Additive generators in interval-valued fuzzy set theory

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Abstract

In this paper we study generators of t-norms on the lattice $\mathcal{L}^I$ of closed subintervals of the unit interval. Starting from a minimal set of axioms that any addition and multiplication operator on $\mathcal{L}^I$ must fulfill, we investigate the properties of the additive and multiplicative generators on $\mathcal{L}^I$ obtained using these arithmetic operators. We also investigate under which conditions the generators on $\mathcal{L}^I$ generate a t-norm.

Keywords: Triangular norm, additive generator, interval-valued fuzzy set.

1 Introduction

Additive generators are very useful in the construction of t-norms: any generator on $([0,1],\leq)$ can be used to generate a t-norm. Generators play also an important role in the representation of continuous Archimedean t-norms on $([0,1],\leq)$. Moreover, some properties of t-norms which have a generator can be related to properties of their generator. See e.g. [13, 14, 15, 17] for more information about generators on the unit interval.

Interval-valued fuzzy set theory [12, 18] is an extension of fuzzy theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. Another extension of fuzzy set theory is intuitionistic fuzzy set theory introduced by Atanassov [1]. In [7] it is shown that intuitionistic fuzzy set theory is equivalent to interval-valued fuzzy set theory and that both are equivalent to L-fuzzy set theory in the sense of Goguen [11] w.r.t. a special lattice $\mathcal{L}^I$. In [3] we introduced additive and multiplicative generators on $\mathcal{L}^I$ based on a special kind of addition introduced in [4]. In [9] another addition was introduced and many more additions can be introduced. Therefore, in this paper we will investigate additive generators on $\mathcal{L}^I$ independently of the addition. For some special additions we will investigate which t-norms can be generated by continuous additive generators which are a natural extension of an additive generator on the unit interval.

2 The lattice $\mathcal{L}^I$

Definition 2.1 *We define* $\mathcal{L}^I = (L^I, \leq_{L^I})$, *where*

$$L^I = \{[x_1,x_2] \mid (x_1,x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\},$$
$$[x_1,x_2] \leq_{L^I} [y_1,y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2),$$
$$\text{for all } [x_1,x_2],[y_1,y_2] \text{ in } L^I.$$

Similarly as Lemma 2.1 in [7] it can be shown that $\mathcal{L}^I$ is a complete lattice.

Definition 2.2 [12, 18] *An interval-valued fuzzy set on* U *is a mapping* $A : U \to L^I$.

Definition 2.3 [1] *An intuitionistic fuzzy set on* U *is a set*

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

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where $\mu_A(u) \in [0,1]$ denotes the membership degree and $\nu_A(u) \in [0,1]$ the non-membership degree of u in A and where for all $u \in U$, $\mu_A(u) + \nu_A(u) \leq 1$.

An intuitionistic fuzzy set A on U can be represented by the $\mathcal{L}^I$-fuzzy set A given by

$$\begin{aligned} A \ : \ U \ &\rightarrow \ L^I \ : \\ u \ &\mapsto \ [\mu_A(u), 1 - \nu_A(u)], \end{aligned}$$

In Figure 1 the set L^I is shown. Note that to each element $x = [x_1, x_2]$ of L^I corresponds a point $(x_1, x_2) \in \mathbb{R}^2$.

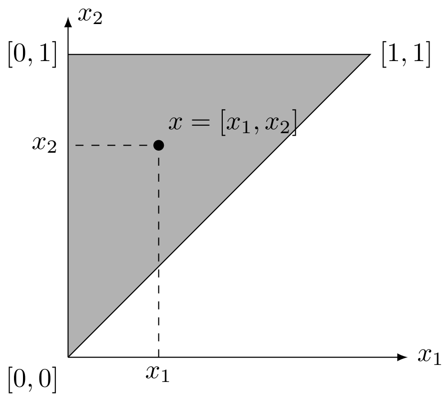

Figure 1: The grey area is L^I.

In the sequel, if $x \in L^I$, then we denote its bounds by x_1 and x_2, i.e. $x = [x_1, x_2]$, or, equivalently, $([x_1, x_2])_1 = x_1$ and $([x_1, x_2])_2 = x_2$. The length $x_2 - x_1$ of the interval $x \in L^I$ is called the degree of uncertainty and is denoted by x_π. The smallest and the largest element of $\mathcal{L}^I$ are given by $0_{\mathcal{L}^I} = [0,0]$ and $1_{\mathcal{L}^I} = [1,1]$. Note that, for x, y in L^I, $x <_{L^I} y$ is equivalent to $x \leq_{L^I} y$ and $x \neq y$, i.e. either $x_1 < y_1$ and $x_2 \leq y_2$, or $x_1 \leq y_1$ and $x_2 < y_2$. We define the relation $\ll_{L^I}$ by $x \ll_{L^I} y \iff x_1 < y_1$ and $x_2 < y_2$, for x, y in L^I. We define for further usage the sets

