Constraints associated with Choquet integrals and other aggregation-free ranking devices

Souhila Kaci
CRIL, Rue de l'Université SP 16
62307 Lens, France
kaci@cril.univ-artois.fr

Henri Prade
IRIT, 118 route de Narbonne
31062 Toulouse Cedex 9 France
prade@irit.fr

Abstract

The paper contrasts Choquet integral-based aggregation method with a rank-ordering approach based on the specification of constraints induced by generic principles (or by examples of ranking of particular vectors of attribute values). The latter approach does not use any aggregation operation for evaluating the vectors. It is based on the minimal specificity principle (which amounts to demote vectors as much as they violate constraints) in order to get a complete preorder on the vectors. This approach is compared on an illustrative example to a Choquet integral-based method. We point out the specificity of constraints expressing relative importance that underly Choquet integral.

1 Introduction

The comparison of items described in terms of attribute values is a problem that takes place in different contexts. It may correspond to a decision problem where one has to choose the best alternative among several potentially possible choices, on the basis of a multiple criteria evaluation. In database querying, one may not only want to retrieve all the tuples that satisfy a collection of specified requirements, but also to rank-order them, partially or completely, in order to provide the user first with the best answers.

In the fuzzy set approach, different aggregation schemes have been advocated including ordered weighted averages [12] and more generally Choquet integrals [5], or Sugeno integrals [1], which can be seen as a qualitative counterpart of Choquet integrals, both of them enabling the possibility to weight the requirements not only individually, but also in subgroups. This enables us to take into account some synergy between the requirements. In a similar spirit Fagin and Wimmers [2] have advocated the use of another general weighted aggregation mode. However, their framework requires a numerical scaling, as for Choquet integrals, while weighted min aggregations, which are particular cases of Sugeno integrals only require an ordinal, linearly ordered evaluation scale.

All these approaches are based on the (often implicit) hypothesis of the commensurateness of the evaluation scales of the different attributes involved in a query. Another approach that does not use any aggregation scheme (even on qualitative scales having a finite number of levels, since the internal operations [9, 3] that can be defined on such scales have a limited discriminating power) has been recently proposed in an artificial intelligence perspective [4] after having been initially suggested in soft constraint satisfaction problems [10].

The problem that we face thus amounts to compare tuples represented by vectors of qualitative criteria evaluations without aggregating them. Apart from the Pareto partial preorder that should constrain any complete pre-

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order between vectors of attribute values, one may have some generic rules that further constrain these complete preorders. For instance, one may state that some criterion is more important than, or equally important as, other criteria (maybe in a limited context). One may also have at our disposal some examples of preferences between fully specified alternatives. The problem is then to complete the Pareto partial preorder in agreement with the constraints in a way that is as little arbitrary as possible.

The paper is organized as follows. Section 2 first gives necessary notation and definitions before it introduces the problem. Section 3 provides a short background on a general family of aggregation functions that can be defined under the form of a Choquet integral, which will be used in the paper as a comparison landmark. Section 4 describes the proposed approach that uses a minimal commitment principle for building a complete preorder in agreement with the constraints. This principle expresses that an alternative is good as much as there is no other alternative that is considered to be better. Section 5 applies this approach to constraints that directly mirror the way the comparative importance of criteria is stated when using a Choquet integral aggregation. This enables a comparison between the two approaches. Lastly we conclude in section 6.

2 Problem description

It is assumed that objects to be rank-ordered are vectors of satisfaction levels belonging to a linearly ordered scale $S = \{s_1, \cdots, s_h\}$ with $s_1 < \cdots < s_h$, each vector component referring to a particular criterion. Thus, it is supposed that there exists a unique scale S on which all the criteria can be estimated (commensurateness hypothesis). Let $\mathcal{U} = S^n$ be the set of all possible vectors $u = a_1 \cdots a_n$, called also vectors of attribute values, such that $a_j \in S$ for all $j = 1, \cdots, n$. Preferences are expressed through comparisons of such vectors $u_i = (a^i_1, \cdots, a^i_n)$ (written $a^i_1 \cdots a^i_n$ for short), with $a^i_j \in S$, under the form of constraints

$$a_1 \cdots a_n \succ a'_1 \cdots a'_n$$

expressing that $u = a_1 \cdots a_n$ is preferred to (or is more satisfactory than) $u' = a'_1 \cdots a'_n$.

