On the distributivity of fuzzy implications over representable uninorms

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Abstract

Recently, many works have appeared dealing with the distributivity of fuzzy implications over t-norms, t-conorms and uninorms (see [2, 3, 4, 5, 12, 13, 14]). These equations have a very important role to play in efficient inferencing in approximate reasoning, especially fuzzy control systems (see [6]). In this work we present some results connected with two functional equations describing the distributivity of fuzzy implications over representable uninorms. Toward this end, some new solutions of the additive Cauchy functional equation on the set $[-\infty, \infty]$ has been obtained.

Keywords: fuzzy implication, functional equations, uninorm.

1 Introduction

Distributivity of fuzzy implications over different fuzzy logic connectives has been studied in the recent past by many authors. This interest, perhaps, was kick started by Combs and Andrews in [6] wherein they exploit the following classical tautology

$$(p \wedge q) \rightarrow r \equiv (p \rightarrow r) \vee (q \rightarrow r)$$

in their inference mechanism towards reduction in the complexity of fuzzy "If-Then" rules. Subsequently, there were many discussions in [7, 11], most of them pointing out the need for a theoretical investigation required for employing such equations in practice.

It was Trillas and Alsina [14], who were the first to investigate the generalized version of the above law for any $x, y, z \in [0, 1]$,

$$I(T(x, y), z) = S(I(x, z), I(y, z)), \qquad (1)$$

where T, S are a t-norm and a t-conorm, respectively, and I is a fuzzy implication. Using similar techniques as above, Balasubramaniam and Rao [5] considered the following dual equations of (1):

$$I(S(x, y), z) = T(I(x, z), I(y, z)), \qquad (2)$$

$$I(x, T_1(y, z)) = T_2(I(x, y), I(x, z)), \qquad (3)$$

$$I(x, S_1(y, z)) = S_2(I(x, y), I(x, z)), \qquad (4)$$

where again T, T_1, T_2 and S, S_1, S_2 are t-norms and t-conorms, respectively and I is a fuzzy implication. In both papers it was shown that when I is either an R-implication obtained from a left-continuous t-norm or an S-implication, in almost all the cases the distributivity equations (1)–(4) hold if and only if $T_1 = T_2 = T = \min$ and $S_1 = S_2 = S = \max$. In fact, the equation (4) for the case when I is an R-implication obtained from a strict t-norm was left unsolved in [5], but this situation was considered by the authors in [4], where we characterized functions I, which satisfy the functional equation (4), when S_1, S_2 are either both strict or nilpotent t-conorms.

Meanwhile, Baczyński in [2, 3] considered the functional equation (3), both independently and along with other equations, and characterized fuzzy implications I in the case when $T_1 = T_2$ is a strict t-norm.

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In this paper we investigate the other possible generalizations, i.e., we concentrate on the following two distributive equations

$$I(x, U_1(y, z)) = U_2(I(x, y), I(x, z)), \quad (5)$$
$$I(U_1(x, y), z) = U_2(I(x, z), I(y, z)), \quad (6)$$

by characterizing fuzzy implications I, which satisfy the above equations, when U_1 and U_2 are given representable uninorms.

It should be noted that the equation (6) was studied by Ruiz and Torrens in [12] for the major part of known classes of uninorms with continuous underlying t-norm and t-conorm and for strong implications derived from uninorms, while in [13], they also studied (6), but with the assumption, that I is a residual implication derived from uninorms.

Finally, we would like to underline, that the general solutions of the distributive equation

$$F(x, G(y, z)) = G(F(x, z), F(y, z)),$$

where F is continuous and G is assumed to be continuous, strictly increasing and associative were presented by Aczél (see [1], Theorem 6, p. 319). Our results can be seen as a generalization of the above mentioned result without any assumptions on the function F and less assumptions on the function G.

2 Fuzzy logic connectives

We assume that the reader is familiar with the classical results concerning uninorms, so we only recall basic definitions and facts which will be useful in the sequel.

Definition 1 (Fodor et al., [9])**.** An associative, commutative and increasing operation $U \colon [0, 1]^2 \to [0, 1]$ is called a uninorm if it has the neutral element $e \in [0, 1]$.

