Tackling Incomparability of Fuzzy Preferences Using Self-dual Uninorms

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Abstract

Under a set of conditions consistency of fuzzy preference relations is to be modelled by uninorms. This result is exploited for tackling incomparability in a fuzzy preference relation. We present a uninorm consistency based method to estimate the unknown preference values in the pair comparison of a set of alternatives based on the known ones.

Keywords: Fuzzy preference relation, consistency, uninorm, missing information

1 Introduction

Given two alternatives an expert either prefers one to the other or is indifferent between them. Obviously, there is another possibility, that of an expert being unable to compare them. The main advantage of pairwise comparison is that of focusing exclusively on two alternatives at a time which facilitates experts when expressing their preferences. However, this way of providing preferences limits experts in their global perception of the alternatives and, as a consequence, the provided preferences could be not rational or consistent.

In a crisp context the concept of consistency has traditionally been defined in terms of acyclicity [1]. This condition is closely related to the transitivity of the corresponding binary preference relation, in the sense that if alternative x_i is preferred to alternative x_j and this one to x_k then alternative x_i should be preferred to x_k. However, the question whether the "degree or strength of preference" of x_i over x_j exceeds, equals, or is less than the "degree or strength of preference" of x_j over x_k cannot be answered by the classical preference modelling. The implementation of the degree of preference between alternatives may be essential in many situations, and this can be done by using fuzzy preference relations [2].

In a fuzzy context, the traditional requirement to characterise consistency has followed the way of extending the classical requirements of binary preference relations. Thus, consistency is also based on the notion of transitivity. We consider the term "consistency" of fuzzy preferences as described by Saaty [3: page 7]:

> not merely the traditional requirement of the transitivity of preferences [...], but the actual intensity with which the preference is expressed transits through the sequence of objects in comparison.

Therefore, although intransitivity implies inconsistency, it is clear that inconsistency does not necessarily imply intransitivity. The problem here is how to model mathematically consistency of fuzzy preference relations.

Another issue to take into account in decision making is when experts do not have an in-depth knowledge of the problem to be solved.

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In such cases, experts may not put their opinion forward about certain aspect of the problem, and as a result they may present incomplete preferences, i.e. some preference values may not be given or may be missing. Indeed, there may be cases where an expert would not be able to efficiently express any kind of preference degree between two or more of the available options. Therefore, it would be of great importance to provide the experts with tools that allow them to express this lack of knowledge in their opinions.

In [4] a set of conditions was put forward for a fuzzy preference relation to be considered 'fully consistent.' Under this set of conditions it was shown that consistency of fuzzy preference relations can be characterised by self-dual almost continuous uninorms. This result is exploited for tackling the presence of incompatibilities in a fuzzy preference relation. We present a uninorm consistency based method to estimate unknown values in the pair comparison of a set of alternatives based on the known ones. By doing this, we assure that the estimated values are compatible with the rest of the information provided by that expert.

The rest of the paper is set out as follows. Section 2 comprises an introduction to fuzzy preferences, as well as some preliminaries on consistency of preferences. In Section 3, the uninorm characterisation of consistency of fuzzy preferences is presented. In Section 4, we present a uninorm consistency based estimation procedure of unknown values in a fuzzy preference relation. Finally, conclusions are drawn in Section 5.

2 Fuzzy Preference Relations: Consistency

In the classical preference modelling, given two alternatives, an expert judges them in one of the following ways: (i) one alternative is preferred to another; (ii) the two alternatives are indifferent to him/her; (iii) he/she is unable to compare them.

Fishburn in [5] defines indifference as the absence of strict preference, and also points out that one of the three possible different ways in which indifference between alternatives might arise is when when both alternatives are considered incomparable on a preference basis by the expert. This obviously implies that Fishburn treats incomparability as indifference. However, we believe this interpretation is not correct. If an expert is unable to compare two alternatives then this situation should be reflected in the preference relation not as an indifference situation, but with a missing entry for that particular pair of alternatives. In other word, a missing value in a preference relation is not equivalent to a lack of preference of one alternative over another. A missing value might be the result of the incapacity of an expert to quantify the degree of preference of one alternative over another, in which case he/she may decide not to 'guess' to maintain the consistency of the values already provided.