$\succeq$ is a preorder on $\mathcal{U}$ if and only if it is reflexive and transitive relation. $u \succ u'$ means that $u \succeq u'$ holds but not $u' \succeq u$. $u \approx u'$ means that both $u \succeq u'$ and $u' \succeq u$ hold, i.e. u and u' are equally preferred, and $u \sim u'$ means that neither $u \succeq u'$ nor $u' \succeq u$ hold, i.e. u and u' are incomparable. $\succeq$ is complete if and only if all pairs of vectors are comparable, i.e. $\forall u, u' \in \mathcal{U}$, either $u \succeq u'$ or $u' \succeq u$ holds. Otherwise $\succeq$ is called partial. A preorder $\succeq$ extends a preorder $\succeq'$ iff $\forall u, u' \in \mathcal{U}$, (if $u \succeq' u'$ then $u \succeq u'$).
The set of best (or undominated) vectors of $\Sigma \subseteq \mathcal{U}$ w.r.t. $\succeq$ is defined by $\max(\Sigma, \succeq) = \{u | u \in \Sigma, \nexists u' \in \Sigma \text{ s.t. } u' \succ u\}$. Similarly the set of worst vectors of Σ w.r.t. $\succeq$ is defined by $\min(\Sigma, \succeq) = \{u | u \in \Sigma, \nexists u' \in \Sigma \text{ s.t. } u \succ u'\}$.

Definition 1 *A* sequence *of sets of vectors of the form* $(E_1, \cdots, E_n)$ *is an* ordered *partition of* $\mathcal{U}$ *iff (i)* $\forall i = 1, \cdots, n$, $E_i \neq \emptyset$, *(ii)* $E_1 \cup \cdots \cup E_n = \mathcal{U}$, *(iii)* $\forall i, j$, $E_i \cap E_j = \emptyset$ *for* $i \neq j$.

An ordered partition of $\mathcal{U}$ is associated with a complete preorder $\succeq$ if and only if ($\forall u, u' \in \mathcal{U}$ with $u \in E_i$ and $u' \in E_j$ we have $i \leq j$ iff $u \succeq u'$). Complete preorders can be compared on the basis of specificity principles [11].

Definition 2 (Specificity-based principles) *Let* $\succeq$ *and* $\succeq'$ *be two complete preorders on* $\mathcal{U}$ *represented by* $(E_1, \cdots, E_n)$ *and* $(E'_1, \cdots, E'_{n'})$ *resp. We say that* $\succeq$ *is less specific than* $\succeq'$*, written as* $\succeq \sqsubseteq \succeq'$*, iff* $\forall u \in \mathcal{U}$*, if* $u \in E_i$ *and* $u \in E'_j$ *then* $i \leq j$. $\succeq$ *belongs to the set of the least (resp. most) specific preorders among a set of preorders* $\mathcal{O}$ *if there is no* $\succeq'$ *in* $\mathcal{O}$ *such that* $\succeq' \sqsubset \succeq$*, i.e.,* $\succeq' \sqsubseteq \succeq$ *holds but* $\succeq \sqsubseteq \succeq'$ *(resp.* $\succeq \sqsubset \succeq'$*) does not.*

Some components may remain unspecified when comparing vectors. They are replaced by a variable x_j if the jth component is free to take any value in the scale. This allows to express generic preferences as for e.g. Pareto ordering, i.e. $\forall x_i \forall x'_i$,

$$x_1 \cdots x_n \succ x'_1 \cdots x'_n \text{ if } \forall i, x_i \geq x'_i \text{ and } \exists k, x_k > x'_k.$$