$$\begin{aligned} D &= \{[x,x] \mid x \in [0,1]\}, \\ \bar{L}^I &= \{[x_1,x_2] \mid (x_1,x_2) \in \mathbb{R}^2 \\ &\qquad \text{and } x_1 \leq x_2\}, \\ \bar{D} &= \{[x,x] \mid x \in \mathbb{R}\}; \\ \bar{L}^I_+ &= \{[x_1,x_2] \mid (x_1,x_2) \in [0,+\infty[^2 \\ &\qquad \text{and } x_1 \leq x_2\}, \\ \bar{D}_+ &= \{[x,x] \mid x \in [0,+\infty[\}, \\ \bar{L}^I_{+,0} &= \{[x_1,x_2] \mid (x_1,x_2) \in]0,+\infty[^2 \\ &\qquad \text{and } x_1 \leq x_2\}, \\ \bar{L}^I_{\infty,+} &= \{[x_1,x_2] \mid (x_1,x_2) \in [0,+\infty]^2 \\ &\qquad \text{and } x_1 \leq x_2\}, \\ \bar{D}_{\infty,+} &= \{[x,x] \mid x \in [0,+\infty]\}. \end{aligned}$$

Note that for any non-empty subset A of L^I it holds that

$$\begin{aligned} \sup A = [&\sup\{x_1 \mid x_1 \in [0,1] \text{ and} \\ &(\exists x_2 \in [x_1,1])([x_1,x_2] \in A)\}, \\ &\sup\{x_2 \mid x_2 \in [0,1] \text{ and} \\ &(\exists x_1 \in [0,x_2])([x_1,x_2] \in A)\}]; \\ \inf A = [&\inf\{x_1 \mid x_1 \in [0,1] \text{ and} \\ &(\exists x_2 \in [x_1,1])([x_1,x_2] \in A)\}, \\ &\inf\{x_2 \mid x_2 \in [0,1] \text{ and} \\ &(\exists x_1 \in [0,x_2])([x_1,x_2] \in A)\}]. \end{aligned}$$

Theorem 2.1 (Characterization of supremum in $\mathcal{L}^I$) [5] *Let A be an arbitrary non-empty subset of L^I and $a \in L^I$. Then $a = \sup A$ if and only if*

$$\begin{aligned} &(\forall x \in A)(x \leq_{L^I} a) \\ &\textit{and } (\forall \varepsilon_1 > 0)(\exists z \in A)(z_1 > a_1 - \varepsilon_1) \\ &\textit{and } (\forall \varepsilon_2 > 0)(\exists z \in A)(z_2 > a_2 - \varepsilon_2). \end{aligned}$$

Definition 2.4 *A t-norm on $\mathcal{L}^I$ is a commutative, associative, increasing mapping $\mathcal{T} : (L^I)^2 \rightarrow L^I$ which satisfies $\mathcal{T}(1_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.*

A t-conorm on $\mathcal{L}^I$ is a commutative, associative, increasing mapping $\mathcal{S} : (L^I)^2 \rightarrow L^I$ which satisfies $\mathcal{S}(0_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.

Example 2.1 [6, 8] We give some special classes of t-norms on $\mathcal{L}^I$. Let T, T_1 and T_2 be t-norms on $([0,1], \leq)$ such that $T_1(x_1, y_1) \leq T_2(x_1, y_1)$ for all x_1, y_1 in $[0,1]$, and let $t \in [0,1]$. Then we have the following classes:

- t-representable t-norms: $\mathcal{T}_{T_1,T_2}(x,y) = [T_1(x_1,y_1), T_2(x_2,y_2)]$, for all x, y in L^I;
- pseudo-t-representable t-norms: $\mathcal{T}_T(x,y) = [T(x_1,y_1),\ \max(T(x_1,y_2),\ T(x_2,y_1))]$, for all x, y in L^I;

- $\mathcal{T}_{T,t}(x,y) = [T(x_1,y_1), \max(T(t,T(x_2,y_2)), T(x_1,y_2), T(x_2,y_1))]$, for all x,y in L^I;
- $\mathcal{T}'_T(x,y) = [\min(T(x_1,y_2), T(x_2,y_1)), T(x_2,y_2)]$, for all x,y in L^I;

If for a mapping f on $[0,1]$ and a mapping F on L^I it holds that $F(D) \subseteq \bar{D}$, and $F([a,a]) = [f(a), f(a)]$, for all $a \in [0,1]$, then we say that F is a natural extension of f to L^I. E.g. $\mathcal{T}_{T,T}$, $\mathcal{T}_T$, $\mathcal{T}_{T,t}$ and $\mathcal{T}'_T$ are all natural extensions of T to L^I.

Example 2.2 Let, for all x,y in $[0,1]$,

$$\begin{aligned} T_W(x,y) &= \max(0, x+y-1), \\ T_P(x,y) &= xy, \\ T_D(x,y) &= \begin{cases} \min(x,y), & \text{if } \max(x,y)=1, \\ 0, & \text{else,} \end{cases} \\ S_W(x,y) &= \min(1, x+y). \end{aligned}$$

Then T_W, T_P and T_D are t-norms, and S_W is a t-conorm on $([0,1], \leq)$. Let now, for all x,y in L^I,

$$\begin{aligned} \mathcal{T}_W(x,y) &= [\max(0, x_1+y_1-1), \\ &\qquad \max(0, x_1+y_2-1, x_2+y_1-1)], \\ \mathcal{T}_P(x,y) &= [x_1y_1, \max(x_1y_2, x_2y_1)], \\ \mathcal{S}_W(x,y) &= [\min(1, x_1+y_2, x_2+y_1), x_2+y_2]. \end{aligned}$$

Then $\mathcal{T}_W = \mathcal{T}_{T_W}$ and $\mathcal{T}_P = \mathcal{T}_{T_P}$ are t-norms, and $\mathcal{S}_W$ is a t-conorm on $\mathcal{L}^I$. Furthermore, $\mathcal{T}_W$, $\mathcal{T}_P$ and $\mathcal{S}_W$ are natural extensions of T_W, T_P and S_W respectively.

