \frac{n}{n+1}\right]$, there is a unique parameter $p \in [-\infty,\infty]$ such that $m(A_p) = \alpha$. Based on the mean value, Dujmovic [4] introduced the global orness/andness, initially called the disjunction/conjunction degree. For example, the global orness of A_p was defined by

$$\begin{aligned}\omega_g(A_p) &= \frac{m(A_p) - m(Min)}{m(Max) - m(Min)} \\ &= \frac{(n+1)m(A_p) - 1}{n-1}. \qquad (3)\end{aligned}$$

3 Unipolar parametric evaluation of aggregation functions

The formula (2) can be considered for any measurable n–ary aggregation function A. Recall that an n–ary aggregation function function $A : [0,1]^n \to [0,1]$ satisfies the properties

(A1) $A(0,\ldots,0) = 0, \qquad A(1,\ldots,1) = 1;$

(A2) $A(x_1,\ldots,x_n) \le A(y_1,\ldots,y_n)$ for all $(x_1,\ldots,x_n)$, $(y_1,\ldots,y_n) \in [0,1]^n$ such that $x_i \le y_i$, $i \in \{1,\ldots,n\}$.

We define the *mean value* $m(A)$ of any measurable n–ary aggregation function A by

$$m(A) = \int\limits_{[0,1]^n} A(x_1,\ldots,x_n)\, dx_1 \ldots dx_n. \qquad (4)$$

For the weakest and strongest aggregation functions A_w and A_s, respectively, which are defined by

$$A_w(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = (1, \ldots, 1), \\ 0 & \text{otherwise,} \end{cases}$$

$$A_s(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} = (0, \ldots, 0), \\ 1 & \text{otherwise,} \end{cases}$$

we have $m(A_w) = 0$ and $m(A_s) = 1$. Obviously, $m(A) = 0$ if, and only if, $A = A_w$ a.e., and $m(A) = 1$ if, and only if, $A = A_s$ a.e.. For each weighted arithmetic mean $M_{\mathbf{w}}$ it holds $m(M_{\mathbf{w}}) = \frac{1}{2}$, further, $m(Min) = \frac{1}{n+1}$, $m(Max) = \frac{n}{n+1}$, the mean value for the drastic product T_D (the weakest t–norm) is $m(T_D) = 0$, and for the drastic sum S_D (the strongest t–conorm) $m(S_D) = 1$. Moreover, for the product t–norm T_P and the Łukasiewicz t–norm T_L it holds $m(T_P) = \frac{1}{2^n}$, $m(T_L) = \frac{1}{(n+1)!}$.

Note that we will respect the notations of functions usual at special classes of aggregation functions and moreover, we will use the same notation for all n–ary forms of aggregation functions.

Formula (3) can also be generalized. In each subclass $\mathcal{A}$ of n–ary aggregation functions with the smallest and greatest elements $\underline{A}$ and $\overline{A}$, respectively, one can introduce the *normalized mean value* by

$$\widetilde{m}(A) = \frac{m(A) - m(\underline{A})}{m(\overline{A}) - m(\underline{A})}, \tag{5}$$

or by

$$\widetilde{m}^*(A) = \frac{m(\overline{A}) - m(A)}{m(\overline{A}) - m(\underline{A})}. \tag{6}$$

Note that if in $\mathcal{A}$ there is only one extremal element then the previous formulae can be modified by using inf or sup of the set $\{m(B) \mid B \in \mathcal{A}\}$ instead of $m(\underline{A})$ and $m(\overline{A})$, respectively.

It holds $\widetilde{m}(\overline{A}) = 1$, $\widetilde{m}(\underline{A}) = 0$, and conversely for $\widetilde{m}^*$. The functions $\widetilde{m}, \widetilde{m}^* : \mathcal{A} \to [0,1]$ defined by (5) and (6) are unipolar measures of similarity, mentioned in Introduction, with prototypes $\overline{A}$ and $\underline{A}$, respectively. Evidently, they are complementary, $\widetilde{m}(A) + \widetilde{m}^*(A) = 1$. If A^d is a standard dual aggregation function to $A \in \mathcal{A}$, i.e., $A^d(x_1, \ldots, x_n) = 1 - A(1 - x_1, \ldots, 1 - x_n)$ for all $(x_1, \ldots, x_n) \in [0,1]^n$, then $m(A^d) = 1 - m(A)$, and if $\mathcal{A}$ is closed under duality, then it holds

$$\widetilde{m}(A^d) = 1 - \widetilde{m}(A) = \widetilde{m}^*(A).$$

Clearly, if $A^d = A$, that is, if A is a self–dual function (symmetric sum), then necessarily $m(A) = \widetilde{m}(A) = \widetilde{m}^*(A) = 0.5$.

