Additive generation of some classes of finitely-valued t–conorms

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Abstract

This paper deals with the problem of the additive generation of triangular conorms defined on a finite totally ordered set. We obtain a generalization of the known result about the existence of an additive generator for any divisible t-conorm by considering ordinal sums of t-conorms belonging to a Łukasiewicz-like class, and after defining a new method to construct t-conorms by a nesting procedure, we study the existence of additive generators for t-conorms obtained from nesting appropriate t-conorms in the basic maximum, drastic and Łukasiewicz t-conorms.

Keywords: additive generator, finitely-valued triangular conorm, ordinal sum, nesting of t-conorms.

1 Introduction

An old problem is whether there exist constructions involving only a one-place real function and the usual addition (or multiplication) which introduce two-place real functions having interesting algebraic properties, in particular, the associativity. More details on this topic can be found in [3], in particular, we point out the following facts:
1) a continuous t-conorm is Archimedean if and only if it has a continuous additive generator;
2) there exist additive generators for the drastic t-conorm (and for other non-continuous t-conorms), but none for the maximum t-conorm.

Fuzzy logic is one of the tools for management of uncertainty; it usually works with a continuous scale, the real interval $[0, 1]$, and the logical connectives are modeled by triangular norms (conjunction) and triangular conorms (disjunction). However, practical applications of fuzzy logic are limited to a finite number of truth values. Thus, technical implementations allow us to work only with a finite (though very large) number of values. On the other hand, when representing vagueness it is usually meaningless to distinguish a high number of truth values; only a small number suffices. In this paper, we deal with triangular conorms defined on a finite ordinal scale.

In full analogy to the representation theorem of continuous t-conorms, there is a characterization of divisible (smooth) finitely-valued t-conorms as ordinal sums of Archimedean finitely-valued t-conorms ([5]). However, some other results concerning discrete t-conorms differ substantially from those obtained for t-conorms on $[0, 1]$. Thus, we know that a t-conorm with nontrivial idempotent elements has not an additive generator; this is not true for finitely-valued t-conorms as we can see in this paper.

Sections 2 and 3 contain the main definitions and results which are the basis of the new ones exposed in sections 4 and 5. The proofs in these sections are only shown for the most important non-trivial results.

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2 Preliminaries

Consider $L = \{0, 1, 2, \ldots, n\}$ equipped with the usual ordering. We begin recalling basic definitions, examples and properties of finitely–valued t–conorms. A complete exposition of this topic can be found in [5, 6].

Definition 1 *A triangular conorm (briefly t–conorm) on L is a binary operation $S : L \times L \to L$ such that for all $x, y, z \in L$ the following axioms are satisfied:*

1) $S(x, y) = S(y, x)$ *(commutativity)*

2) $S(S(x, y), z) = S(x, S(y, z))$ *(associativity)*

3) $S(x, y) \le S(x', y')$ *whenever* $x \le x'$, $y \le y'$ *(monotonicity)*

4) $S(x, 0) = x$ *(boundary condition)*

A triangular norm (t–norm for short) is a binary operation $T : L \times L \to L$ which, for all $x, y, z \in L$, satisfies 1)-3) and $T(x, n) = x$.

Example 1 We can consider as basic t–conorms: the drastic

$$S_D(x, y) = \begin{cases} x & \text{if } y = 0 \\ y & \text{if } x = 0 \\ n & \text{otherwise} \end{cases}$$

the maximum $S_M(x, y) = max(x, y)$ and the bounded sum or Łukasiewicz t–conorm $S_Ł(x, y) = min(x + y, n)$.

Remark 1 The only strong negation N on L (N is an involutive and order-reversing function from L into itself) is $N(x) = n - x$. For each t–conorm S on L one obtains a t–norm T on L which is the dual to S in the following sense: $T(x, y) = N(S(N(x), N(y))$ $\forall x, y \in L$. Observe that applying this construction to the t–norm T, we get back the t–conorm S we started with. Due to this duality all the results in this paper can be translated to t-norms.