3 Arithmetic operators on $\bar{L}^I$

We start from two arithmetic operators $\oplus : (\bar{L}^I)^2 \to \bar{L}^I$ and $\otimes : (\bar{L}^I_+)^2 \to \bar{L}^I$ satisfying the following properties,

(ADD-1) $\oplus$ is commutative,

(ADD-2) $\oplus$ is associative,

(ADD-3) $\oplus$ is increasing,

(ADD-4) $[\alpha,\alpha] \oplus b = [\alpha + b_1, \alpha + b_2]$, for all $\alpha \in [0, +\infty[$ and $b \in \bar{L}^I$,

(MUL-1) $\otimes$ is commutative,

(MUL-2) $\otimes$ is associative,

(MUL-3) $\otimes$ is increasing,

(MUL-4) $[\alpha,\alpha] \otimes b = [\alpha b_1, \alpha b_2]$, for all $\alpha \in [0, +\infty[$ and $b \in \bar{L}^I_+$.

The conditions (ADD-1)–(ADD-3) and (MUL-1)–(MUL-3) are natural conditions for any addition and multiplication operators. The conditions (ADD-4) and (MUL-4) ensure that these operators are natural extensions of the addition and multiplication of real numbers to $\bar{L}^I$.

Note that from (ADD-3) and (ADD-4) it follows that, for all a,b in $\bar{L}^I$, $a \oplus b \geq_{L^I} a$, if $b \geq_{L^I} 0_{\mathcal{L}^I}$. Similarly, we find that $a \otimes b \geq_{L^I} a$, if $b \geq_{L^I} 1_{\mathcal{L}^I}$, for all a,b in $\bar{L}^I_+$.

Define the mapping $\ominus$ by, for all x,y in $\bar{L}^I$,

$$1_{\mathcal{L}^I} \ominus x = [1 - x_2, 1 - x_1], \tag{1}$$

and

$$x \ominus y = 1_{\mathcal{L}^I} \ominus ((1_{\mathcal{L}^I} \ominus x) \oplus y). \tag{2}$$

Define finally the mapping $\oslash$ by, for all x,y in $\bar{L}^I_{+,0}$,

$$1_{\mathcal{L}^I} \oslash x = \left[\frac{1}{x_2}, \frac{1}{x_1}\right], \tag{3}$$

and

$$x \oslash y = 1_{\mathcal{L}^I} \oslash ((1_{\mathcal{L}^I} \oslash x) \otimes y). \tag{4}$$

Clearly, $(\bar{L}^I, \leq_{L^I}, \oplus)$ and $(\bar{L}^I_+, \leq_{L^I}, \otimes)$ are commutative partially ordered monoids (in the sense of Birkhoff [2]) with identity element $0_{\mathcal{L}^I}$ and $1_{\mathcal{L}^I}$ respectively. On the other hand, $(\bar{L}^I, \oplus, 0_{\mathcal{L}^I})$ and $(\bar{L}^I_{+,0}, \otimes, 1_{\mathcal{L}^I})$ are not groups [9].

Example 3.1 We give some examples of arithmetic operators satisfying the conditions (ADD-1)–(ADD-4), (MUL-1)–(MUL-4), (1), (2), (3) and (4).

- In the interval calculus (see e.g. [16]) the following operators are defined: for all x,y in $\bar{L}^I$,

$$\begin{aligned} x \oplus y &= [x_1 + y_1, x_2 + y_2], \\ x \ominus y &= [x_1 - y_2, x_2 - y_1], \end{aligned}$$

$$x \otimes y = [x_1 y_1, x_2 y_2], \quad \text{if } x, y \text{ in } \bar{L}^I_+,$$
$$x \oslash y = \Big[\frac{x_1}{y_2}, \frac{x_2}{y_1}\Big], \quad \text{if } x, y \text{ in } \bar{L}^I_{+,0}.$$

- In [4] the following operators are defined: for all x, y in $\bar{L}^I$,

$$x \oplus_{\mathcal{L}^I} y = [\min(x_1 + y_2, x_2 + y_1), x_2 + y_2],$$
$$x \ominus_{\mathcal{L}^I} y = [x_1 - y_2, \max(x_1 - y_1, x_2 - y_2)],$$
$$x \otimes_{\mathcal{L}^I} y = [x_1 y_1, \max(x_1 y_2, x_2 y_1)], \quad \text{if } x, y \text{ in } \bar{L}^I_+,$$
$$x \oslash_{\mathcal{L}^I} y = \Big[\min\Big(\frac{x_1}{y_1}, \frac{x_2}{y_2}\Big), \frac{x_2}{y_1}\Big], \quad \text{if } x, y \text{ in } \bar{L}^I_{+,0}.$$

Other examples can be found in [9].