If $e = 0$ then U is a t-conorm and if $e = 1$ then U is a t-norm. One can easily observe, that $U(0, 1) = U(1, 0) \in \{0, 1\}$ for any uninorm U. A uninorm U such that $U(0, 1) = 0$ is called conjunctive and if $U(0, 1) = 1$ it is called disjunctive.

Analogously to the representation theorems for continuous Archimedean t-norms and t-conorms, we have the following result.

Theorem 2 (Fodor et al. [9], Theorem 3)**.** *For a function $U \colon [0, 1]^2 \to [0, 1]$ the following statements are equivalent:*

(i) U is continuous on $(0, 1)^2$ and a strictly increasing uninorm with the neutral element $e \in (0, 1)$ such that U is self-dual with respect to a strong negation N with the fixed point e.

(ii) There exists a continuous and strictly increasing function $h \colon [0, 1] \to [-\infty, \infty]$ with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = \infty$ such that

$$U(x, y) = h^{-1}(h(x) + h(y)), \quad (7)$$

for all $(x, y) \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\}$ and either $U(0, 1) = U(1, 0) = 0$ or $U(0, 1) = U(1, 0) = 1$.

Moreover, the generator h of the uninorm U is uniquely determined up to a positive multiplicative constant.

Uninorms that can be characterized as above are called representable uninorms.

Example 3. For $h(x) = \ln\left(\frac{x}{1-x}\right)$ we get the following conjunctive and representable uninorm with $e = \frac{1}{2}$:

$$U(x, y) = \begin{cases} 0, & \text{if } (x, y) \in \{(0, 1), (1, 0)\}, \\ \dfrac{xy}{(1-x)(1-y) + xy}, & \text{otherwise.} \end{cases}$$

Remark 4. One can easily observe, that if a representable uninorm U is conjunctive, then the representation (7) is true for all $x, y \in [0, 1]$ with the assumption, that

$$(-\infty) + \infty = \infty + (-\infty) = -\infty. \quad (8)$$

Similarly, if a representable uninorm U is disjunctive, then the representation (7) is true for all $x, y \in [0, 1]$ with the assumption, that

$$(-\infty) + \infty = \infty + (-\infty) = \infty. \quad (9)$$

In the literature we can find several diverse definitions of fuzzy implications. In this article we will use the following one, which is equivalent to the well accepted definition introduced by Fodor and Roubens (see [8], Definition 1.15).

Definition 5. A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:

$$I \text{ is decreasing in the first variable,} \tag{I1}$$

$$I \text{ is increasing in the second variable,} \tag{I2}$$

$$I(0,0) = 1, \quad I(1,1) = 1, \quad I(1,0) = 0. \tag{I3}$$

3 Preliminary results

Our main goal in this paper is to present the representations of some classes of fuzzy implications that satisfy equation (5) when U_1, U_2 are representable uninorms. Within this context, we firstly show, that only some cases are needed to be investigated.

Lemma 6. *Let a function* $I\colon [0,1]^2 \to [0,1]$ *satisfy* (I3) *and also* (5) *for some uninorms* U_1, U_2. *Then* U_1 *is conjunctive if and only if* U_2 *is conjunctive.*

Proof. Firstly, substituting $x = y = 1$ and $z = 0$ into (5), we get

$$I(1, U_1(1,0)) = U_2(I(1,1), I(1,0)). \tag{10}$$

Now, if U_1 is conjunctive, then $U_1(1,0) = 0$ and by (I3) we get from (10), $I(1,0) = U_2(1,0) = 0$, i.e., U_2 is also a conjunctive uninorm.

Instead, if U_1 is disjunctive, then $U_1(1,0) = 1$ and we get from (10), $I(1,1) = U_2(1,0) = 1$, i.e., U_2 is a disjunctive uninorm. □

On the other side, quite contrastingly, we have the following fact which is again easier to prove as above.

Lemma 7. *Let a function* $I\colon [0,1]^2 \to [0,1]$ *satisfy* (I3) *and also* (6) *for some uninorms* U_1, U_2. *Then* U_1 *is conjunctive if and only if* U_2 *is disjunctive.*

By the above results it is enough, in our context, to consider the functional equation (5) only when both uninorms U_1, U_2 are either representable conjunctive uninorms or representable disjunctive uninorms and the functional equation (6) only when either U_1 is conjunctive and U_2 is disjunctive, or U_1 is disjunctive and U_2 is conjunctive.