Proposition 1 *Let S be a t–conorm on L. Then we have:*

1. *$S \le S_D$. Thus, S_D is the largest t–conorm.*
2. *$S(x, y) \ge max(x, y)$ $\forall x, y \in L$. Thus S_M is the smallest t–conorm*
3. *$S(x, n) = n$ $\forall x \in L$ (n is an annihilator)*
4. *$S(x, x) = x$ $\forall x \in L$ if and only if $S = S_M$.*

The property of divisibility can be considered for a t–conorm. This condition is the proper equivalent of the continuity of ordinary t–conorms and it plays a crucial role in our approach.

Definition 2 *A t–conorm S on L is divisible if the following condition holds:*

For all $x, y \in L$ with $x \le y$ there is $z \in L$ such that $y = S(x, z)$

Observe that the divisibility condition is equivalent to the smoothness condition ([1, 2]): $0 \le S(x + 1, y) - S(x, y) \le 1$ for all $x, y \in L$, $x < n$.

Given a t–conorm S on L, we say that $x \in L$ is an idempotent element of S if $S(x, x) = x$. Observe that 0 and n are idempotent elements for any t–conorm. A t–conorm S on L is Archimedean if and only if it has as unique idempotent elements the trivial ones 0 and n.

Proposition 2 *$S_Ł$ is the only divisible Archimedean t–conorm on L.*

Now, we recall a method for constructing a new t–conorm from two given t–conorms

Proposition 3 *Let S_1 be a t–conorm on $L_m = \{0, 1, \ldots, m\}$ and S_2 a t–conorm on $L_n = \{0, 1, \ldots, n\}$, with $m, n \ge 1$. Consider the binary operation S defined on $L_{m+n} = \{0, 1, \ldots, m + n\}$ as follows*

$$S(x, y) = \begin{cases} S_1(x, y) \text{ if } (x, y) \in L_m^2 \\ m + S_2(x - m, y - m) \\ \text{if } (x, y) \in \{m, m + 1, \ldots, m + n\}^2 \\ \max(x, y) \text{ otherwise} \end{cases}$$

Then, S is a t–conorm on L_{m+n} that we call the ordinal sum of S_1 and S_2. We will denote $S = \langle S_1, S_2\rangle$

Next we characterize the class of divisible t-conorms as ordinal sums of Łukasiewicz t-conorms.

Proposition 4 *A t–conorm S on $L = \{0, 1, 2, \ldots, n\}$ is divisible (smooth) if and only if there exists a set $I = \{0 = a_0 < a_1 <$*

$\ldots < a_r < a_{r+1} = n\}$, $0 \leq r \leq n-1$, *of elements of* L *such that*

$$S(x,y) = \begin{cases} \min(a_{i+1}, x+y-a_i) \\ \textit{if } (x,y) \in (a_i, a_{i+1})^2,\ 0 \leq i \leq r \\ \max(x,y) \ \textit{otherwise} \end{cases}$$

Remark 2 In case $r = 0$, that is $I = \{0, n\}$, we obtain $S = S_{Ł}$. In case $r = n-1$, that is $I = L$, we obtain $S = S_M$. On the other hand, we have $S_M \leq S \leq S_{Ł}$ for any divisible t–conorm S.

The correspondance $\Psi : Div(L) \to \mathcal{P}(L - \{0, n\})$ between the set Div(L) of all divisible t–conorms on L and the power set of $\{1, 2, \ldots, n-1\}$ defined by $\Psi(S) = I - \{0, n\}$ (the set of non–trivial idempotent elements of S) is a bijection. Thus, there are exactly 2^{n-1} divisible t–conorms on L.

Example 2 There are 2386 t–conorms on $L = \{0, 1, 2, 3, 4, 5, 6, 7\}$, 471 of them are Archimedean. Between the 1915 non–Archimedean t–conorms there are 1021 ordinal sums. On the other hand, there are 64 t–conorms that are divisible (only one of them is Archimedean and the other 63 are ordinal sums).

3 Additive generators of finitely–valued t–conorms

In this section we consider the pseudoinverse of appropriate monotone functions from $L = \{0, 1, 2, \ldots, n\}$ to $[0, +\infty)$, and we introduce a construction similar to that given in case of ordinary t–conorms.

