# On the law of importation for some kinds of fuzzy implications derived from uninorms


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## Abstract

The law of importation for fuzzy implications is studied and it is solved for S, R, QL and D-implications derived from uninorms. Along this study many new solutions of this property appears different from those already known for implications derived from t-norms and t-conorms.



**Keywords:** Law of importation, fuzzy implications, uninorms.


## 1 Introduction

It is well known that fuzzy implication functions play an essential role in fuzzy logic and approximate reasoning, as well as in most of fields where this theory is applied (see the recent survey [15]). Many properties of these implications have been extensively studied by several authors along the time. One of them is the so-called *law of importation*,

$$(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r),$$

or equivalently in the context of fuzzy logic

$$I(T(x, y), z) = I(x, I(y, z)) \qquad (1)$$

for all $x, y \in [0, 1]$, with $T$ a t-norm and $I$ a fuzzy implication.

Due to the commutativity of $T$, the law of importation directly implies that the corresponding implication must satisfy also the exchange principle that is a crucial property for fuzzy implications. Moreover, some possible applications of the law of importation were also pointed out in [2].

Although their interest only few results about it are known in the literature. Specifically, a particular case (taking $T$ as the product) was initially studied in [19] where it was proposed as an axiom for the definition of implication. In [1], it was studied jointly with the distributive law

$$I(x, T(y, z)) \equiv T(I(x, y), I(x, z))$$

and those implications satisfying both properties were characterized. Recently, in [2] the law of importation was studied in detail for some kinds of implications including S, R and QL-implications derived from t-norms and t-conorms.

In this paper we want to extend such a study to implications derived from uninorms and we will prove that new solutions appear in this context. In fact, we want to solve the equation

$$I(U_c(x, y), z) = I(x, I(y, z)) \qquad (2)$$

for all $x, y, z \in [0, 1]$ where $U_c$ is a conjunctive uninorm and $I$ is an implication function derived from a uninorm $U$ by one of the methods stated in the preliminaries.

## 2 Preliminaries

We will suppose the reader to be familiar with the theory of t-norms and t-conorms (see [11]). We also assume that some basic facts about uninorms and their different classes are known (see for instance [6]). We recall here only some facts on their derived implications.

**Definition 1** *A binary operator $I : [0,1] \times [0,1] \to [0,1]$ is said to be an implication operator, or an implication, if it satisfies:*

*I1) I is decreasing in the first variable and increasing in the second one.*

*I2) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.*

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not derived from the definition.

Let $U, U_d, U_c$, and $N$ denote an arbitrary uninorm, a disjunctive uninorm, a conjunctive uninorm and a strong negation, respectively. The four most usual ways to define implication functions from uninorms are:

i) *S-implications* defined by

$$I_{U_d,N}(x,y) = U_d(N(x),y) \tag{3}$$

for all $x, y \in [0,1]$.

ii) *R-implications* defined by

$$I_U(x,y) = \sup\{z \in [0,1] \mid U(x,z) \le y\} \tag{4}$$

for all $x, y \in [0,1]$.

iii) *QL-implications* defined by

$$I_{QL}(x,y) = U_d(N(x), U_c(x,y)) \tag{5}$$

for all $x, y \in [0,1]$.

iv) *D-implications*, that are the contraposition with respect to the strong negation $N$ of QL-implications, and are given by

$$I_D(x,y) = U_d(U_c(N(x),N(y)),y) \tag{6}$$

for all $x, y \in [0,1]$.

It is clear that S-implications $I_{U_d,N}$ are always implications, whereas R-implications $I_U$ are implications if and only if $U(0,x) = 0$ for all $x < 1$ (see [4]). With respect to $I_{QL}$ and $I_D$ the results are more complicated and they can be found for instance in [14]. We will recall along the text the necessary details on these kinds of implications.

Obviously, S, R, QL and D-implications can be obtained also from t-norms and t-conorms. With respect to S-implications and R-implications derived from t-norms the following results on the law of importation are known (see [2]).