In any case, we only consider preorders that extend Pareto ordering. Besides other generic constraints of particular interest include those pertaining to the expression of the relative importance of criteria. The greater importance of criterion j w.r.t. criterion k can be expressed under different forms. One way to state it is by exchanging x_j and x_k and writing $x_1 \cdots x_j \cdots x_k \cdots x_n \succ x_1 \cdots x_k \cdots x_j \cdots x_n$ when $x_j > x_k$.
One may think of other ways of expressing that j is more important than k. For instance, one may restrict the above preferences to extreme values of S for the x_i's such that $i \neq j$ and $i \neq k$, since weights of importance in conjunctive aggregation can be obtained in this way for a large family of operators (e.g., [1]). A more drastic way for expressing relative importance would be to use a lexicographic ordering of the vector evaluations based on a linear order of the levels of importance for the criteria. In this case, the problem of ordering the vectors would be immediately solved.
Note that the first above view of relative importance, which is used in the following, is a ceteris paribus preference [7] of subvector (x_j, x_k) w.r.t. (x_k, x_j) for $x_j > x_k$, where the first (resp. second) component refers to criterion j (resp. k), all other vector components being equal.

Another way to relate criteria is to express an equal importance between them. Equal importance can be expressed by stating that any two vectors where x_j and x_k are exchanged, and otherwise identical, have the same levels of satisfaction. Formally, we write:

$$x_1 \cdots x_j \cdots x_k \cdots x_n \approx x_1 \cdots x_k \cdots x_j \cdots x_n.$$

In addition to generic constraints we may also have particular examples of preferences between some specific vectors.

Given a set of constraints (generic constraints and examples) $\mathcal{C}$ of the form $\{u^i \succ u^{i'} | i = 1, \cdots, m\}$, where u^i and $u^{i'}$ are instantiated on S, our aim is to aggregate these constraints and compute a *complete preorder* $\succeq$ over $\mathcal{U}$ that satisfies all constraints of $\mathcal{C}$. Such a preorder should not add any additional constraint. One may wonder why we look for a complete preorder from the set $\mathcal{C}$ of partially specified preferences. This is a debatable question and the answer depends on queries we intend to perform. Firstly, incomparability is not always appropriate. For example if $\mathcal{C}$ describes preferences over cars then we may allow that two cars be incomparable, but if $\mathcal{C}$ describes preferences over students' grades, as it is the case in our running examples later in the paper, then we would like to have a complete preorder over students in case of a competitive examination. Secondly, sometimes we do not need to compare pairs of vectors but to determine the best ones, then those that are immediately less preferred, and so on. Note that some preorders, such as those induced by the minimum aggregation operator, are excluded as soon as Pareto constraints are considered.

It is worth noticing that transitivity is required between vectors only and not between generic constraints. More precisely if it holds that $u \succ u'$ and $u' \succ u''$ w.r.t. some generic constraints then we have necessarily $u \succ u''$. However if we have two generic constraints $X \succ Y$ and $Y \succ Z$, where X, Y and Z are three criteria, representing that X (resp. Y) is more important than Y (resp. Z) then we do not have necessarily $X \succ Z$.

Example 1 *Let X, Y and Z be evaluated on a scale $\{a, b, c\}$ s.t. $a > b > c$. $X \succ Y$ and $Y \succ Z$ are relative importance constraints defined by $xyz \succ yxz$ for $x > y$ and $xyz \succ xzy$ for $y > z$ respectively. Let us now check whether we have $X \succ Z$ i.e. $xyz \succ zyx$ for $x > z$. We have $abc \succ cba$ obtained by transitivity from $abc \succ bac$ (w.r.t. $X \succ Y$), $bac \succ bca$ (w.r.t. $Y \succ Z$) and $bca \succ cba$ (w.r.t. $X \succ Y$). However acb is not preferred to bca since we cannot reach bca from acb by transitivity using the generic constraints.*

Indeed generic constraints require to express explicitly each constraint we would like to have, for e.g. $X \succ Z$ in the above example.

3 Numerical aggregation approach

Aggregation of object attribute values in the presence of interaction between criteria is essential in many decision making problems. For this purpose, several multicriteria aggregation approaches have been proposed in literature [5, 6, 1]. In this section we focus on discrete Choquet integral [5, 6].