Definition 3 *An additive generator* $f : L \to [0, +\infty)$ *of a t–conorm* S *on* L *is a strictly increasing function with* $f(0) = 0$ *such that*

$$S(x,y) = f^{(-1)}(f(x) + f(y)) \quad \forall x, y \in L, \quad (1)$$

where $f^{(-1)} : [0, +\infty) \to L$ *is the pseudoinverse of* f*, defined by* $f^{(-1)}(t) = max\{z \in L; f(z) \leq t\}$.

If S is a t–conorm on L of the form (1) for some f, we say that S is additively generated by f. We often indicate $S = \langle(a_0, a_1, \ldots, a_n)\rangle$ where $a_x = f(x)$, $x \in L$. Of course $0 = a_0 < a_1 < \ldots < a_{n-1} < a_n$.

Remark 3 If $S = \langle(a_0, a_1, \ldots, a_n)\rangle$ then defining $a_x * a_y = max\{a_z \ ; \ a_z \leq a_x + a_y\}$ we can write $S(x,y) = f^{(-1)}(f(x) + f(y)) = f^{(-1)}(a_x + a_y) = f^{-1}(a_x * a_y)$ for all $x, y \in L$.

Example 3 The basic t–conorms quoted above have additive generator:
$S_{Ł} = \langle(0, 1, \ldots, n-1, n)\rangle$,
$S_M = \langle(0, 1, 3, 7, 2^{n-1} - 1, 2^n - 1)\rangle$,
$S_D = \langle(0, n-1, n, \ldots, 2n-3, 2n-2)\rangle$.

A known result about characterization of those t–conorms having additive generator is the following. For more details and results about additive generation of binary operations see [4].

Proposition 5 *A t–conorm* S *on* L *has an additive generator if and only if there exists a continuous non–strict Archimedean t–conorm* $\hat{S}$ *on the real interval* $[0, n]$ *such that* $S(x,y) = \lfloor \hat{S}(x,y) \rfloor$ *for all* $x, y \in L$*, where* $\lfloor z \rfloor$ *stands for the floor of* z *(the greatest integer which is less than or equal to* z*).*

On the other hand, we have proved through an exhaustive computation that any t–conorm on $L = \{0, 1, \ldots, n\}$ with $n \leq 7$ can be additively generated.

Next we prove that any t–conorm that is an ordinal sum of additively generated t–conorms is also an additively generated t–conorm.

Proposition 6 *Let* $f_1 = (a_0, a_1, \ldots, a_m)$ *be an additive generator of a t–conorm* S_1 *on* $L_m = \{0, 1, \ldots, m\}$ *and* $f_2 = (b_0, b_1, \ldots, b_m)$ *be an additive generator of a t–conorm* S_2 *on* $L_n = \{0, 1, \ldots, n\}$*. Then* $f = (a_0, a_1, \ldots, a_m, (2a_m + 1)b_1, (2a_m + 1)b_2, \ldots, (2a_m + 1)b_n)$ *is an additive generator of the ordinal sum* $\langle S_1, S_2 \rangle$*.*

According to previous results we can now establish the following result.

Proposition 7 *Any divisible t–conorm on* $L = \{0, 1, \ldots, n\}$ *has an additive generator.*

4 Ordinal sums of a class of Archimedean t–conorms

We have just proved that any divisible t–conorm on $L = \{0, 1, 2, \ldots, n\}$ is additively generated by using the fact that they are ordinal sums of additively generated t–conorms (Łukasiewicz t–conorms). In this section we generalize this result by introducing a family of Łukasiewicz-like t–conorms.

Given $n \geq 2$, we consider the class of binary operations defined on $L = \{0, 1, 2, \ldots, n\}$ as follows:

$$S_k(x, y) = \begin{cases} \max(x, y) & if\ \min(x, y) = 0 \\ \min(n, x + y + k) & otherwise \end{cases}$$

where $k = 0, 1, \ldots, n-2$.

Note that S_0 and S_{n-2} are the Łukasiewicz and the drastic t–conorms repectively.