4 The arithmetic operators and t-norms and t-conorms on $\mathcal{L}^I$

Theorem 4.1 [9] *The mapping $\mathcal{S}_\oplus : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathcal{S}_\oplus(x, y) = \inf(1_{\mathcal{L}^I}, x \oplus y), \tag{5}$$

is a t-conorm on $\mathcal{L}^I$ if and only if $\oplus$ satisfies the following condition:

$$(\forall(x, y, z) \in (L^I)^3)$$
$$(((\inf(1_{\mathcal{L}^I}, x \oplus y) \oplus z)_1 < 1 \text{ and } (x \oplus y)_2 > 1)$$
$$\Longrightarrow (\inf(1_{\mathcal{L}^I}, x \oplus y) \oplus z)_1 = (x \oplus \inf(1_{\mathcal{L}^I}, y \oplus z))_1). \tag{6}$$

Furthermore $\mathcal{S}_\oplus$ is a natural extension of S_W to L^I.

Theorem 4.2 [9] *The mapping $\mathcal{T}_\oplus : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathcal{T}_\oplus(x, y) = \sup(0_{\mathcal{L}^I}, x \ominus (1_{\mathcal{L}^I} \ominus y)), \tag{7}$$

is a t-norm on $\mathcal{L}^I$ if and only if $\oplus$ satisfies (6). Furthermore, $\mathcal{T}_\oplus$ is a natural extension of T_W to L^I.

Theorem 4.3 [9] *The mapping $\mathcal{T}_\otimes : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathcal{T}_\otimes(x, y) = x \otimes y,$$

is a t-norm on $\mathcal{L}^I$. Furthermore $\mathcal{T}_\otimes$ is a natural extension of T_P to L^I.

Theorem 4.4 [9] *Assume that $\oplus$ satisfies the following condition:*

$$(\forall(x, y) \in \bar{L}^I_+ \times L^I)$$
$$((([x_1, 1] \oplus y)_1 < 1 \text{ and } x_2 \in \,]1, 2])$$
$$\Longrightarrow ([x_1, 1] \oplus y)_1 = (x \oplus y)_1). \tag{8}$$

Then the mappings $\mathcal{T}_\oplus, \mathcal{S}_\oplus : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,

$$\mathcal{T}_\oplus(x, y) = \sup(0_{\mathcal{L}^I}, x \ominus (1_{\mathcal{L}^I} \ominus y)),$$
$$\mathcal{S}_\oplus(x, y) = \inf(1_{\mathcal{L}^I}, x \oplus y),$$

are a t-norm and a t-conorm on $\mathcal{L}^I$ respectively. Furthermore $\mathcal{T}_\oplus$ is a natural extension of T_W to L^I, and $\mathcal{S}_\oplus$ is a natural extension of S_W to L^I.

Example 4.1 [9] We give t-norms $\mathcal{T}_\oplus$, $\mathcal{T}_\otimes$ and t-conorms $\mathcal{S}_\oplus$ on $\mathcal{L}^I$ defined using the examples for $\oplus$ and $\ominus$ given in the previous section.

- Let $\oplus$, $\ominus$ and $\otimes$ be the addition, subtraction and multiplication used in the interval calculus, then, for all x, y in L^I, $\mathcal{T}_\oplus = \mathcal{T}_{T_W, T_W}$, $\mathcal{T}_\otimes = \mathcal{T}_{T_P, T_P}$ and $\mathcal{S}_\oplus = \mathcal{S}_{S_W, S_W}$. Thus the t-norms $\mathcal{T}_\oplus$, $\mathcal{T}_\otimes$ and the t-conorm $\mathcal{S}_\oplus$ obtained using the arithmetic operators from the interval calculus are t-representable.
- Using $\oplus_{\mathcal{L}^I}$, $\ominus_{\mathcal{L}^I}$ and $\otimes_{\mathcal{L}^I}$ we obtain, for all x, y in L^I, $\mathcal{T}_{\oplus_{\mathcal{L}^I}} = \mathcal{T}_W$, $\mathcal{T}_{\otimes_{\mathcal{L}^I}} = \mathcal{T}_P$ and $\mathcal{S}_{\oplus_{\mathcal{L}^I}} = \mathcal{S}_W$. Thus the t-norm $\mathcal{T}_{\oplus_{\mathcal{L}^I}}$ and the t-conorm $\mathcal{S}_{\oplus_{\mathcal{L}^I}}$ are the Łukasiewicz t-norm and t-conorm on $\mathcal{L}^I$, and $\mathcal{T}_\otimes$ is the product t-norm on $\mathcal{L}^I$, which are pseudo-t-representable.

5 Additive generators on $\mathcal{L}^I$

Definition 5.1 [10, 13, 14] *A mapping $f : [0, 1] \to [0, +\infty]$ satisfying the following conditions:*

(ag.1) *f is strictly decreasing;*
(ag.2) *$f(1) = 0$;*
(ag.3) *f is right-continuous in 0;*
(ag.4) *$f(x) + f(y) \in \mathrm{rng}(f) \cup [f(0), +\infty]$, for all $x, y \in [0, 1]$;*

is called an additive generator on $([0,1],\leq)$.