Asymmetry is considered by Fishburn [5] as an "obvious" condition for preferences: if an expert prefers x to y ($x \succ y$), then he[/she] should not simultaneously prefers y to x. Using a numerical representation of crisp preferences on a set of alternatives X, we have [6]:

$$\begin{array}{lcl} r_{ij} = 1 & \Leftrightarrow & x_i \succ x_j \\ r_{ij} = 0 & \Leftrightarrow & x_j \succ x_i \end{array}$$

Clearly, this can be extended by adding the indifference case:

$$r_{ij} = 0.5 \quad \Leftrightarrow \quad x_i \sim x_j$$

A fuzzy preference relation R on a set of alternatives X is a fuzzy set on the product set $X \times X$, that is characterized by a membership function $\mu_R : X \times X \longrightarrow [0,1]$.

When cardinality of X is small, the preference relation may be conveniently represented by the $n \times n$ matrix $R = (r_{ij})$ being $r_{ij} = \mu_R(x_i, x_j)$ $(\forall i, j \in \{1, \ldots, n\})$. The asymmetry property is usually modelled by imposing the reciprocity property $r_{ij} + r_{ji} = 1$ $(\forall i, j \in \{1, \ldots, n\})$.

In this approach, given two alternatives an experts provides [7]:

(i) a value in the range $(0.5, 1]$ to quantify the "degree or strength of preference" of an alternative when preferred to another;

(ii) the value 0.5 when the two alternatives are indifferent to him/her;

(iii) no value when he/she is uncertain as to his/her preference between the alternatives or he/she is unable to compare them.

The main advantage of pairwise comparison is that of focusing exclusively on two alternatives at a time which facilitates experts when expressing their preferences. However, this way of providing preferences limits experts in their global perception of the alternatives and, as a consequence, the provided preferences could not be rational.

There are three fundamental and hierarchical levels of rationality assumptions when dealing with preference relations [8]:

- The first level of rationality requires indifference between any alternative and itself.
- The second one assumes the property of reciprocity in the pairwise comparison between any two alternatives.
- Finally, the third one is associated with the transitivity in the pairwise comparison among any three alternatives.

A preference relation verifying the third level of rationality is usually called a *consistent preference relation* and any property that guarantees the transitivity of the preferences is called a consistency property. The lack of consistency in decision making can lead to inconsistent conclusions; that is why it is important, in fact crucial, to study conditions under which consistency is satisfied [3, 9, 10].

In a crisp context, the concept of consistency it has traditionally been defined in terms of acyclicity [11], that is the absence of sequences such as $x_1, x_2, \ldots, x_k (x_{k+1} = x_1)$ with $x_j P x_{j+1} \forall j = 1, \ldots, k$. In a fuzzy context, the traditional requirement to characterise consistency has followed the way of extending the classical requirements of binary preference relations. Thus, consistency is also based on the notion of transitivity, in the sense that if alternative x_i is preferred to alternative x_j ($r_{ij} \geq 0.5$) and this one to x_k ($r_{jk} \geq 0.5$) then alternative x_i should be preferred to x_k ($r_{ik} \geq 0.5$), which is normally referred to as *weak transitivity* [12]. However, the main difference in this case with respect to the classical one is that consistency has been modelled in many different ways due to the role the intensity of preference has [3, 10, 12].

3 Uninorm Characterisation of Consistency of Fuzzy Preferences

The assumption of experts being able to quantify their preferences in the domain [0,1] instead of $\{0, 1\}$ underlies unlimited computational abilities and resources from the experts. Taking these unlimited computational abilities and resources into account, consistency of preferences may be formulated as follows:

$$r_{ik} = f(r_{ij}, r_{jk}) \quad \forall i, j, k \tag{1}$$

being f a function $f\colon [0,1] \times [0,1] \to [0,1]$.

In practical cases, expression (1) might obviously not be verified even when the preference values of a preference relation are transitive (*weak transitivity*): $r_{ij} \geq 0.5$ and $r_{jk} \geq 0.5$ then $r_{ik} \geq 0.5$. However, the assumption of modelling consistency using the expression (1) can be used to introduce levels of consistency, which in group decision making situations could be exploited by assigning a relative importance weight to each one of the experts in arriving to a collective preference opinion. Also, as we will see in the following section, it can be used for tackling the presence of incompatibilities in a fuzzy preference relation by estimating unknown values in the pair comparison of a set of alternatives based on the known ones.