Proposition 8 *Each S_k is an Archimedean t–conorm on L. All of them are smooth on $L^* = L - \{0\}$.*

Proposition 9 *Let S be a t–conorm on $L = \{0, 1, 2, \ldots, n\}$ that is Archimedean, smooth on $L^* = L - \{0\}$ and strictly increasing out of the n-region. Then there exists $k \in \{0, 1, \ldots, n-2\}$ such that $S = S_k$.*

Proposition 10 *The t–conorms S_k , $k = 0, 1, \ldots, n-2$, have an additive generator.*

Proof: It is sufficient to consider $(0 = a_0, a_1, \ldots, a_n)$ where $a_1, \ldots, a_n$ is an arithmetical progression with common diference d such that $\lfloor \frac{a_1}{d} \rfloor = k + 1$.

Proposition 11 *Any ordinal sum of t–conorms described above has an additive generator.*

Remark 4 If we denote by S_k^n the t–conorm on $L = \{0, 1, 2, \ldots, n\}$ corresponding to the value k ($k = 0, 1, \ldots, n-2$) then, fixed $i_0 = 0 < i_1 < \ldots < i_r < i_{r+1} = n$, we can consider t–conorms on $L = \{0, 1, 2, \ldots, n\}$ which are ordinal sums $S = \langle S_{k_1}^{n_1}, S_{k_2}^{n_2}, \ldots, S_{k_{r+1}}^{n_{r+1}} \rangle$ where $n_j = i_j - i_{j-1} \geq 2$, $j = 1, \ldots, r+1$, and $k_j = 0, 1, \ldots, n_j - 2$.

Thus, we can construct as many t–conorms as $N = \prod_{j \in J}(n_j - 1)$ where $J = \{j; n_j \geq 2\}$. Only one of them is smooth. All of these t–conorms have an additive generator.

Example: $S = \langle S_0^3, S_1^5 \rangle$ is the t–conorm on $L = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ obtained from S_0^3 and S_1^5. See figure below

S	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	3	3	4	5	6	7	8
2	2	3	3	3	4	5	6	7	8
3	3	3	3	3	4	5	6	7	8
4	4	4	4	4	6	7	8	8	8
5	5	5	5	5	7	8	8	8	8
6	6	6	6	6	8	8	8	8	8
7	7	7	7	7	8	8	8	8	8
8	8	8	8	8	8	8	8	8	8

S is additively generated by $(0, 1, 2, 3, 8, 12, 16, 20, 24)$.

In the next section we introduce and study a new method to construct t–conorms.

5 Nesting of t–conorms

Definition 4 *Given a t–conorm S_2 on $L_n = \{0, 1, \ldots, k, k+1, \ldots, n\}$ with $0 < k < n$ and a t–conorm S_1 on $\{0, 1, \ldots, k\}$ we define a binary operation S on L_n as follows:*

$$S(x, y) = \begin{cases} S_1(x, y) & if\ 0 \leq x, y \leq k \\ S_2(x, y) & otherwise \end{cases} \quad (2)$$

We say that S is the nesting of S_1 in S_2 (fixed k) and we denote $S = [S_1, S_2]$.

For any t–conorms S_1 and S_2, $S = [S_1, S_2]$ is commutative, non–decreasing in each place with 0 as neutral element. We are interested in obtaining by this method a new t–conorm.

Proposition 12 *Consider a t–conorm S_2 on $L_n = \{0, 1, \ldots, k, k+1, \ldots, n\}$ with $0 < k < n$ and a t–conorm S_1 on $\{0, 1, \ldots, k\}$. The nesting $S = [S_1, S_2]$ is associative (is a t–conorm) if and only if the following condition holds*

$$S_2(S_1(x, y), z) = S_2(S_2(x, y), z) \quad \forall x, y \leq k, \forall z > k \quad (3)$$

A remarkable particular case is when $S_2(k, x) = \max(k, x)$ for all x. Under this hypothesis the condition (3) is trivially satisfied and $S = [S_1, S_2]$ is just the ordinal sum $\langle S_1, S_2' \rangle$, where S_2' is the t–conorm on $\{0, 1, \dots, n-k\}$ defined by $S_2'(x, y) = S_2(x+k, y+k) - k$.

It is also worth observing that if $S = [S_1, S_2]$ is a t–conorm then it is non–Archimedean (k is a non trivial idempotent of S). Reciprocally, if S is a non–Archimedean t–conorm on $\{0, 1, \dots, n\}$ with k as non–trivial idempotent, then S is the nesting $[S_1, S]$ where S_1 is the restriction of S to $\{0, 1, \dots, k\}$. Thus the class of non–Archimedean t–conorms on $\{0, 1, \dots, n\}$ is equal to the class of nestings on the same domain $\{0, 1, \dots, n\}$ that satisfy condition (3).