3.1 Discrete Choquet integral

Choquet integrals [5, 6] are very popular aggregation operators as they allow to model interactions between criteria and thus to represent preferences that cannot be captured by a simple weighted arithmetic mean. Using a particular measure, they aggregate valued attributes describing vectors into a unique value. A Choquet integral is based on a fuzzy measure defined by

Definition 3 *Let $\mathcal{A}$ be the set of attributes and $I(\mathcal{A})$ be the set of all possible subsets of $\mathcal{A}$. A fuzzy measure is a function μ from $I(\mathcal{A})$ to $[0,1]$ such that: (i) $\forall X, Y \in I(\mathcal{A})$ if $X \subseteq Y$ then $\mu(X) \leq \mu(Y)$, (ii) $\mu(\emptyset) = 0$ and (iii) $\mu(\mathcal{A}) = 1$.*

A discrete Choquet integral w.r.t. a fuzzy measure μ is defined as follows:

Definition 4 *Let μ be a fuzzy measure on $\mathcal{A} = \{a_1, \cdots, a_n\}$. The discrete Choquet integral w.r.t. μ is defined by*
$Ch_\mu(a_1 \cdots a_n) = \sum_{i=1,\cdots,n}(a_{(i)} - a_{(i-1)}) \cdot \mu_{\mathcal{A}_{(i)}}$, *where $a_{(i)}$ indicates that the indices have been permuted so that $0 \leq a_{(1)} \leq \cdots \leq a_{(n)}$, and $\mathcal{A}_{(i)} = \{a_{(i)}, \cdots, a_{(n)}\}$ with $a_{(0)} = 0$.*

Example 2 *(borrowed from [5, 6, 8])*
Let A, B and C be three students evaluated w.r.t. three subjects: mathematics (M), physics (P) and literature (L). Students' grades are summarized in Table 1. Using

student	M	P	L
A	18	16	10
B	10	12	18
C	14	15	15

Table 1: Students' grades.

Choquet integral with a fuzzy measure μ, the global grade for each student is computed as follows:

- *student A: $Ch_\mu(A) = Ch_\mu(18, 16, 10) = 10 \cdot \mu_{MPL} + (16-10) \cdot \mu_{PM} + (18-16) \cdot \mu_M$,*
- *student B: $Ch_\mu(B) = Ch_\mu(10, 12, 18) = 10 \cdot \mu_{MPL} + (12-10) \cdot \mu_{PL} + (18-12) \cdot \mu_L$,*
- *student C: $Ch_\mu(C) = Ch_\mu(14, 15, 15) = 14 \cdot \mu_{MPL} + (15-14) \cdot \mu_{PL}$,*

where μ_X, μ_{XY} and μ_{XYZ} with $X, Y, Z \in \{M, P, L\}$ denote the values of the fuzzy measure μ for the corresponding set of subjects. The school gives the same importance to mathematics and physics and is more scientifically than literary oriented. Moreover the school wants to favor well equilibrated students without weak grades so we should have: C preferred to A and A preferred to B, i.e. $C \succ A \succ B$ [1]. As indicated before, the fuzzy measure μ models interaction between subjects. Since mathematics and physics have the same importance and they are more important than literature we have $\mu_M = \mu_P$, $\mu_M > \mu_L$ and $\mu_P > \mu_L$. Moreover since both mathematics and physics are scientific subjects, and thus considered close, while literature is not then the interaction between mathematics (resp. physics) and literature is higher than the interaction between mathematics and physics. Then $\mu_{ML} = \mu_{PL} > \mu_{PM}$. Therefore we have the following set of constraints on μ: $\mathcal{M} = \{\mu_M = \mu_P, \mu_M > \mu_L, \mu_P > \mu_L, \mu_{ML} = \mu_{PL}, \mu_{ML} > \mu_{PM}, \mu_{PL} > \mu_{PM}\}$.
In addition to this set we consider the constraints $Ch_\mu(C) > Ch_\mu(A)$ and $Ch_\mu(A) > Ch_\mu(B)$ corresponding to the order between the students A, B and C.
Table 2 gives an example of μ given in [8].
Using discrete Choquet integral w.r.t. μ

μ_M	μ_P	μ_L	μ_{PM}	μ_{ML}	μ_{PL}	μ_{MPL}
0.45	0.45	0.3	0.5	0.9	0.9	1

Table 2: Fuzzy measure.

given in Table 2 we get $Ch_\mu(A) = 13.9$, $Ch_\mu(B) = 13.6$ and $Ch_\mu(C) = 14.9$. Thus C is preferred to A and A is preferred to B.

[1]It has been shown in [8] that there is no weighted arithmetic mean that gives this order over A, B and C.