The normalized mean value of aggregation functions given by (5) in the classes of averaging, conjunctive and disjunctive aggregation functions are in detail discussed in [10]. Let us briefly recall several facts.

■ In the class $\mathcal{A}_{av}$ of *averaging aggregation functions*, i.e., aggregation functions characterized by the property $Min \le A \le Max$, formula (5) leads to the *global disjunction measure* which, in what follows, will be denoted by GDM, i.e., for each $A \in \mathcal{A}_{av}$,

$$GDM(A) = \frac{(n+1) \int\limits_{[0,1]^n} A(\mathbf{x})\, d\mathbf{x} - 1}{n-1}. \tag{7}$$

The parameter $GDM(A)$ expresses the degree of similarity between A and Max, the basic operator for disjunction. The function GDM is a unipolar measure of similarity in the class $\mathcal{A}_{av}$ with prototype Max.
Note that Dujmovic's concept of global orness/andness has been extended to the Choquet integrals (forming a subclass of $\mathcal{A}_{av}$) by Marichal in [11] and in the class $\mathcal{A}_{av}$ has been studied by Salido and Murakami in [14], and by Dujmovic in [5]. As shown in [14], Yager's OWA orness coincides with (7).

Note that the complementary function $\widetilde{m}^*_{av}$ in the class $\mathcal{A}_{av}$ is a unipolar measure of similarity with prototype Min. It will be called the *global conjunction measure.*

■ In the classes of conjunctive and disjunctive aggregation functions, in notation $\mathcal{A}_c$ and $\mathcal{A}_d$, respectively, this approach to evaluation leads to some new global parameters, e.g., to global idempotency measure.
Recall that an aggregation function A is idempotent if for all $x \in [0,1]$ it holds $A(x, \ldots, x) = x$. For aggregation functions

the idempotency of A is equivalent to the property $Min \leq A \leq Max$, which means that all averaging aggregation functions are idempotent. In the class $\mathcal{A}_c$ of all *conjunctive aggregation functions*, i.e., aggregation functions bounded from above by Min, the only idempotent function is Min. Similarly, in the class $\mathcal{A}_d$ of all *disjunctive aggregation functions*, i.e., aggregation functions bounded from below by Max, the only idempotent function is just Max. To obtain the degree of idempotency, one can compare conjunctive (disjunctive) aggregation functions to Min (Max).
In the class $\mathcal{A}_c$ the smallest element is the function A_w, the greatest Min. Since $m(Min) = \frac{1}{n+1}$ and $m(A_w) = 0$, the normalized mean value of a conjunctive aggregation function A is given by

$$\widetilde{m}_c(A) = \frac{m(A) - m(A_w)}{m(Min) - m(A_w)} = (n+1)\, m(A). \tag{8}$$

The number $\widetilde{m}_c(A)$ expresses the degree of similarity between a conjunctive aggregation function A and Min, and can be interpreted as the degree of the idempotency of A. It holds $\widetilde{m}_c(A) = 1$ if, and only if, $A = Min$, i.e., a conjunctive aggregation function A is idempotent if, and only if, the value $\widetilde{m}_c(A) = 1$. Therefore we define the *global idempotency measure of a conjunctive aggregation function* A, with notation $GIM_c(A)$ instead of $\widetilde{m}_c(A)$, by:

$$GIM_c(A) = (n+1) \int\limits_{[0,1]^n} A(\mathbf{x})\, d\mathbf{x}. \tag{9}$$

In fuzzy logics, and consequently, in fuzzy set theory, important conjunctive aggregation functions are t–norms. In the class $\mathcal{T}$ of all (measurable) t–norms (with extremal elements T_M and T_D) formula (5) also leads to the global idempotency measure given by (9). For t-norms this parameter has already been introduced and studied in [8], compare also [15, 18].

Example 1 **(i)** Let $n = 2$. It can be shown that the global idempotency measure of the product t–norm T_P is $GIM_c(T_P) = 0.75$, for the Łukasiewicz t–norm T_L we obtain $GIM_c(T_L) = 0.5$ and for the nilpotent minimum T^{nM} we have $GIM_c(T^{nM}) = 0.75$.