**Theorem 1** *An S-implication derived from a strong negation $N$ and a t-conorm $S$ satisfies the law of importation with a t-norm $T$ if and only if $T$ is the $N$-dual t-norm of $S$.*

**Theorem 2** *An R-implication derived from a left-continuous t-norm $T$ satisfies the law of importation with a t-norm $T_1$ if and only if $T = T_1$.*

For the case of QL-implications only some partial results on the law of importation are known (see [2]). Before we give them we need some notations. Recall that any strong negation is isomorphic to the standard one $x \to 1 - x$ (see [18]) and thus, it must be given by

$$N_\varphi(x) = \varphi^{-1}(1 - \varphi(x)) \text{ for all } x \in [0,1]$$

where $\varphi : [0,1] \to [0,1]$ is an increasing bijection.

Given a binary operation $S : [0,1]^2 \to [0,1]$, we will denote also by $S_\varphi$ the $\varphi$-conjugate of $S$, that is,

$$S_\varphi(x,y) = \varphi^{-1}(S(\varphi(x),\varphi(y)))$$

for all $x, y \in [0,1]$. With these notations, when $S$ is a continuous t-conorm, a necessary condition for the operation $I_{QL}$ to be an implication is that there must exist an increasing bijection $\varphi$ such that $S = W^*_\varphi$, where $W^*$ is the Łukasiewicz t-conorm, and $N \ge N_\varphi$ (see [13]). In the special case when $N = N_\varphi$ the following results on the law of importation are known (see [2]).

**Theorem 3** *Let $\varphi : [0,1] \to [0,1]$ be an increasing bijection, $S = W^*_\varphi$, $N = N_\varphi$ and $T$ a t-norm. If $T$ is the minimum or the $\varphi$-conjugate of the product or of the Łukasiewicz t-norm, the corresponding $I_{QL}$ is always a QL-implication and,* *i) If $T$ is the $\varphi$-conjugate of the Łukasiewicz t-norm, then $I_{QL}$ satisfies the law of importation with a t-norm $T'$ if and only if $T'$ is the minimum t-norm.*

*ii) If $T$ is the $\varphi$-conjugate of the product t-norm, then $I_{QL}$ satisfies the law of importation with a t-norm $T'$ if and only if $T' = T$.*

*iii) If $T$ is the minimum t-norm, then $I_{QL}$ satisfies the law of importation with a t-norm $T'$ if and only if $T'$ is the $\varphi$-conjugate of the Łukasiewicz t-norm.*

The law of importation for D-implications derived from t-norms and t-conorms has not been studied yet, but it is similar to the case of QL-implications and it will be included in this paper in the corresponding section before studying the general case of D-implications derived from uninorms.

## 3 Main results

Our main goal in this paper is to solve the law of importation (equation (2)) where $I$ is an implication derived from some uninorms and $U_c$ is a conjunctive uninorm. We divide our study in several subsections one for each kind of implication that we want to deal with. That is, S-implications, R-implications, QL-implications and D-implications derived from uninorms.

### 3.1 S-implications

In this section we deal with equation (2) where $I = I_{U,N}$ is an S-implication derived from a strong negation $N$ and a disjunctive uninorm $U$. This case has a trivial solution identical to the case of S-implications derived from t-conorms (see Theorem 1).

Before we state the result, let us recall that given any strong negation $N$, the $N$-dual of a uninorm $U$ is also a uninorm $U_N$, given by

$$U_N(x, y) = N(U(N(x), N(y)))$$

for all $x, y \in [0, 1]$. It is well known and clear from the equation above that the $N$-dual of a conjunctive uninorm is a disjunctive one and vice versa. Then we have the following result.

**Theorem 4** *Let $U_d$ be a disjunctive uninorm, $N$ a strong negation and $I_{U_d,N}$ its associated S-implication. Then $I_{U_d,N}$ satisfies (2) for a conjunctive uninorm $U_c$ if and only if $U_c$ is the $N$-dual of $U_d$.*

### 3.2 R-implications

In this section we deal with equation (2) where $I = I_U$ is an R-implication derived from a uninorm $U$ with neutral element $e \in ]0, 1[$. Recall that in this case the uninorm $U$ must satisfy $U(x, 0) = 0$ for all $x < 1$. In the case that $U$ is left-continuous, the situation is similar to the case of R-implications derived from t-norms and we have the following result.