Next subsections show how we obtain new t–conorms by nesting in the basic t–conorms: maximum, drastic and Łuckasiewicz.

5.1 Nesting in the maximum t–conorm

First we note that making nestings in the maximum t–conorm we obtain an ordinal sum of t–conorms.

Proposition 13 *Let S_1 be a t–conorm on $\{0, 1, \dots, k\}$, and let S_M the maximum t–conorm on $\{0, 1, \dots, k, \dots, n\}$ ($k < n$), and consider S_M' the maximum t–conorm on $\{0, 1, \dots, n-k\}$. Then $[S_1, S_M]$ is a t–conorm satisfying $[S_1, S_M] = \langle S_1, S_M' \rangle$.*

Thus, nesting a t–conorm S_1 in the maximum t–conorm we obtain a new additively generated t–conorm whenever S_1 also is.

5.2 Nesting in the drastic t–conorm

Using the drastic t–conorm, we can state:

Proposition 14 *The nesting $[S_1, S_D]$ of a t–conorm S_1 in the drastic t–conorm S_D is a t–conorm. Moreover, if $(0 = a_0, a_1, \dots, a_k)$ is an additive generator of S_1 then $(0 = b_0, b_1, \dots, b_n)$ is an additive generator of $[S_1, S_D]$, where $b_i = (n-k)a_i$ $i = 1, \dots, k$, $b_{k+1} = 2b_k + 1$, $b_j = b_{j-1} + 1$ $j = k+2, \dots, n$.*

Observe that, except in trivial cases, $[S_1, S_D]$ is a non–Archimedean t–conorm that is not an ordinal sum. We can also observe that this construction can be iterated. Thus, we can consider $[[S_1, S_D], S_D]$, … , $[\dots[[S_1, S_D], S_D], \dots, S_D]$, obtaining new t–conorms additively generated (if S_1 also is).

5.3 Nesting in the Łuckasiewicz t–conorm

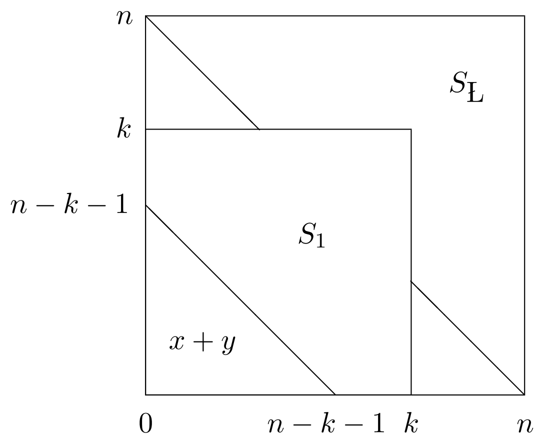
This is a different case from the two previous ones. Now S_1 needs to satisfy some conditions in order to get a new t-conorm.

Proposition 15 *The nesting $[S_1, S_Ł]$ of a t–conorm S_1 in the Łuckasiewicz t–conorm $S_Ł$ is a t–conorm if and only if*

i) $k > \frac{n-2}{2}$

ii) $S_1(x, y) = x + y$ *if* $x + y < n - k - 1$

iii) $S_1(x, y) \geq n - k - 1$ *if* $x + y \geq n - k - 1$.



Proof: First we observe that condition (3) can be written in the form

$$\min(S_1(x, y) + z, n) = \min(x + y + z, n) \quad \forall x, y \leq k \quad \forall z > k \tag{4}$$

Suppose that this condition is satisfied. Taking $x = k$, $y = 1$ and $z = k + 1$ then $\min(S_1(k, 1) + k + 1, n) = \min(2k + 2, n)$. Suppose first $k < \frac{n-2}{2}$, then $S_1(k, 1) + k + 1 = 2k + 2$ which is a contradiction because $S_1(k, 1) = k$. Suppose now $k = \frac{n-2}{2}$, then $S_1(k, 1) + k + 1 \geq n$. Thus $S_1(k, 1) \geq n - k - 1 = k + 1$ which is a contradiction too. Hence $k > \frac{n-2}{2}$

Now we proof *ii)*. Taking $z = k+1$, if $x+y+k+1 < n$ then from (4) we have $S_1(x,y) + k+1 = x+y+k+1$ and so $S_1(x,y) = x+y$ for all x, y such that $x+y < n-k-1$.