(ii) Consider the Sugeno-Weber family of t–norms $\left(T_\lambda^{SW}\right)_{\lambda \in [-1,\infty]}$, where

$$T_\lambda^{SW}(x,y) = \begin{cases} \max\left\{0, \frac{x+y-1+\lambda xy}{1+\lambda}\right\} & \lambda \in]-1,\infty[, \\ T_D & \lambda = -1, \\ T_P & \lambda = \infty. \end{cases}$$

For $\lambda \in]-1,0[\cup]0,\infty[$ we obtain

$$GIM_c(T_\lambda^{SW}) = \frac{3}{4} - \frac{3}{4\lambda} - \frac{3}{2\lambda^2} + \frac{3\lambda+3}{\lambda^2} \log(1+\lambda).$$

and

$$GIM_c(T_{-1}^{SW}) = 0\,, \quad GIM_c(T_0^{SW}) = 0.5\,,$$
$$GIM_c(T_\infty^{SW}) = 0.75.$$

For ordinal sums of t–norms [7] we have the following result.

Proposition 1 *Let $n = 2$ and let $T = (\langle a_k, b_k, T_k\rangle)_{k\in\mathcal{K}}$ be an ordinal sum of t–norms. Then*

$$GIM_c(T) = 1 - \sum_{k\in\mathcal{K}} (b_k - a_k)^3 (1 - GIM_c(T_k))\,.$$

Since idempotency is in fact the property of A on the diagonal of the unit cube $[0,1]^n$, it has sense to define the *diagonal idempotency measure of a conjunctive aggregation function* $A \in \mathcal{A}_c$ by

$$DIM_c(A) = 2 \int\limits_{[0,1]} \delta_A(x) dx,$$

where $\delta_A(x) = A(x,\ldots,x)$ is the diagonal section of A.
Note that this formula can naturally be derived from (5), where instead of the mean value m over $[0,1]^n$ the mean value over the diagonal is considered. Clearly, $DIM_c(Min) = 1$, $DIM_c(A_w) = 0$. In this sense, the function DIM_c is a unipolar measure of similarity with Min as the prototype.

Example 2 For $n = 2$ the values of the diagonal idempotency measure of t–norms T_D, T_L, T^{nM} and T_P are

$$DIM_c(T_D) = 0\,, \quad DIM_c(T_L) = 0.5,$$

$$DIM_c(T^{nM}) = 0.75, \quad \text{but} \quad DIM_c(T_P) = \frac{2}{3},$$

compare with the previous example.

The *global idempotency measure* $GIM_d(A)$ *of a disjunctive aggregation function* $A \in \mathcal{A}_d$ can be defined directly following (6) or, as the global idempotency measure of the corresponding dual conjunctive aggregation function, $GIM_d(A) = GIM_c(A^d)$.
Finally, let us note that the parameter introduced for t-norms by Yager et al. in [17] (in the normalized form) can also be explained as a unipolar measure of similarity with prototype T_M [10].

4 Global and local unipolar parametric evaluations

A common property of unipolar measures of similarity is that they measure the degree of some property of aggregation functions by comparing with a prototype. From the definition of a unipolar measure of similarity μ the monotonicity of μ can be deduced. Let us consider the case that for all $A, B \in \mathcal{A}$, $A \leq B \;\Rightarrow\; \mu(A) \leq \mu(B)$. Next, a natural required property of μ is the validity of the property
$\mu\left(\sum\limits_i \lambda_i A_i\right) = \sum\limits_i \lambda_i \, \mu(A_i)$
for all convex combinations of aggregation functions. Two solutions with expected properties are the global and local parametric evaluations which for n–ary aggregation functions are given by

$$\mu_G(A) = f\left(\int\limits_{[0,1]^n} A(\mathbf{x}) dP(\mathbf{x})\right), \qquad (10)$$

$$\mu_L(A) = \int\limits_{[0,1]^n} f_{\mathbf{x}}\left(A(\mathbf{x})\right) dP(\mathbf{x}), \qquad (11)$$

respectively, where P is a probability measure on the Borel subsets of $[0,1]^n$ and $f : \mathbb{R} \to \mathbb{R}$ is an increasing function with the property
$f\left(\sum\limits_i \lambda_i \, x_i\right) = \sum\limits_i \lambda_i \, f(x_i)$
for all convex combinations of arguments. By [1] the only functions of this property are those of the form $f(u) = b + c\,u$, where $b \in \mathbb{R}$, $c > 0$. The same holds for functions $f_{\mathbf{x}}$, see below.
■ First, let us consider a global parametric evaluation μ_G,

$$\mu_G(A) = b + c \int\limits_{[0,1]^n} A(\mathbf{x}) dP(\mathbf{x}). \qquad (12)$$