4 Some new results pertaining to functional equations

Here we show some new results related to the additive Cauchy functional equation:

$$f(x+y) = f(x) + f(y). \tag{11}$$

The presented facts, which are new and crucial in the proofs of the main theorems, can be seen as the generalizations of the classical theorems from the theory of functional equations (see [1], [10]).

Proposition 8. *For a function* $f\colon [-\infty,\infty] \to [-\infty,\infty]$ *the following statements are equivalent:*

(i) f *satisfies the additive Cauchy functional equation* (11) *for all* $x, y \in [-\infty,\infty]$, *with the assumption* (8).

(ii) Either $f = -\infty$, *or* $f = 0$, *or* $f = \infty$, *or*

$$f(x) = \begin{cases} 0, & \textit{if } x \in (-\infty,\infty], \\ -\infty, & \textit{if } x = -\infty, \end{cases} \tag{12}$$

or

$$f(x) = \begin{cases} 0, & \textit{if } x \in (-\infty,\infty], \\ \infty, & \textit{if } x = -\infty, \end{cases} \tag{13}$$

or

$$f(x) = \begin{cases} \infty, & \textit{if } x \in (-\infty,\infty], \\ -\infty, & \textit{if } x = -\infty, \end{cases} \tag{14}$$

or

$$f(x) = \begin{cases} \infty, & \textit{if } x \in \mathbb{R}, \\ -\infty, & \textit{if } x \in \{-\infty,\infty\}, \end{cases} \tag{15}$$

or there exists a unique additive function $g\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$f(x) = \begin{cases} -\infty, & \textit{if } x \in \{-\infty,\infty\}, \\ g(x), & \textit{if } x \in \mathbb{R}, \end{cases} \tag{16}$$

or

$$f(x) = \begin{cases} \infty, & \textit{if } x \in \{-\infty,\infty\}, \\ g(x), & \textit{if } x \in \mathbb{R}, \end{cases} \tag{17}$$

or

$$f(x) = \begin{cases} -\infty, & \textit{if } x = -\infty, \\ g(x), & \textit{if } x \in \mathbb{R}, \\ \infty, & \textit{if } x = \infty. \end{cases} \tag{18}$$

Proof. *(ii)* $\Longrightarrow$ *(i)* It is a direct calculation that all the above functions satisfy the equation (11) with the assumption (8).

(i) $\Longrightarrow$ *(ii)* Let $f\colon [-\infty,\infty] \to [-\infty,\infty]$ satisfy (11) for all $x, y \in [-\infty,\infty]$ with the assumption (8). Firstly, observe that the situation when $f(-\infty) = \infty$ and $f(\infty) = -\infty$ is not possible now. Indeed, if we assume the above, we get

$$\begin{aligned}\infty &= f(-\infty) = f((-\infty) + \infty) \\ &= f(-\infty) + f(\infty) = \infty + (-\infty) = -\infty,\end{aligned}$$

a contradiction.

Setting $x = y = -\infty$ in (11) we get $f(-\infty) = f(-\infty) + f(-\infty)$. Therefore $f(-\infty) = -\infty$, or $f(-\infty) = 0$, or $f(-\infty) = \infty$. If $f(-\infty) = 0$, then for any $x \in [-\infty,\infty]$ we have

$$\begin{aligned}0 &= f(-\infty) = f((-\infty) + x) \\ &= f(-\infty) + f(x) = 0 + f(x) = f(x),\end{aligned}$$

thus we obtain the solution $f = 0$.

Setting $x = y = \infty$ in (11) we get $f(\infty) = f(\infty) + f(\infty)$. Therefore $f(\infty) = -\infty$, or $f(\infty) = 0$, or $f(\infty) = \infty$. If $f(\infty) = 0$, then for any $x \in (-\infty,\infty]$ we have

$$\begin{aligned}0 &= f(\infty) = f(\infty + x) \\ &= f(\infty) + f(x) = 0 + f(x) = f(x),\end{aligned}$$

thus considering the other possible values for $-\infty$ we get two possible solutions (12) or (13).