3.2 Discussion of Choquet integral

We have seen that the main component in Choquet integral is the fuzzy measure μ. Also the computation of μ is based on the set $\mathcal{M}$ together with the ordering on A, B and C. This makes Choquet integral very sensitive to the constraints from which it is computed as it is shown in the following example.

Example 3 *Let us consider another student D having 15 in physics, 15 is mathematics and 12 in literature. Using discrete Choquet integral w.r.t. μ given in Table 2 we get $Ch_\mu(D) = 13.5$. Then we have the following ordering $C \succ A \succ B \succ D$. Let us now use another fuzzy measure μ' which is equal to μ except for μ_{PL} and μ_{ML}. Instead we have $\mu'_{PL} = \mu'_{ML} = 0.8$. We can check that μ' satisfies the set of constraints on μ. Using discrete Choquet integral w.r.t. μ' we have $C \succ A \succ D \succ B$. So we still have $C \succ A \succ B$ but the ordering over B and D is reversed.*

Indeed the way μ is computed requires to compute a new fuzzy measure each time an example or a constraint between criteria is added. In section 4 we present a purely qualitative approach to rank-order the vectors on the basis of generic constraints and examples without resorting to a numerical aggregation.

4 A qualitative ranking approach

An elementary preference has generally the following form:

$$u \succ u', \text{ with } u, u' \in \mathcal{U}. \tag{1}$$

For e.g. given three criteria X, Y and Z, a relative importance constraint of X over Y is written as:

$$xyz_0 \succ yxz_0 \text{ for } x > y, \forall z_0.$$

A set of constraints of the form (1) can be written in a compact form as a set of the following constraints:

$$\text{if } u \in \min(\mathcal{U}_1, \succeq) \text{ and } u' \in \max(\mathcal{U}_2, \succeq) \text{ then } u \succ u', \tag{2}$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are subsets of $\mathcal{U}$.
For the sake of readability, we denote constraints of the form (2) as $C(\mathcal{U}_1,\mathcal{U}_2)$.

We may also have equality constraints i.e.

$$u \approx u', \text{ with } u, u' \in \mathcal{U}. \tag{3}$$

For e.g. given criteria X, Y, and Z; X and Y having the same importance is written by

$$xyz_0 \approx yxz_0, \forall z_0.$$

In our framework, constraints are gathered in two sets $\mathcal{C}$ and $\mathcal{EQ}$ where $\mathcal{C} = \{C(\mathcal{U}^i,\mathcal{U}^j)\}$[2] and $\mathcal{EQ} = \{u^k \approx u^l\}$.

A set of constraints $\mathcal{C} \cup \mathcal{EQ}$ generates a partial preorder on the set of object attribute values provided that the set of constraints is consistent. In the rest of this paper we suppose that $\mathcal{C} \cup \mathcal{EQ}$ is consistent.

Example 4 *Let us consider two subjects "mathematics" and "literature" that are evaluated on "a" for good, "b" for medium and "c" for bad with $a > b > c$. Thus a student having "ac" is good in mathematics and bad in literature. Pareto ordering forces to have $xy \succ x'y'$ as soon as $x > x'$ and $y \geq y'$ or $x \geq x'$ and $y > y'$ for x, y, x', y' ranging in $\{a, b, c\}$. Pareto principle generates the following set of constraints $\mathcal{C} = \{C(\{aa\}, \{ab, ba, ca\}), C(\{aa, ab\}, \{ac, bb, bc, cb, cc\}), C(\{ac, ba\}, \{bc, cc\}), C(\{bb\}, \{cb, cc, bc\}), C(\{ba\}, \{ca, bb, cb\}), C(\{bc, cb, ca\}, \{cc\}), C(\{ca\}, \{cb\})\}$. The partial preorder $\succeq_p$ associated to $\mathcal{C}$ is depicted in Figure 1.a.*

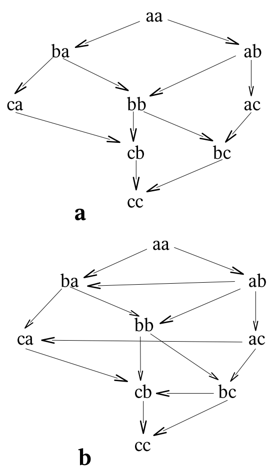

Figure 1: Partial preorders $\succeq_p$ and $\succeq'_p$ associated to $\mathcal{C}$ and $\mathcal{C} \cup \mathcal{C}'$ respectively.