According to the chosen probability measure P on $\mathcal{B}([0,1]^n)$ and the considered class of aggregation functions we obtain various types of measures of similarity. For example:
(i) Let $\mathcal{A} = \mathcal{A}_{av}$ and let P be the probability measure uniformly distributed on $[0,1]^n$. From the requirements $\mu_G(Max) = 1$, $\mu_G(Min) = 0$ and the values

$$\int\limits_{[0,1]^n} Max(\mathbf{x}) d\mathbf{x} = \frac{n}{n+1}, \quad \int\limits_{[0,1]^n} Min(\mathbf{x}) d\mathbf{x} = \frac{1}{n+1},$$

we can compute the constants c and b and conclude that in the class $\mathcal{A}_{av}$ formula (12) is of the form

$$\mu_G(A) = \frac{(n+1) \int\limits_{[0,1]^n} A(\mathbf{x})\, d\mathbf{x} - 1}{n-1},$$

which coincides with the formula for the global disjunction measure GDM in the class $\mathcal{A}_{av}$.
(ii) Let $\mathcal{A} = \mathcal{A}_c$ and let P be the probability measure uniformly distributed over the diagonal of the unit cube $[0,1]^n$. Then

$$\mu_G(A) = b + c \int\limits_{[0,1]} A(x, \ldots, x)\, dx.$$

The constants b and c can be determined from the conditions $\mu_G(Min) = 1$ and $\mu_G(A_w) = 0$. It holds $b = 0$, $c = 2$, i.e.,

$$\mu_G(A) = 2 \int\limits_{[0,1]} A(x, \ldots, x)\, dx, \quad A \in \mathcal{A}_c,$$

which is the formula for the diagonal idempotency measure DIM_c for conjunctive aggregation functions introduced in Section 3.
■ Now, let us consider a local parametric evaluation given by (11), where $f_{\mathbf{x}}(u) = b_{\mathbf{x}} + c_{\mathbf{x}}\, u$ with local constants $b_{\mathbf{x}} \in \mathbb{R}$, $c_{\mathbf{x}} > 0$ corresponding to the points $\mathbf{x} \in [0,1]^n$. In a class

$\mathcal{A}$ of aggregation functions with the greatest element $\overline{A}$ as the prototype and the smallest element $\underline{A}$, a pointwise fitting gives the following equations for computing $b_{\mathbf{x}}$, $c_{\mathbf{x}}$:

$$b_{\mathbf{x}} + c_{\mathbf{x}}\,\underline{A}(\mathbf{x}) = 0, \quad b_{\mathbf{x}} + c_{\mathbf{x}}\,\overline{A}(\mathbf{x}) = 1.$$

After a short computation we obtain

$$\mu_L(A) = \int\limits_{[0,1]^n} \frac{A(\mathbf{x}) - \underline{A}(\mathbf{x})}{\overline{A}(\mathbf{x}) - \underline{A}(\mathbf{x})}\, dP(\mathbf{x}), \qquad (13)$$

where the convention $\frac{0}{0} = 0$ is considered.

(i) If $\mathcal{A} = \mathcal{A}_{av}$ and P is the probability measure uniformly distributed on $\mathcal{B}([0,1]^n)$, the right-hand side of the previous formula is of the form

$$\int\limits_{[0,1]^n} \frac{A(\mathbf{x}) - Min(\mathbf{x})}{Max(\mathbf{x}) - Min(\mathbf{x})}\, d\mathbf{x}, \qquad (14)$$

and defines the *mean local disjunction measure* of A, in notation $LDM(A)$. This value coincides with orness average value studied by Fernández Salido and Murakami in [14].
In [14] the authors proved that for all OWA operators $GDM(A) = LDM(A)$. Marichal [12] extended this result for any discrete Choquet integral. As the following example shows the class of aggregation functions fulfilling this property is certainly larger.