Setting $x = y = 0$ in (11) we get $f(0) = f(0) + f(0)$, so $f(0) = -\infty$, or $f(0) = 0$, or $f(0) = \infty$. If $f(0) = -\infty$, then for any $x \in [-\infty,\infty]$ we get

$$\begin{aligned}f(x) &= f(x + 0) = f(x) + f(0) \\ &= f(x) + (-\infty) = -\infty,\end{aligned}$$

thus we obtain the solution $f = -\infty$. If $f(0) = \infty$, then for any $x \in [-\infty,\infty]$ we get

$$\begin{aligned}f(x) &= f(x + 0) = f(x) + f(0) \\ &= f(x) + \infty,\end{aligned}$$

so for every fixed $x \in [-\infty,\infty]$ we get that either $f(x) = -\infty$ or $f(x) = \infty$. On the other side, for all $x \in \mathbb{R}$ we have

$$\infty = f(0) = f(x + (-x)) = f(x) + f(-x),$$

hence, by our assumption (8) we obtain, that $f(x) = \infty$ for all $x \in \mathbb{R}$. By the first step of our proof we get the next three possible solutions: $f = \infty$, or (15), or (14).

Let us assume now that $f(0) = 0$. See that for $x \in \mathbb{R}$ we have

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x),$$

so $f(x) \in \mathbb{R}$ for every $x \in \mathbb{R}$. Therefore there exists a unique additive function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f(x) = g(x)$ for $x \in \mathbb{R}$. Taking again into account the previous steps of our proof we obtain the last three possible solutions, i.e., f has the form either (16), or (17), or (18). □

By the well known solutions of the additive Cauchy functional equation for real numbers (see [1] or [10], Theorem 5.2.1) we get

Corollary 9. *For a continuous function $f\colon [-\infty,\infty] \to [-\infty,\infty]$ the following statements are equivalent:*

(i) f satisfies the additive Cauchy functional equation (11) for all $x, y \in [-\infty,\infty]$, with the assumption (8).

(ii) Either $f = -\infty$, or $f = 0$, or $f = \infty$, or there exists a unique constant $c \in (0,\infty)$ such that

$$f(x) = cx, \tag{19}$$

for all $x \in [-\infty,\infty]$.

Using similar techniques as above we can prove the next results.

Proposition 10. *For a function $f\colon [-\infty,\infty] \to [-\infty,\infty]$ the following statements are equivalent:*

(i) f satisfies the additive Cauchy functional equation (11) for all $x, y \in [-\infty,\infty]$, with the assumption (9).

(ii) Either $f = -\infty$, or $f = 0$, or $f = \infty$, or

$$f(x) = \begin{cases} 0, & \text{if } x \in [-\infty,\infty), \\ -\infty, & \text{if } x = \infty, \end{cases} \tag{20}$$

or

$$f(x) = \begin{cases} 0, & \text{if } x \in [-\infty,\infty), \\ \infty, & \text{if } x = \infty, \end{cases} \tag{21}$$

or

$$f(x) = \begin{cases} -\infty, & \text{if } x \in [-\infty, \infty), \\ \infty, & \text{if } x = \infty, \end{cases} \tag{22}$$

or

$$f(x) = \begin{cases} -\infty, & \text{if } x \in \mathbb{R}, \\ \infty, & \text{if } x \in \{-\infty, \infty\}, \end{cases} \tag{23}$$

or there exists a unique additive function $g\colon \mathbb{R} \to \mathbb{R}$ *such that* f *has the form* (16)*, or* (17)*, or* (18)*.*

Corollary 11. *For a continuous function* $f\colon [-\infty, \infty] \to [-\infty, \infty]$ *the following statements are equivalent:*

(i) f *satisfies the additive Cauchy functional equation* (11) *for all* $x, y \in [-\infty, \infty]$*, with the assumption* (9)*.*

(ii) Either $f = -\infty$*, or* $f = 0$*, or* $f = \infty$*, or there exists a unique constant* $c \in (0, \infty)$ *such that* f *has the representation* (19) *for all* $x \in [-\infty, \infty]$*.*

Proposition 12. *Let* $X = Y = [-\infty, \infty]$*. For a function* $f\colon X \to Y$ *the following statements are equivalent:*

(i) f *satisfies the additive Cauchy functional equation* (11) *for all* $x, y \in [-\infty, \infty]$*, with the assumption* (8) *in the set* X *and the assumption* (9) *in the set* Y*.*