[2]Constraints of the form (1) are a special case of constraints of the form (2).

Since our aim is to associate a complete preorder to $\mathcal{C}$ we need to use a completion principle in order to extend the partial preorder $\succeq_p$ associated to $\mathcal{C}$. We distinguish two completion principles [11]:

- *minimal specificity principle:* it first computes the best/undominated vectors w.r.t. $\succeq_p$. Let E_1 be this set. After that we select the set of immediately preferred vectors, i.e. those that are only dominated by vectors in E_1, and so on. Indeed, the complete preorder that extends $\succeq_p$ following the minimal specificity principle is $\succeq = (E_1, E_2, \cdots, E_n)$.
  In Example 4 we have $E_1 = \{aa\}$, $E_2 = \{ab, ba\}$, $E_3 = \{ac, ca, bb\}$, $E_4 = \{bc, cb\}$ and $E_5 = \{cc\}$. So $\succeq = (\{aa\}, \{ab, ba\}, \{ac, ca, bb\}, \{bc, cb\}, \{cc\})$.
- *maximal specificity principle:* it first computes the least preferred vectors i.e. those that do not dominate any other tuple. Let E'_1 be this set. After that we compute vectors that are immediately more preferred. These vectors are those that only dominate vectors in E'_1. Indeed the complete preorder that extends $\succeq_p$ following the maximal specificity principle is $\succeq' = (E'_m, \cdots, E'_1)$.
  In Example 4 we have $E'_1 = \{cc\}$, $E'_2 = \{bc, cb\}$, $E'_3 = \{ca, ac, bb\}$, $E'_4 = \{ab, ba\}$ and $E'_5 = \{aa\}$. Then $\succeq' = (E'_5, E'_4, E'_3, E'_2, E'_1) = (\{aa\}, \{ab, ba\}, \{ca, ac, bb\}, \{bc, cb\}, \{cc\})$.

The two preorders $\succeq$ and $\succeq'$ obtained in Example 4 following minimal and maximal specificity principles respectively are the same but this is not always the case as it will be shown in the next example which extends Example 4 with relative importance constraints:

Example 5 *(Example 4 continued)*
Suppose that mathematics is more important than literature. This is translated by the following relative importance constraint: $xy \succ yx$ *for* $x > y$. *The instantiation of this constraint provides a new set of constraints:* $\mathcal{C}' = \{ab \succ ba, ac \succ ca, bc \succ cb\}$.
The partial preorder associated to $\mathcal{C} \cup \mathcal{C}'$ *is depicted in Figure 1.b. Following the minimal specificity principle we have* $\succeq = (\{aa\}, \{ab\}, \{ba, ac\}, \{ca, bb\}, \{bc\}, \{cb\}, \{cc\})$. *Now following the maximal specificity principle we have* $\succeq' = (\{aa\}, \{ab\}, \{ba\}, \{ac, bb\}, \{bc, ca\}, \{cb\}, \{cc\})$.

The minimal specificity principle amounts to put a vector all the higher in the ranking as it is dominated by less vectors according to the constraints. The maximal specificity principle would rather amount to put a vector all the lower in the ranking as it is dominated by more vectors according to the constraints. In the following, we will use minimal specificity principle. Algorithm 1 gives a formal approach to compute a complete preorder which extends a partial preorder following this principle.
Let $\mathcal{C} = \{C(\mathcal{U}^i, \mathcal{U}^j)\}$ and $\mathcal{EQ} = \{u^k \approx u^l\}$. ¿From $\mathcal{C}$ we define the following set $\mathcal{L}_\mathcal{C} = \{(\mathcal{U}^i, \mathcal{U}^j) | C(\mathcal{U}^i, \mathcal{U}^j) \in \mathcal{C}\}$.

Algorithm 1: Completion following the minimal specificity principle.