Definition 5.2 [13, 14] *Let* $f : [0,1] \to [0,+\infty]$ *be a strictly decreasing function. The pseudo-inverse* $f^{(-1)} : [0,+\infty] \to [0,1]$ *of* f *is defined by, for all* $y \in [0,+\infty]$, $f^{(-1)}(y) = \sup(\{0\} \cup \{x \mid x \in [0,1]$ *and* $f(x) > y\})$.

We extend these definitions to L^I as follows.

Definition 5.3 *Let* $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ *be a strictly decreasing function. The pseudo-inverse* $\mathfrak{f}^{(-1)} : \bar{L}^I_{\infty,+} \to L^I$ *of* $\mathfrak{f}$ *is defined by, for all* $y \in \bar{L}^I_{\infty,+}$,

$$\mathfrak{f}^{(-1)}(y) = \begin{cases} \sup\{x \mid x \in L^I \text{ and } \mathfrak{f}(x) \gg_{L^I} y\}, \\ \quad \text{if } y \ll_{L^I} \mathfrak{f}(0_{\mathcal{L}^I}); \\ \sup(\{0_{\mathcal{L}^I}\} \cup \{x \mid x \in L^I \text{ and } (\mathfrak{f}(x))_1 > y_1 \\ \quad \text{and } (\mathfrak{f}(x))_2 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_2\}), \\ \quad \text{if } y_2 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_2; \\ \sup(\{0_{\mathcal{L}^I}\} \cup \{x \mid x \in L^I \text{ and } (\mathfrak{f}(x))_2 > y_2 \\ \quad \text{and } (\mathfrak{f}(x))_1 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_1\}), \\ \quad \text{if } y_1 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_1. \end{cases}$$

Note that if $\mathfrak{f}(0_{\mathcal{L}^I}) \in \bar{D}_{\infty,+}$, then, for all $y \in \bar{L}^I_{\infty,+}$, $\mathfrak{f}^{(-1)}(y) = \sup \Phi_y$, where

$$\Phi_y = \begin{cases} \{x \mid x \in L^I \text{ and } \mathfrak{f}(x) \gg_{L^I} y\}, \\ \quad \text{if } y \ll_{L^I} \mathfrak{f}(0_{\mathcal{L}^I}); \\ \{0_{\mathcal{L}^I}\} \cup \{x \mid x \in L^I \text{ and } (\mathfrak{f}(x))_1 > y_1 \\ \quad \text{and } (\mathfrak{f}(x))_2 = (\mathfrak{f}(0_{\mathcal{L}^I}))_2\}, \\ \quad \text{if } y_2 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_2. \end{cases} \tag{9}$$

Definition 5.4 *A mapping* $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ *satisfying the following conditions:*

(AG.1) $\mathfrak{f}$ *is strictly decreasing;*

(AG.2) $\mathfrak{f}(1_{\mathcal{L}^I}) = 0_{\mathcal{L}^I}$;

(AG.3) $\mathfrak{f}$ *is right-continuous in* $0_{\mathcal{L}^I}$;

(AG.4) $\mathfrak{f}(x) \oplus \mathfrak{f}(y) \in \mathcal{R}(\mathfrak{f})$, *for all* x, y *in* L^I, *where* $\mathcal{R}(\mathfrak{f}) = \mathrm{rng}(\mathfrak{f}) \cup \{x \mid x \in \bar{L}^I_{\infty,+}$ *and* $[x_1, (\mathfrak{f}(0_{\mathcal{L}^I}))_2] \in \mathrm{rng}(\mathfrak{f})$ *and* $x_2 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_2\} \cup \{x \mid x \in \bar{L}^I_{\infty,+}$ *and* $[(\mathfrak{f}(0_{\mathcal{L}^I}))_1, x_2] \in \mathrm{rng}(\mathfrak{f})$ *and* $x_1 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_1\} \cup \{x \mid x \in \bar{L}^I_{\infty,+}$ *and* $x \geq_{L^I} \mathfrak{f}(0_{\mathcal{L}^I})\}$;

(AG.5) $\mathfrak{f}^{(-1)}(\mathfrak{f}(x)) = x$, *for all* $x \in L^I$;

is called an additive generator on $\mathcal{L}^I$.

If $\mathfrak{f}(0_{\mathcal{L}^I}) \in \bar{D}_{\infty,+}$, then $\mathcal{R}(\mathfrak{f}) = \mathrm{rng}(\mathfrak{f}) \cup \{x \mid x \in \bar{L}^I_{\infty,+}$ and $[x_1, (\mathfrak{f}(0_{\mathcal{L}^I}))_2] \in \mathrm{rng}(\mathfrak{f})$ and $x_2 \geq (\mathfrak{f}(0_{\mathcal{L}^I}))_2\} \cup \{x \mid x \in \bar{L}^I_{\infty,+}$ and $x \geq_{L^I} \mathfrak{f}(0_{\mathcal{L}^I})\}$.