The following properties are imposed to function f [4]:

Monotonicity

$$f(x, y) \geq f(x', y') \text{ if } x \geq x' \wedge y \geq y'$$

Associativity

$$f(f(x,y),z) = f(x,f(y,z)) \quad \forall x,y,z$$

Reciprocity

$$f(x,y) + f(1-y,1-x) = 1$$
$$\forall (x,y) \in [0,1] \times [0,1] \backslash \{(0,1),(1,0)\}$$

Identity element

$$f(0.5,x) = f(x,0.5) = x \; \forall x \in [0,1]$$

Cancellative

$$f(x,y) = f(x,z) \wedge f(y,x) = f(z,x)$$
$$\forall x \in]0,1[\implies y = z$$

Continuity f is continuous in $[0,1] \times [0,1] \backslash \{(0,1),(1,0)\}$

Clearly, function f shares all properties of a uninorm [13] except perhaps commutativity, which cannot be directly derived from the above set of properties. However, commutativity of f can be derived indirectly from associativity, cancellativity and continuity of f. Indeed, the following result was proved by Aczél in [14]:

Theorem 1. *Let I be a (closed, open, half-open, finite or infinite) proper interval of real numbers. Then $F\colon I^2 \to I$ is a continuous operation on I^2 which satisfies the associativity equation*

$$F(F(x,y),z) = F(x,F(y,z)) \; \forall x,y,z \in I$$

and is cancellative, that is,

$$F(x_1,y) = F(x_2,y) \text{ or } F(y,x_1) = F(y,x_2)$$
$$\text{implies } x_1 = x_2 \text{ for any } y \in I$$

if, and only if, there exists a continuous and strictly monotonic function $\phi\colon J \longrightarrow I$ such that

$$F(x,y) = \phi\left[\phi^{-1}(x) + \phi^{-1}(y)\right] \; \forall x,y \in I \quad (2)$$

Here J is one of the real intervals

$$]-\infty,\gamma],\;]-\infty,\gamma[,\; [\delta,\infty[,\;]\delta,\infty[,\; \text{or }]-\infty,\infty[$$

for some $\gamma \leq 0 \leq \delta$. Accordingly I has to be open at least from one side.

The function in (2) is unique up to a linear transformation of the variable ($\phi(x)$ may be replaced by $\phi(Cx)$, $C \neq 0$ but by no other function.)

The representation of function F given by (2) coincides with Fodor, Yager and Rybalov representation theorem for almost continuous uninorms U, i.e. uninorms with identity element in $]0,1[$ continuous on $[0,1]^2 \backslash \{(0,1),(1,0)\}$ [15].

We note that indifference and reciprocity of preferences in $[0,1]$ are based on the use of the strong negation $N(x) = 1-x$. Therefore, the assumption of modelling consistency of reciprocal preferences in $[0,1]$ using the functional expression (1) has solution f an almost continuous self-dual uninorm, with respect to the strong negator $N(x) = 1-x$ [13]. Following this result, we propose the following definition of 'consistent fuzzy reference relation':

Definition 1 (U-Consistent Fuzzy Preference Relation)**.** *A fuzzy preference relation $R = (r_{ij})$ on a finite set of alternatives is consistent with respect to U (U-consistent) if the following conditions are verified:*

1. *U is an almost continuous self-dual uninorm, with respect to the strong negator $N(x) = 1-x$.*

2. *$r_{ij} = U(r_{ik}, r_{kj})$ $\forall i,j,k$ such that $(r_{ik}, r_{kj}) \notin \{(0,1),(1,0)\}$.*

Tanino's multiplicative transitivity property [12] under reciprocity is the restriction to the region $[0,1]^2 \backslash \{(0,1),(1,0)\}$ of the following well known almost continuous andlike uninorm

$$U(x,y) = \begin{cases} 0, & (x,y) \in \{(0,1),(1,0)\} \\ \dfrac{xy}{xy + (1-x)(1-y)}, & \text{otherwise} \end{cases} \quad (3)$$

This 'multiplicative' almost continuous uninorm is self-dual, with respect to $N(x) = 1-x$, with $\phi^{-1}(x) = \ln \frac{x}{1-x}$ its generator function [16]. The behaviour of uninorms on the squares $[0, 0.5] \times [0, 0.5]$ and $[0.5, 1] \times [0.5, 1]$ is closely related to t-norms and t-conorms [15]. For the above multiplicative uninorm (3), we have that

$$U(x, y) = \frac{T_U(2x, 2y)}{2} \; \forall x, y \in [0, 0.5]$$

with

$$T_U(x, y) = \frac{xy}{2 - (x + y - xy)} \; \forall x, y \in [0, 1]$$

being the well known Einstein product.

4 Estimating Missing Preference Values

Missing information is a problem that has to be addressed because experts are not always able to provide preference degrees between every pair of possible alternatives. As aforementioned, expression (1) might not be verified even when the preference values of a preference relation are transitive. However, expression (1) can be used as a principle for deriving missing values. Indeed, using just those preference values provided by an expert, expression (1) could be used to estimate those preference values which were not given by that expert because he/she was uncertain as to his/her preference between the alternatives or he/she is unable to compare them. By doing this, we assure that the estimated values are 'compatible' with the rest of the information provided by that expert [7, 10].