Data: A set of generic constraints and examples $\mathcal{C} \cup \mathcal{EQ}$.
Result: A complete preorder satisfying $\mathcal{C} \cup \mathcal{EQ}$.
begin
  $r = 0$
  while $\mathcal{U} \neq \emptyset$ **do**
    - $r \leftarrow r + 1$, $E_r = \{u | \forall (\mathcal{U}^i, \mathcal{U}^j) \in \mathcal{L}_\mathcal{C}, u \notin \mathcal{U}^j\}$,
    α = true
    while $\alpha = true$ **do**
      α = false
      for $u^k \approx u^l$ *in* $\mathcal{EQ}$ *s.t.* $u^k \notin E_r$ *or* $u^l \notin E_r$ **do** α = true, $E_r = E_r \backslash \{u^k, u^l\}$
    if $E_r = \emptyset$ **then** Stop (inconsistent constraints)
    - $\mathcal{U} = \mathcal{U} \backslash E_r$
    - Replace each $(\mathcal{U}^i, \mathcal{U}^j)$ in $\mathcal{L}_\mathcal{C}$ by $(\mathcal{U}^i \backslash E_r, \mathcal{U}^j)$
    - From $\mathcal{L}_\mathcal{C}$ remove $(\mathcal{U}^i, \mathcal{U}^j)$ with empty $\mathcal{U}^i$
    - From $\mathcal{EQ}$ remove $u^k \approx u^l$ s.t. $u^k \in E_r$.
  return $(E_1, \cdots, E_r)$
end

5 Comparison with Choquet integral

In contrast to Choquet integral which is sensitive to the numerical values of criteria and coefficient of fuzzy measure whose adjustment is not obvious, our approach relies on *qualitative* values of criteria. This qualitative aspect makes that the approach is general, i.e. independent of the values of criteria, which provides more robust results compared with Choquet integral. In fact constraints over coefficients of Choquet integral's fuzzy measure as well as ranking over specific vectors can be encoded in our framework by means of generic constraints and examples respectively. Applying the minimal specificity algorithm computes a complete preorder on $\mathcal{U}$ that satisfies all generic constraints and examples.

Let us consider again Example 2 and show how it can be encoded in our framework. First, we use a qualitative scale $S = \{a, b, c, d, e, f\}$ (with $a > b > c > d > e > f$) to encode students' grades $18, 16, 15, 14, 12$ and 10 respectively given in Table 1. Let x, y, z be students' grades in mathematics, physics and literature respectively so $x, y, z \in \{a, b, c, d, e, f\}$. Then we encode the constraints on μ namely $\mu_M > \mu_L$, $\mu_P > \mu_L$, $\mu_{ML} > \mu_{PM}$, $\mu_{PL} > \mu_{PM}$ and $\mu_P = \mu_M$ by means of generic constraints.

i) *M is more important than L:* At first sight we may encode this constraint by

$$xyz \succ zyx \text{ for } x > z, \forall y. \tag{4}$$

However this encoding, apparently natural, is incorrect since it doesn't recover the ranking on $\mathcal{U}$ induced by Choquet integral. Let us consider the following vectors dfe, efd, ead and dae. Following equation (4) we have $dfe \succ efd$ and $dae \succ ead$. However following Choquet integral we have $Ch_\mu(dfe) = 12.7$, $Ch_\mu(efd) = 12.4$, $Ch_\mu(dae) = 14.8$ and $Ch_\mu(ead) = 15.6$. So we have well dfe preferred to efd but dae is not preferred to ead. This means that constraint (4) is too weak to encode $\mu_M > \mu_L$. The reason is that the constraint $\mu_M > \mu_L$ is more requirying than what it appears. Thus y should be constrained rather than free to take any value in S. Let mpl and $m'p'l'$ two vectors. Note that $Ch_\mu(mpl) > Ch_\mu(m'p'l')$ reduces into $\mu_M > \mu_L$ when $Ch_\mu(mpl) = p + (l - p) \cdot \mu_{ML} + (m - l) \cdot \mu_M > Ch_\mu(m'p'l') = p' + (m' - p') \cdot \mu_{ML} + (l' - m') \cdot \mu_L$. This supposes $p \leq l < m$ and $p' \leq m' < l'$. We put $p = p' = y$, $l = m' = z$, $m = l' = x$. Thus $\mu_M > \mu_L$ is encoded in our framework by