(ii) Either $f = -\infty$*, or* $f = 0$*, or* $f = \infty$*, or* f *has the form* (12)*, or* (13)*, or*

$$f(x) = \begin{cases} -\infty, & \text{if } x \in (-\infty, \infty], \\ \infty, & \text{if } x = -\infty, \end{cases} \tag{24}$$

or

$$f(x) = \begin{cases} -\infty, & \text{if } x \in \mathbb{R}, \\ \infty, & \text{if } x \in \{-\infty, \infty\}, \end{cases} \tag{25}$$

or there exists a unique additive function $g\colon \mathbb{R} \to \mathbb{R}$ *such that* f *has the form* (16)*, or* (17)*, or*

$$f(x) = \begin{cases} \infty, & \text{if } x = -\infty, \\ g(x), & \text{if } x \in \mathbb{R}, \\ -\infty, & \text{if } x = \infty. \end{cases} \tag{26}$$

Corollary 13. *Let* $X = Y = [-\infty, \infty]$*. For a continuous function* $f\colon X \to Y$ *the following statements are equivalent:*

(i) f *satisfies the additive Cauchy functional equation* (11) *for all* $x, y \in [-\infty, \infty]$*, with the assumption* (8) *in the set* X *and the assumption* (9) *in the set* Y*.*

(ii) Either $f = -\infty$*, or* $f = 0$*, or* $f = \infty$*, or there exists a unique constant* $c \in (-\infty, 0)$ *such that* f *has the representation* (19) *for all* $x \in [-\infty, \infty]$*.*

Proposition 14. *Let* $X = Y = [-\infty, \infty]$*. For a function* $f\colon X \to Y$ *the following statements are equivalent:*

(i) f *satisfies the additive Cauchy functional equation* (11) *for all* $x, y \in [-\infty, \infty]$*, with the assumption* (9) *in the set* X *and the assumption* (8) *in the set* Y*.*

(ii) Either $f = -\infty$*, or* $f = 0$*, or* $f = \infty$*, or* f *has the form* (20) *or* (21)*, or there exists a unique additive function* $g\colon \mathbb{R} \to \mathbb{R}$ *such that* f *has the form* (16)*, or* (17)*, or* (26)*.*

Corollary 15. *Let* $X = Y = [-\infty, \infty]$*. For a continuous function* $f\colon X \to Y$ *the following statements are equivalent:*

(i) f *satisfies the additive Cauchy functional equation* (11) *for all* $x, y \in [-\infty, \infty]$*, with the assumption* (9) *in the set* X *and the assumption* (8) *in the set* Y*.*

(ii) Either $f = -\infty$*, or* $f = 0$*, or* $f = \infty$*, or there exists a unique constant* $c \in (-\infty, 0)$ *such that* f *has the representation* (19) *for all* $x \in [-\infty, \infty]$*.*

5 On the Equation (5) when U_1, U_2 are representable conjunctive uninorms

Due to the page limit, we only show some main results for the first equation (5) when U_1, U_2 are both representable conjunctive uninorms. It should be noted, that using the results from the previous section, we are able to obtain description of the solutions I of the equations (5) and (6) for all the cases presented in Section 3.

Theorem 16. *For representable, conjunctive uninorms U_1, U_2 with the neutral elements $e_1, e_2 \in (0,1)$ and a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) The triple of functions U_1, U_2, I satisfies the functional equation (5) for all $x, y, z \in [0,1]$.

(ii) There exist continuous, strictly increasing functions $h_1, h_2\colon [0,1] \to [-\infty, \infty]$ with $h_1(0) = h_2(0) = -\infty$, $h_1(e_1) = h_2(e_2) = 0$ and $h_1(1) = h_2(1) = \infty$, which are uniquely determined up to positive multiplicative constants, such that U_1, U_2 admit the representation (7) for all $x, y \in [0,1]$ with the assumption (8) and h_1, h_2, respectively, and for every fixed $x \in [0,1]$, the vertical section $I(x,\cdot)$ has, for all $y \in [0,1]$, one of the following forms:

$$I(x,y) = 0,$$
$$I(x,y) = e_2,$$
$$I(x,y) = 1,$$
$$I(x,y) = \begin{cases} 0, & \text{if } y = 0, \\ e_2, & \text{if } y \in (0,1], \end{cases}$$
$$I(x,y) = \begin{cases} 1, & \text{if } y = 0, \\ e_2, & \text{if } y \in (0,1], \end{cases}$$
$$I(x,y) = \begin{cases} 0, & \text{if } y = 0, \\ 1, & \text{if } y \in (0,1], \end{cases}$$
$$I(x,y) = \begin{cases} 0, & \text{if } y = 0 \text{ or } y = 1, \\ 1, & \text{if } y \in (0,1), \end{cases}$$
$$I(x,y) = \begin{cases} 0, & \text{if } y \in \{0,1\}, \\ h_2^{-1}\left(g_x(h_1(y))\right), & \text{if } y \in (0,1), \end{cases}$$
$$I(x,y) = \begin{cases} 1, & \text{if } y \in \{0,1\}, \\ h_2^{-1}\left(g_x(h_1(y))\right), & \text{if } y \in (0,1), \end{cases}$$
$$I(x,y) = \begin{cases} 0, & \text{if } y = 0, \\ h_2^{-1}\left(g_x(h_1(y))\right), & \text{if } y \in (0,1), \\ 1, & \text{if } y = 1, \end{cases}$$

with an additive function $g_x\colon \mathbb{R} \to \mathbb{R}$, which depends on constants for h_1 and h_2.

Proof. $(ii) \Longrightarrow (i)$ The proof in this direction can be easily checked.

$(i) \Longrightarrow (ii)$ Let us assume that uninorms U_1, U_2 and a function I are the solutions of the functional equation (5) satisfying the required properties. By the definition of a representable uninorm and Remark 4 the uninorms U_1 and U_2 admit the representation (7) for some continuous, strictly increasing functions $h_1, h_2\colon [0,1] \to [-\infty,\infty]$ with $h_1(0) = h_2(0) = -\infty$, $h_1(e_1) = h_2(e_2) = 0$ and $h_1(1) = h_2(1) = \infty$. Moreover, both generators are uniquely determined up to positive multiplicative constants. Now the equation (4) becomes,

$$\begin{aligned} I(x,&h_1^{-1}(h_1(y) + h_1(z))) \\ &= h_2^{-1}(h_2(I(x,y)) + h_2(I(x,z))), \end{aligned}$$

for all $x, y, z \in [0,1]$. Let $x \in [0,1]$ be arbitrary but fixed. Define a function $I_x\colon [0,1] \to [0,1]$ by the formula

$$I_x(y) = I(x,y), \qquad y \in [0,1].$$

By routine substitutions, $h_x = h_2 \circ I_x \circ h_1^{-1}$, $u = h_1(y)$, $v = h_1(z)$, for $y, z \in [0,1]$, we obtain the additive Cauchy functional equation

$$h_x(u+v) = h_x(u) + h_x(v), \quad u, v \in [-\infty,\infty],$$

where $h_x\colon [-\infty,\infty] \to [-\infty,\infty]$, with the assumption (8). Proposition 8 describes all solutions h_x. Because of the definition of the function h_x we get all our formulas.

We show that in the last three cases the additive function g depends on constants for h_1 and h_2. Let $h_1'(y) = a \cdot h_1(y)$ and $h_2'(y) = b \cdot h_2(y)$ for all $y \in (0,1)$ and some $a, b \in (0,\infty)$. If we assume that

$$h_2^{-1}\left(g_x(h_1(y))\right) = h_2'^{-1}\left(g_x'(h_1'(y))\right),$$

for all $y \in (0,1)$ and with some additive functions g_x and g_x', then we get

$$h_2^{-1}\left(g_x(h_1(y))\right) = h_2^{-1}\left(\frac{g_x'(h_1'(y))}{b}\right),$$

so

$$g_x(h_1(y)) = \frac{g'_x(a \cdot h_1(y))}{b},$$

thus, for every $u \in \mathbb{R}$ we obtain

$$g'_x(u) = b \cdot g_x\left(\frac{u}{a}\right).$$

□

Example 17. If U_1 and U_2 are representable conjunctive uninorms, then the greatest solution which is a fuzzy implication is the greatest fuzzy implication:

$$I_1(x,y) = \begin{cases} 0, & \text{if } x = 1 \text{ and } y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

The vertical sections are the following: for $x \in [0,1)$ this is the third solution and for $x = 1$ this is the sixth solution in Theorem 16.