Given a fuzzy preference relation R and U a self-dual uninorm with respect to the strong negation $N(x) = 1 - x$, the preference value r_{ik} $(i \neq k)$ can be partially U-estimated using an intermediate alternative x_j such that $(r_{ij}, r_{jk}) \notin \{(0, 1), (1, 0)\}$ as follows

$$ur_{ik}^{j} = U(r_{ij}, r_{jk}) \tag{4}$$

The average of the partially U-estimated values obtained using all possible intermediate alternatives can be seen as the global consistency based estimated value

$$ur_{ik} = \frac{\sum_{j \in R_{ik}^{01}} ur_{ik}^{j}}{\# R_{ij}^{01}} \tag{5}$$

where

$$R_{ik}^{01} = \{j \neq i, k | (r_{ij}, r_{jk}) \notin \{(0, 1), (1, 0)\}\}.$$

Example 1. *For*

$$R = \begin{pmatrix} 0.5 & 0.55 & 0.7 & 0.95 \\ 0.45 & 0.5 & 0.65 & 0.9 \\ 0.3 & 0.35 & 0.5 & 0.75 \\ 0.05 & 0.1 & 0.25 & 0.5 \end{pmatrix}$$

using the multiplicative uninorm (3) we get the following estimated preference values:

$$UR = \begin{pmatrix} 0.5 & 0.62 & 0.78 & 0.9 \\ 0.38 & 0.5 & 0.7 & 0.89 \\ 0.22 & 0.3 & 0.5 & 0.86 \\ 0.01 & 0.11 & 0.14 & 0.5 \end{pmatrix}$$

Expression (5) needs to be extended to include the case when working with an incomplete fuzzy preference relation. To do this, the following sets are introduced:

$$\begin{aligned} A &= \{(i, k) \mid i, k \in \{1, \ldots, n\} \wedge i \neq k\} \\ MV &= \{(i, k) \in A \mid r_{ik} \wedge \; r_{ki} \text{ are unknown}\} \\ EV &= A \setminus MV \\ H_{ik}^{01} &= \{j \in R_{ik}^{01} \; | (i, k) \in MV \wedge \\ & \qquad\qquad (i, j), (j, k) \in EV\} \end{aligned}$$

MV is the set of pairs of alternatives for which the preference degree between them are unknown or missing; EV is the set of pairs of alternatives for which the expert provides preference values; H_{ik}^{01} is the set of intermediate alternatives x_j $(j \neq i,k)$ that can be used to estimate the unknown preference value r_{ik} $(i \neq k)$ via a a self-dual uninorm, U, with respect to the strong negation $N(x) = 1 - x$. The final overall estimated preference value of a missing one, ur_{ik}, can be calculated when $\#H_{ik} \neq 0$, and will be defined as the average of the estimated values obtained using all the possible intermediate alternatives x_j

$$ur_{ik} = \frac{\sum_{j \in H_{ik}^{01}} ur_{ik}^j}{\#H_{ik}^{01}} \; if \; \#H_{ik}^{01} \neq 0. \qquad (6)$$

Note that when the preference relation is complete we have that $MV = \emptyset$ and $H_{ik}^{01} = R_{ik}^{01}$, with means that expression (6) is more general than (5).

An iterative procedure with the first task at each step of it being the identification of the missing preference values r_{ik} $(i \neq k)$ for which there exists at least one intermediate alternative x_j that allows to apply expression (3) can be designed [7]. The overall estimated values obtained at each step of this iterative procedure are added to the already known for the next step, with the procedure ending when no more missing values can be estimated.

5 Conclusions

The assumption of experts being able to quantify their preferences in the domain [0,1] instead of $\{0, 1\}$ underlies unlimited computational abilities and resources from the experts. Taking these unlimited computational abilities and resources into account we have shown that consistency of reciprocal fuzzy preference relations can be mathematically modelled via self-dual uninorms with respect to the strong negation $N(x) = 1 - x$.

In practical cases, this consistency property might obviously not be verified even when the preference values of a preference relation are transitive. However, we have shown that this result can be exploited to design an iterative procedure to estimate missing preference values using only the rest of the preference values provided by a particular expert. By doing this, it is assured that the reconstruction of the incomplete fuzzy preference relation is compatible with the rest of the information provided by the expert.

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