$$xyz \succ zyx \text{ for } x > z \geq y. \tag{5}$$

ii) *P is more important than L:* The same reasoning can be made for $\mu_P > \mu_L$. It is encoded by

$$xyz \succ xzy \text{ for } y > z \geq x. \tag{6}$$

iii) *The interaction between M and L is higher than the interaction between P and M:* The inequality $\mu_{ML} > \mu_{PM}$ is equivalent to the following inequality between the two Choquet integrals $Ch_\mu(mpl) = p + (l - p) \cdot \mu_{ML} + (m - l) \cdot \mu_M > Ch_\mu(m'p'l') = l' + (p' - l') \cdot \mu_{PM} + (m' - p') \cdot \mu_M$. This supposes $p < l \leq m$ and $l' < p' \leq m'$. Letting $p = l' = y$, $l = p' = z$ and $m = m' = x$, then $\mu_{ML} > \mu_{PM}$ is encoded by

$$xyz \succ xzy \text{ for } x \geq z > y. \tag{7}$$

iv) *The interaction between P and L is higher than the interaction between P and M:* Similarly $\mu_{PL} > \mu_{PM}$ is encoded by

$$xyz \succ zyx \text{ for } y \geq z > x. \tag{8}$$

v) *M and P have the same importance:*

$$xyz \approx yxz \text{ for all } x, y, z. \tag{9}$$

vi) *As previously said, we suppose that Pareto oredering holds.* Namely

$$xyz \succ x'y'z' \tag{10}$$

for $x \geq x', y \geq y', z \geq z'$ and $x > x'$ or $y > y'$ or $z > z'$.

vii) Lastly C preferred to A and A preferred to B is encoded by

$$dcc \succ abf \succ fea \tag{11}$$

In sum we have the following set of generic constraints and examples:

$$\mathcal{C} = \begin{cases} xyz \succ zyx & \text{for } x > z \geq y \\ xyz \succ xzy & \text{for } y > z \geq x \\ xyz \succ xzy & \text{for } x \geq z > y \\ xyz \succ zyx & \text{for } y \geq z > x \\ xyz \approx yxz & \text{for all } x, y, z \\ xyz \succ x'y'z' & \text{for } x \geq x', y \geq y', z \geq z' \\ & x > x' \text{ or } y > y' \text{ or } z > z' \\ dcc \succ abf \succ fea & \end{cases}$$

Applying Algorithm 1 on $\mathcal{C}$ returns a complete preorder with 26 strata $\succeq = (E_1, \cdots, E_{26})$ with $E_1 = \{aaa\}$, $E_2 = \{baa, aba\}$, $E_3 = \{caa, aca, aab\}$, $\cdots$, $E_{26} = \{fff\}$. The vectors dcc, abf and fea belong to E_{12}, E_{13} and E_{15} respectively. Using Choquet integral we get 77 different levels: $Ch_\mu(aaa) = 18$, $Ch_\mu(aba) = Ch_\mu(baa) = 17.8$, $Ch_\mu(aca) = Ch_\mu(caa) = Ch_\mu(fea) = 13.6, \cdots, Ch_\mu(fff) = 10$. Thus our approach gives a more compact preorder.

6 Conclusion

A qualitative method has been proposed for building a complete preorder that agrees with a set of constraints in a qualitative way. The approach is fairly general, and agrees with the way humans state their preferences in a granular manner, either in terms of generic rules or by means of examples.

Besides, a comparative discussion on an example suggests that the proposed approach may be more robust, more flexible, and is more transparent to the user (who can control precisely what is expressed by means of the constraints) than an aggregation-based method, which moreover requires the use of a numerical scale in order to have a sufficiently discriminative scale.

The use of Choquet integral is based on a strong hypothesis that the ranking on specific objective attribute values ($dcc \succ abf \succ fea$ in our example) does not contradict generic constraints otherwise there is no fuzzy measure that satisfies a set of contradictory constraints on μ. Our Algo. 1 is also based on this hypothesis. However a ranking on specific objective attribute values may be added for two reasons: (i) it may give an additional preference which is not stated by generic constraints, as it is the case with Choquet integral and well treated by Algo. 1, or (ii) to express an exception of generic constraints. The handling of contradictory examples together with generic constraints has been outlined in [4].

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