Example 3 Let $e \in]0,1[$. Consider the function $U_e : [0,1]^2 \to [0,1]$ given by

$$U_e(x,y) = \begin{cases} \min\{x,y\} & \text{if } y \le f_e(x) \\ \max\{x,y\} & \text{if } y > f_e(x), \end{cases}$$

where

$$f_e(x) = \begin{cases} 1 - \frac{1-e}{e}x & \text{if } x \in [0,e] \\ \frac{e}{1-e}(1-x) & \text{if } x \in]e,1]\,. \end{cases}$$

The function f_e is piecewise linear, its graph links the points $(0,1)$, (e,e) and $(1,0)$. For each $e \in]0,1[$ the function U_e is a non-trivial conjunctive uninorm. It is only a matter of computation to show that the mean value

$$m(U_e) = \int\limits_{[0,1]^2} U_e(x,y)dxdy = \frac{2-e}{3},$$

and consequently, the global disjunction measure of U_e is $GDM(U_e) = 3\,m(U_e) - 1 = 1 - e$. On the other hand, since for all $(x,y) \in [0,1]^2$, $x \ne y$,

$$\frac{U_e(x,y) - Min(x,y)}{Max(x,y) - Min(x,y)} = \\ = \begin{cases} 0 & \text{if } x \in [0,1],\ y \le f_e(x) \\ 1 & \text{if } x \in [0,1],\ y > f_e(x), \end{cases}$$

the mean local disjunction measure of U_e computed by formula (14) is $LDM(U_e) = 1 - e$, i.e., $GDM(U_e) = LDM(U_e)$.

The question for which types of averaging aggregation functions GDM and LDM coincide is still open.

(ii) If we consider the class $\mathcal{A}_c$ of all conjunctive aggregation functions and the probability uniformly distributed over $[0,1]^n$, the local parametric evaluation (13) gives for $A \in \mathcal{A}_c$ the value

$$\int\limits_{[0,1]^n} \frac{A(\mathbf{x})}{Min(\mathbf{x})} d\mathbf{x}, \qquad (15)$$

which expresses the mean local idempotency measure of a conjunctive aggregation function A. It will be denoted by $LIM_c(A)$, i.e., $LIM_c(A) = \int\limits_{[0,1]^n} \frac{A(\mathbf{x})}{Min(\mathbf{x})} d\mathbf{x}$. The parameter $LIM_c(A)$ coincides with the idempotency average value of a conjunctive aggregation function A introduced by Marichal in [12]. For example, by (15) for the product t–norm T_P (binary form) we have $LIM_c(T_P) = \frac{2}{3}$. As mentioned above, $GIM_c(T_P) = \frac{3}{4}$ and $DIM_c(T_P) = \frac{2}{3}$. In general, for n–ary form of the product t–norm it holds $GIM_c(T_P) = \frac{n+1}{2^n}$, $DIM_c(T_P) = \frac{2}{n+1}$ and by [12], $LIM_c(T_P) = \frac{2^{n-1}}{\binom{2n-1}{n}}$.

5 A mixed approach

The concepts of local and global parametric evaluations can be unified into a more general mixed approach assigning to n–ary aggregation functions parameters defined by

$$\mu(A) = f\left(\int\limits_{[0,1]^n} f_{\mathbf{x}}\left(A(\mathbf{x})\right) dP(\mathbf{x})\right), \qquad (16)$$

with the same general requirements on f, $f_{\mathbf{x}}$ and P as in the previous part. Evidently, if f is the identity function, μ coincides with μ_L and μ_G is a special case of μ for $f_{\mathbf{x}} = id$. Using the linear form of functions f and $f_{\mathbf{x}}$ and the requirements $\mu(\underline{A}) = 0$ and $\mu(\overline{A}) = 1$ we obtain the following formula for $\mu(A)$:

$$\mu(A) = \frac{\int\limits_{[0,1]^n} c_{\mathbf{x}} A(\mathbf{x}) dP(\mathbf{x}) - \int\limits_{[0,1]^n} c_{\mathbf{x}} \underline{A}(\mathbf{x}) dP(\mathbf{x})}{\int\limits_{[0,1]^n} c_{\mathbf{x}} \overline{A}(\mathbf{x}) dP(\mathbf{x}) - \int\limits_{[0,1]^n} c_{\mathbf{x}} \underline{A}(\mathbf{x}) dP(\mathbf{x})}. \tag{17}$$