From the previous results we are in a position to describe the continuous solutions I of (5).

Theorem 18. *Let U_1, U_2 be representable conjunctive uninorms with the neutral elements $e_1, e_2 \in (0,1)$. For a continuous function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) The triple of functions U_1, U_2, I satisfies the functional equation (5) *for all $x, y, z \in [0,1]$.*

(ii) There exist continuous, strictly increasing functions $h_1, h_2\colon [0,1] \to [-\infty, \infty]$ with $h_1(0) = h_2(0) = -\infty$, $h_1(e_1) = h_2(e_2) = 0$ and $h_1(1) = h_2(1) = \infty$, which are uniquely determined up to positive multiplicative constants, such that U_1, U_2 admit the representation (7) *for all $x, y \in [0,1]$ with the assumption* (8) *and h_1, h_2, respectively, and either $I = 0$, or $I = e_2$, or $I = 1$, or there exists a continuous function $c\colon [0,1] \to (0,\infty)$, uniquely determined up to a positive multiplicative constant depending on constants for h_1 and h_2, such that I has the following form*

$$I(x,y) = h_2^{-1}\left(c(x) \cdot h_1(y)\right), \qquad (27)$$

for all $x, y \in [0,1]$.

Since the equation (5) is the generalization of a tautology from the classical logic involving boolean implication, it is reasonable to expect that the solution I of (5) is also a fuzzy implication. But from Theorem 18 we get

Corollary 19. *If U_1, U_2 are representable conjunctive uninorms, then there are no continuous solutions I of* (5) *which satisfy* (I3).

Proof. Let a continuous function I satisfy (I3) and (5) with some representable conjunctive t-conorms U_1, U_2 with continuous additive generators h_1, h_2, respectively. Then I has to have the form (27) with a continuous function $c\colon [0,1] \to (0,\infty)$, but in this case we get

$$\begin{aligned} I(0,0) &= h_2^{-1}\left(c(0) \cdot h_1(0)\right) \\ &= h_2^{-1}\left(c(0) \cdot (-\infty)\right) \\ &= h_2^{-1}(-\infty) = 0, \end{aligned}$$

so I does not satisfy the first condition in (I3). □

From Corollary 19 it is obvious that in our situation we need to look for solutions which are not continuous at the point $(0,0)$. This case has also been investigated by us and the function I has the following form:

$$I(x,y) = \begin{cases} 1, & \text{if } x = y = 0, \\ h_2^{-1}\left(c(x) \cdot h_1(y)\right), & \text{otherwise,} \end{cases}$$

where $c\colon [0,1] \to (0,\infty]$ is a continuous decreasing function, with $c(x) < \infty$ for $x \in (0,1]$ and $c(0) = \infty$.

Example 20. One specific example is the function $c(x) = \frac{1}{x}$ defined for all $x \in [0,1]$, with the assumption that $\frac{1}{0} = \infty$. For example, when we consider the conjunctive representable uninorm given in Example 3, then one fuzzy implication which satisfies (5) with this U has the following form:

$$I(x,y) = \begin{cases} 1, & \text{if } x, y \in \{(0,0), (0,1), (1,1)\}, \\ \frac{\left(\frac{y}{1-y}\right)^{\frac{1}{x}}}{1+\left(\frac{y}{1-y}\right)^{\frac{1}{x}}}, & \text{otherwise.} \end{cases}$$

6 Summary

In this paper we have investigated the distributivity of fuzzy implications over representable uninorms. To obtain all solutions, we have firstly shown, in Section 3, that some cases are not possible in our context. Next, in Section 4, we have obtained new results dealing with the additive Cauchy functional equation (11), when the domain and range of the function f is equal to $[-\infty, \infty]$. Finally, in Section 5, we have presented some solutions of the equation (5), when U_1 and U_2 are both representable conjunctive uninorms.

From our proof of Theorem 16, one can observe that for the equation (5) and the situation when U_1 and U_2 are both representable disjunctive uninorms we will be using Proposition 10. Further, for the equation (6) when U_1 is a representable conjunctive uninorm and U_2 is a representable disjunctive uninorm we will use Proposition 12, while for the equation (6) when U_1 is a representable disjunctive uninorm and U_2 is a representable conjunctive uninorm we shall use Proposition 14.

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