Let us prove now *iii)*. Taking $z = k+1$ and x, y such that $x+y+k+1 = n$ condition (4) implies $S_1(x,y)+k+1 \geq n$, hence $S_1(x,y) \geq n-k-1$ for all x, y with $x+y = n-k-1$ and finally from monotonicity *iii)* follows.

Reciprocally suppose that *i)*, *ii)* and *iii)* hold. Consider $x, y \leq k$ and $z > k$. If $x+y+z < n$ then $x+y+k+1 < n$ and $x+y < n-k-1$, hence from *ii)* $S_1(x,y) = x+y$ and condition (4) is satisfied.
In case $x+y+z \geq n$, we need to prove $S_1(x,y)+z \geq n$. If $x+y < n-k-1$ then $S_1(x,y)+z = x+y+z \geq n$, and if $x+y \geq n-k-1$ then $S_1(x,y)+z \geq n-k-1+k+1 = n$ and condition (4) holds.

The next proposition illustrates how we can get an additive generator for $[S_1, S_{\text{Ł}}]$ from one of S_1.

Proposition 16 *Let S_1 be a t–conorm on $L = \{0,1,\ldots,k\}$ having $(0 = a_0, a_1, \ldots, a_k)$ as an additive generator. Then $(0 = b_0, b_1, \ldots, b_k, b_{k+1} \ldots, b_n)$ is an additive generator of $[S_1, S_{\text{Ł}}]$, where $b_i = a_i$ $i = 0, \ldots, k$ and $b_{k+1} = 2a_k + 1$, $b_{k+2} = 2a_k + 2 + a_0$, $b_{k+3} = 2a_k+2+a_1$, ..., $b_n = 2a_k+2+a_{n-k-2}$.*

Proof: Since $b_i = a_i$ $i = 0, \ldots, k$ we only need to show that

$$\begin{array}{c} b_i * b_r = b_{\min(i+r,n)} \\ \forall\ i, r:\ 0 \leq i \leq k < r \leq n \end{array} \qquad (5)$$

where $*$ is the binary operation considered in Remark 3.

Observe also that $b_i * b_j = b_{i+j}$ for all $0 \leq i, j \leq n-k-2$, because S_1 satisfies conditions described above. This means $b_{i+j} \leq b_i + b_j < b_{i+j+1}$ $\forall i, j \leq n-k-2$.

We have to study the following three cases:

1) Let's see that $b_i * b_{k+1} = b_{i+k+1}$ whenever $1 \leq i \leq n-k-2$. We can clearly see that $b_1 * b_{k+1} = b_{k+2}$ because $b_1 + b_{k+1} = b_{k+2}$. And if we suppose $i \geq 2$ then $b_i * b_{k+1} = b_{k+i+1}$ if and only if $b_{k+i+1} \leq b_i * b_{k+1} < b_{k+i+2}$. Since $1 - b_{i-1} \leq b_1 + b_{i-1} \leq b_i$ then $1 + b_{i-1} \leq b_i < b_i + 1$ and the condition (5) holds.

2) In order to see that $b_i * b_{k+p} = b_{\min(k+i+p,n)}$ $p \geq 2$, we only need to observe that $b_{i+p-2} \leq b_i + b_{p-2} < b_{i+p-1}$ (last inequality will be only considered when $i+p-1 \leq n$). From this we obtain $b_{k+i+p} \leq b_i + b_{k+p} < b_{i+k+p+1}$ and condition (5) is satisfied.

3) Finally, we have to see $b_{n-k-1} * b_{k+p} = b_n$ $\forall p \geq 1$. This is true because $b_{n-k-1} + b_{k+p} \geq b_{n-k-1} + b_{k+1} = b_{n-k-1} + 2b_k + 1 \geq b_{n-k-2} + 2b_k + 2 = b_n$ and condition (5) holds.

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