Theorem 5.1 *Let* f *be an additive generator on* $([0,1],\leq)$ *and let* $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ *the mapping defined by, for all* $x \in L^I$,

$$\mathfrak{f}(x) = [f(x_2), f(x_1)]. \tag{10}$$

Then, for all $y \in \bar{L}^I_{\infty,+}$,

$$\mathfrak{f}^{(-1)}(y) = [f^{(-1)}(y_2), f^{(-1)}(y_1)]. \tag{11}$$

Lemma 5.2 *Let* $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ *be a continuous mapping satisfying (AG.1), (AG.2), (AG.3), (AG.5) and* $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$. *Then there exists a continuous additive generator* f *on* $([0,1],\leq)$ *such that, for all* $x \in L^I$,

$$\mathfrak{f}(x) = [f(x_2), f(x_1)].$$

Lemma 5.3 *Let* $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ *be a continuous mapping satisfying (AG.1), (AG.2), (AG.3), (AG.5) and* $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$. *Then* $\mathcal{R}(\mathfrak{f}) = \bar{L}^I_{\infty,+}$ *and* $\mathfrak{f}$ *satisfies (AG.4).*

Theorem 5.4 *A mapping* $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ *is a continuous additive generator on* $\mathcal{L}^I$ *for which* $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$ *if and only if there exists a continuous additive generator* f *on* $([0,1],\leq)$ *such that, for all* $x \in L^I$,

$$\mathfrak{f}(x) = [f(x_2), f(x_1)]. \tag{12}$$

Theorem 5.4 shows that no matter which operator $\oplus$ satisfying (ADD-1)–(ADD-4) is used in (AG.4), a continuous additive generator $\mathfrak{f}$ on $\mathcal{L}^I$ for which $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$ can be represented using an additive generator on $([0,1], \leq)$.

Theorem 5.5 *Let* f *be an additive generator on* $([0,1],\leq)$. *Then the mapping* $\mathfrak{f} : L^I \to \bar{L}^I_{\infty,+}$ *defined by, for all* $x \in L^I$,

$$\mathfrak{f}(x) = [f(x_2), f(x_1)],$$

is an additive generator on $\mathcal{L}^I$ *associated to* $\oplus$ *if and only if, for all* x, y *in* L^I,

$$\mathfrak{f}(x) \oplus \mathfrak{f}(y) \in (\mathrm{rng}(f) \cup [f(0), +\infty])^2.$$

6 Additive generators and t-norms on $\mathcal{L}^I$

Lemma 6.1 *Let $\mathfrak{f}$ be an additive generator on $\mathcal{L}^I$ associated to $\oplus$. Then the mapping $\mathcal{T}_\mathfrak{f} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathcal{T}_\mathfrak{f}(x,y) = \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus \mathfrak{f}(y)),$$

is commutative, increasing and $\mathcal{T}_\mathfrak{f}(1_{\mathcal{L}^I}, x) = x$, for all $x \in L^I$.

Theorem 6.2 *Let $\mathfrak{f}$ be a continuous additive generator on $\mathcal{L}^I$ associated to $\oplus$ for which $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$. The mapping $\mathcal{T}_\mathfrak{f} : (L^I)^2 \to L^I$ defined by, for all x, y in L^I,*

$$\mathcal{T}_\mathfrak{f}(x,y) = \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus \mathfrak{f}(y)),$$

is a t-norm on $\mathcal{L}^I$ if and only if $\oplus$ satisfies the following condition:

$$\begin{aligned}&(\forall(x,y,z) \in A^3)\\ &(((\inf(\alpha, x \oplus y) \oplus z)_1 < \alpha_1 \text{ and } (x \oplus y)_2 > \alpha_1)\\ &\implies (\inf(\alpha, x \oplus y) \oplus z)_1 = (x \oplus \inf(\alpha, y \oplus z))_1),\end{aligned} \tag{13}$$

where $\alpha = \mathfrak{f}(0_{\mathcal{L}^I})$ and $A = \{x \mid x \in \bar{L}^I_{\infty,+}$ and $x \leq_{L^I} \mathfrak{f}(0_{\mathcal{L}^I})\}$.

The condition (13) is very similar to (6). In the following lemma we show that in some cases both conditions are equivalent.

Lemma 6.3 *Let for all $\alpha \in \bar{D}_+ \setminus \{0_{\mathcal{L}^I}\}$,*

$$\begin{aligned}&P_\alpha \iff (\forall(x,y,z) \in A_\alpha^3)\\ &(((\inf(\alpha, x \oplus y) \oplus z)_1 < \alpha_1 \text{ and } (x \oplus y)_2 > \alpha_1)\\ &\implies (\inf(\alpha, x \oplus y) \oplus z)_1 = (x \oplus \inf(\alpha, y \oplus z))_1),\end{aligned} \tag{14}$$

where $A_\alpha = \{x \mid x \in \bar{L}^I_+$ and $x \leq_{L^I} \alpha\}$. Assume that

(ADDMULDISTR) $[c,c] \otimes (x \oplus y) = ([c,c] \otimes x) \oplus ([c,c] \otimes y)$, *for all $c \in \,]0, +\infty[$ and x, y in $\bar{L}^I_+$.*

Then, for any α, β in $\bar{D}_+ \setminus \{0_{\mathcal{L}^I}\}$, $P_\alpha \iff P_\beta$.

Lemma 6.3 shows that if (ADDMULDISTR) holds, then (13) and (6) are equivalent to each other.

The following theorem shows that there is a close relationship between t-norms generated by a continuous additive generator and $\mathcal{S}_\oplus$.