For example, in the class of averaging aggregation functions the previous formula is of the form
$\mu(A) =$

$$= \frac{\int\limits_{[0,1]^n} c_{\mathbf{x}} A(\mathbf{x}) dP(\mathbf{x}) - \int\limits_{[0,1]^n} c_{\mathbf{x}} Min(\mathbf{x}) dP(\mathbf{x})}{\int\limits_{[0,1]^n} c_{\mathbf{x}} Max(\mathbf{x}) dP(\mathbf{x}) - \int\limits_{[0,1]^n} c_{\mathbf{x}} Min(\mathbf{x}) dP(\mathbf{x})}. \tag{18}$$

Not to prefer any of the coordinates, we will always suppose that $c_{\mathbf{x}}$ is symmetric.

The next parametric class of parametric evaluations is motivated by the moments of random variables in statistics.

Example 4 Using (18), let us introduce in the class of averaging aggregation functions a moment parametric evaluation $\mu^{(p)}$, $p \in]-1, \infty[$, based on the weighting function $c_{\mathbf{x}}^{(p)} = \left(\prod_{i=1}^{n} x_i\right)^p$ and on the uniform probability P on Borel subsets of $[0,1]^n$.
Then, for example, for $n = 3$ we have
$\int\limits_{[0,1]^3} x^p y^p z^p \min\{x,y,z\} dxdydz$
$= 6 \int\limits_0^1 \int\limits_0^x \int\limits_0^y x^p y^p z^{p+1} dxdydz$
$= \frac{6}{(p+2)(2p+3)(3p+4)}$.
Similarly, $\int\limits_{[0,1]^3} x^p y^p z^p \max\{x,y,z\} dxdydz$
$= \frac{6}{(p+1)(2p+2)(3p+4)}$, and
$\int\limits_{[0,1]^3} x^p y^p z^p \mathrm{med}\{x,y,z\} dxdydz$
$= \frac{6}{(p+1)(2p+3)(3p+4)}$.

Substituting these values into (18) we obtain the parametric evaluation of median,

$$\mu^{(p)}(Med) = \frac{2p+2}{3p+4}.$$

Therefore, for ternary OWA operators which are convex combination of Min, Max and Med, whose parameters are $\mu^{(p)}(Min) = 0$, $\mu^{(p)}(Max) = 1$ and $\mu^{(p)}(Med) = \frac{2p+2}{3p+4}$, it holds

$$\mu^{(p)}(OWA) = w_2 \frac{2p+2}{3p+4} + w_3.$$

For the arithmetic mean M it can easily be computed that
$\int\limits_0^1 \int\limits_0^1 \int\limits_0^1 x^p y^p z^p \frac{x+y+z}{3} dxdydz$
$= \frac{1}{(p+1)^2(p+2)}$ and next, by (18),

$$\mu^{(p)}(M) = \frac{5p+6}{9p+12}.$$

Observe that M can be seen as an OWA operator with weighting vector $\mathbf{w} = (1/3, 1/3, 1/3)$, thus

$$\mu^{(p)}(M) = \frac{1}{3}\frac{2p+2}{3p+4} + \frac{1}{3} = \frac{5p+6}{9p+12},$$

which confirms the previous result.
For the geometric mean G it holds

$$\int\limits_0^1 \int\limits_0^1 \int\limits_0^1 x^p y^p z^p \sqrt[3]{x\,y\,z}\, dxdydz = \frac{27}{(3p+4)^3},$$

thus by (18)

$$\mu^{(p)}(G) = \frac{(p+1)^2(15p+22)}{(3p+4)^3}.$$

Note that for $p = 1$ it holds
$\mu^{(1)}(OWA) = \frac{4}{7}w_2 + w_3$, $\quad \mu^{(1)}(M) = \frac{11}{21}$,
$\mu^{(1)}(G) = \frac{148}{343}$,
and the limit values are

$\lim\limits_{p \to -1} \mu^{(p)}(OWA) = w_3$, $\quad \lim\limits_{p \to -1} \mu^{(p)}(M) = \frac{1}{3}$,
$\lim\limits_{p \to -1} \mu^{(p)}(G) = 0$;
and further,
$\lim\limits_{p \to \infty} \mu^{(p)}(OWA) = \frac{2}{3}w_2 + w_3$,
$\lim\limits_{p \to \infty} \mu^{(p)}(M) = \frac{5}{9}$, $\quad \lim\limits_{p \to \infty} \mu^{(p)}(G) = \frac{5}{9}$.

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