Theorem 6.4 *Let $\mathfrak{f}$ be a continuous additive generator on $\mathcal{L}^I$ associated to $\oplus$ for which $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$ and $\mathfrak{f}(0_{\mathcal{L}^I}) \in \bar{L}^I_+$. Define the mappings $\mathcal{T}_\mathfrak{f}, S_\oplus : (L^I)^2 \to L^I$ by, for all x, y in L^I,*

$$\begin{aligned}\mathcal{T}_\mathfrak{f}(x,y) &= \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus \mathfrak{f}(y)),\\ \mathcal{S}_\oplus(x,y) &= \inf(1_{\mathcal{L}^I}, x \oplus y).\end{aligned}$$

If (ADDMULDISTR) *holds, then $\mathcal{T}_\mathfrak{f}$ is a t-norm on $\mathcal{L}^I$ if and only if $\mathcal{S}_\oplus$ is a t-conorm on $\mathcal{L}^I$.*

Taking into consideration the above mentioned similarity between the conditions (6) and (13), we consider a condition which is similar to (8) and prove that it is a sufficient condition for $\oplus$ so that a (not necessarily continuous) additive generator associated to $\oplus$ generates a t-norm. First we give a lemma.

Lemma 6.5 *Let $\mathfrak{f}$ be an additive generator on $\mathcal{L}^I$ associated to $\oplus$. Assume that $\oplus$ satisfies the following conditions:*

$$\begin{aligned}&(\forall(x,y) \in \bar{L}^I_+ \times A)\\ &((([x_1, \alpha_2] \oplus y)_1 < \alpha_1 \text{ and } x_2 \in \,]\alpha_2, 2\alpha_2])\\ &\quad\implies ([x_1, \alpha_2] \oplus y)_1 = (x \oplus y)_1),\end{aligned} \tag{15}$$

and

$$\begin{aligned}&(\forall(x,y) \in \bar{L}^I_+ \times A)\\ &((([\alpha_1, x_2] \oplus y)_2 < \alpha_2 \text{ and } x_1 \in \,]\alpha_1, 2\alpha_1])\\ &\quad\implies ([\alpha_1, x_2] \oplus y)_2 = (x \oplus y)_2),\end{aligned} \tag{16}$$

where $\alpha = \mathfrak{f}(0_{\mathcal{L}^I})$ and $A = \{x \mid x \in \bar{L}^I_{\infty,+}$ and $x \leq_{L^I} \mathfrak{f}(0_{\mathcal{L}^I})\}$. Then, for all $x \in L^I$ and $y \in \mathcal{R}(\mathfrak{f})$ such that $y \leq_{L^I} \mathfrak{f}(0_{\mathcal{L}^I}) \oplus \mathfrak{f}(0_{\mathcal{L}^I})$,

$$\mathfrak{f}(x) \oplus \mathfrak{f}(\mathfrak{f}^{(-1)}(y)) \in \mathcal{R}(\mathfrak{f})$$

and

$$\mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus \mathfrak{f}(\mathfrak{f}^{(-1)}(y))) = \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus y). \tag{17}$$

Using Lemma 6.5, the following theorem can be shown.

Theorem 6.6 *Let $\mathfrak{f}$ be an additive generator on $\mathcal{L}^I$ associated to $\oplus$. If $\oplus$ satisfies (15) and*

(16), *then the mapping* $\mathcal{T}_{\mathfrak{f}} : (L^I)^2 \to L^I$ *defined by, for all* x, y *in* L^I,

$$\mathcal{T}_{\mathfrak{f}}(x, y) = \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus \mathfrak{f}(y)),$$

is a t-norm on $\mathcal{L}^I$.

Theorem 5.4 and Theorem 5.1 show that no matter which operator $\oplus$ is used in (AG.4), a continuous additive generator $\mathfrak{f}$ on $\mathcal{L}^I$ satisfying $\mathfrak{f}(D) \subseteq \bar{D}_{\infty,+}$ is representable and has a representable pseudo-inverse. Therefore it depends on the operator $\oplus$ which classes of t-norms on $\mathcal{L}^I$ can have continuous additive generators that extend additive generators on $([0, 1], \leq)$.

We show that in some cases the conditions (15) and (16) are equivalent with (8). Note first that if $\alpha \in \bar{D}_+ \setminus \{0_{\mathcal{L}^I}\}$, then, for any $x \in \bar{L}_{\infty,+}$ and $y \in A$ (where A is defined as in Lemma 6.5), $([\alpha_1, x_2] \oplus y)_2 \geq ([\alpha_1, x_2] \oplus y)_1 \geq \alpha_1 = \alpha_2$, so (16) holds.

Lemma 6.7 *Let for all* $\alpha \in \bar{D}_+ \setminus \{0_{\mathcal{L}^I}\}$,

$$\begin{aligned} &Q_\alpha \iff (\forall (x, y) \in \bar{L}^I_+ \times A_\alpha) \\ &(((([x_1, \alpha_1] \oplus y)_1 < \alpha_1 \text{ and } x_2 \in]\alpha_1, 2\alpha_1]) \\ &\quad \implies ([x_1, \alpha_1] \oplus y)_1 = (x \oplus y)_1), \end{aligned} \tag{18}$$

where $A_\alpha = \{x \mid x \in \bar{L}^I_+$ *and* $x \leq_{L^I} \alpha\}$. *If* (ADDMULDISTR) *holds, then, for any* α, β *in* $\bar{D}_+ \setminus \{0_{\mathcal{L}^I}\}$, $Q_\alpha \iff Q_\beta$.

The following theorem shows that if (ADDMULDISTR) holds, then it is sufficient to prove the property Q_α mentioned in Lemma 6.7 for just one $\alpha \in \bar{D}_+ \setminus \{0_{\mathcal{L}^I}\}$ in order to obtain that the mappings $\mathcal{T}_\oplus$, $\mathcal{S}_\oplus$ and $\mathcal{T}_{\mathfrak{f}}$ are t-(co)norms on $\mathcal{L}^I$.

Corollary 6.8 *Assume that* (ADDMULDISTR) *holds, and that for an arbitrary* $\alpha \in \bar{D}_+ \setminus \{0_{\mathcal{L}^I}\}$, $\oplus$ *satisfies* Q_α, *where* $A_\alpha = \{x \mid x \in \bar{L}^I_+$ *and* $x \leq_{L^I} \alpha\}$. *Then the following properties hold.*

- *The mappings* $\mathcal{T}_\oplus, \mathcal{S}_\oplus : (L^I)^2 \to L^I$ *defined by, for all* x, y *in* L^I,

  $$\mathcal{T}_\oplus(x, y) = \sup(0_{\mathcal{L}^I}, x \ominus (1_{\mathcal{L}^I} \ominus y)),$$
  $$\mathcal{S}_\oplus(x, y) = \inf(1_{\mathcal{L}^I}, x \oplus y),$$

  are a t-norm and a t-conorm on $\mathcal{L}^I$ *respectively. Furthermore,* $\mathcal{T}_\oplus$ *and* $\mathcal{S}_\oplus$ *are a natural extension of* T_W *and* S_W *to* L^I.
- *For any additive generator* $\mathfrak{f}$ *on* $\mathcal{L}^I$ *associated to* $\oplus$ *for which* $\mathfrak{f}(0_{\mathcal{L}^I}) \in \bar{D}_{\infty,+}$, *the mapping* $\mathcal{T}_{\mathfrak{f}} : (L^I)^2 \to L^I$ *defined by, for all* x, y *in* L^I,

  $$\mathcal{T}_{\mathfrak{f}}(x, y) = \mathfrak{f}^{(-1)}(\mathfrak{f}(x) \oplus \mathfrak{f}(y)),$$

  is a t-norm on $\mathcal{L}^I$.

7 Examples of additive generators of specific t-norms on $\mathcal{L}^I$

7.1 The t-norm $\mathcal{T}_\oplus$

The mapping $f_W : [0, 1] \to [0, +\infty]$ defined by, for all $x \in [0, 1]$, $f_W(x) = 1 - x$, is an additive generator of T_W (see [13, 14]). Define the mapping $\mathfrak{f}_W : L^I \to \bar{L}^I_{\infty,+}$ by $\mathfrak{f}_W(x) = 1_{\mathcal{L}^I} \ominus x = [f_W(x_2), f_W(x_1)]$, for all $x \in L^I$. Then $\mathfrak{f}_W$ is an additive generator on $\mathcal{L}^I$ of $\mathcal{T}_\oplus$.

7.2 The t-norm $\mathcal{T}_\otimes$

The mapping $f_P : [0, 1] \to [0, +\infty]$ defined by, for all $x \in [0, 1]$, $f_P(x) = -\ln(x)$, is an additive generator of T_P (see [13, 14]). Define the mapping $\mathfrak{f}_P : L^I \to \bar{L}^I_{\infty,+}$ by $\mathfrak{f}_P(x) = 0_{\mathcal{L}^I} \ominus \mathfrak{ln}(x) = [f_P(x_2), f_P(x_1)]$, for all $x \in L^I$. Then $\mathfrak{f}_P$ is an additive generator on $\mathcal{L}^I$ of $\mathcal{T}_\otimes$ if and only if for all a, b in $\bar{L}^I_{\infty,+}$,

$$\mathfrak{exp}(0_{\mathcal{L}^I} \ominus a) \otimes \mathfrak{exp}(0_{\mathcal{L}^I} \ominus b) = \mathfrak{exp}(0_{\mathcal{L}^I} \ominus (a \oplus b)).$$

8 Conclusion

In this paper we investigated additive generators on $\mathcal{L}^I$ based on any arithmetic operators satisfying the axioms proposed in [9]. We showed that independently of the choice of the addition, any continuous additive generator which is a natural extension of an additive generator on the unit interval, can be represented by this generator in a componentwise way. Conversely, we gave a necessary and sufficient condition such that any mapping which is defined componentwisely using an additive generator on the unit interval, is an additive generator on $\mathcal{L}^I$. We gave a necessary and sufficient condition such that an additive generator on $\mathcal{L}^I$ generates a t-norm on $\mathcal{L}^I$. When a weakened form of the distributivity of $\oplus$ and $\otimes$ is imposed, the fact that an addition operator $\oplus$ generates a t-conorm on $\mathcal{L}^I$ which extends the Łukasiewicz t-conorm is equivalent to the fact that an additive generator based on $\oplus$ generates a t-norm on $\mathcal{L}^I$. Finally, we extended some additive generators of well-known t-norms on the unit interval to additive generators on $\mathcal{L}^I$.

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