**Theorem 5** *Let $U$ be a left-continuous conjunctive uninorm*[1] *and $I_U$ its associated R-implication. Then $I_U$ satisfies (2) with a conjunctive uninorm $U_c$ if and only if $U_c = U$.*

Note that $I_U$ satisfies the residuation property:

$$U(x, y) \leq z \iff y \leq I_U(x, z)$$

if and only if $U$ is precisely left-continuous and conjunctive (see [8]). However, among the known classes of uninorms from which R-implications can be derived, there are a lot of non-left-continuous uninorms. In the case when $U$ is not left-continuous we will see that the corresponding $I_U$ can satisfy (2) with the same $U$ or not. Specifically, we will see that there are non-left-continuous uninorms $U$ such that $I_U$ satisfies (2) with the same uninorm $U$, and others such that $I_U$ satisfies it with another conjunctive uninorm. We do this by discussing the situation depending on the class of the uninorm $U$.

#### 3.2.1 $U$ is a representable uninorm

Any representable uninorm $U$ with neutral element $e$ and additive generator $h$ will be represented by $U = \langle h, e \rangle_{\text{rep}}$. But with the same

[1] If $U$ is left continuous and $U(x, 0) = 0$ for all $x < 1$ then necessarily $U(1, 0) = 0$ and $U$ is conjunctive.

$e$ and $h$ there are two different representable uninorms, one conjunctive and the other disjunctive. To distinguish between them we will denote by $\langle h, e, \wedge\rangle_{\text{rep}}$ the conjunctive version and by $\langle h, e, \vee\rangle_{\text{rep}}$ the disjunctive one.

Now, it is easy to prove the following result using Theorem 5.

**Theorem 6** *Let $U_c$ be a conjunctive uninorm, $U = \langle h, e\rangle_{\text{rep}}$ a representable uninorm and $I_U$ its R-implication. Then $I_U$ satisfies the law of importation with $U_c$ if and only if $U_c$ coincides with $\langle h, e, \wedge\rangle_{\text{rep}}$ (the conjunctive version of $U$).*

Note that from the previous theorem we have examples of not left-continuous uninorms for which the corresponding $I_U$ satisfies (2) with a uninorm $U_c$ different from the same $U$ (just take any disjunctive representable uninorm). However, given $U = \langle h, e\rangle_{\text{rep}}$, there is only a uninorm $U_c$ for which $I_U$ satisfies (2) with $U_c$.

### 3.2.2 $U$ is an idempotent uninorm

Let $e \in ]0, 1[$ and let $g : [0, 1] \to [0, 1]$ be a decreasing function with $g(e) = e$ such that $g(x) = 0$ for all $x > g(0)$, $g(x) = 1$ for all $x < g(1)$ and

$$\inf\{y \mid g(y) = g(x)\} \leq g^2(x) \leq \sup\{y \mid g(y) = g(x)\}$$

for all $x \in [0, 1]$. An idempotent uninorm with neutral element $e$ and associated function $g$ will be denoted by $U = \langle g, e\rangle_{\text{ide}}$ and it is given by $U(x, y) =$

$$\begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or} \\ & \big(y = g(x) \text{ and } x < g^2(x)\big) \\ \max(x,y) & \text{if } y > g(x) \text{ or} \\ & \big(y = g(x) \text{ and } x > g^2(x)\big) \\ \min(x,y) & \\ \text{or} & \text{if } y = g(x) \text{ and } x = g^2(x) \\ \max(x,y) & \end{cases}$$

being commutative in the points $(x, g(x))$ with $g^2(x) = x$ (see [12]). For details on this kind of uninorms see [3] and [12], and see also [4] and [16] for details on their R-implications.

Note however that there can be a lot of different idempotent uninorms with the same $e$ and $g$, because the values on points $(x, g(x))$ with $g^2(x) = x$ can vary. Recall also that R-implications can be obtained from idempotent uninorms if and only if $g(0) = 1$ and thus, along this section, we will suppose that $U$ satisfies this condition.

Let us begin studying equation (2) in the particular case when $U_c = U$.

**Proposition 1** *Let $U = \langle g, e\rangle_{\text{ide}}$ be an conjunctive idempotent uninorm and $I_U$ its residual implication. Then $I_U$ satisfies the law of importation with the same $U$ if and only if $U$ satisfies the following condition:*

$$U(x, g(x)) = \min(x, g(x)) \textit{ when } g^2(x) = x.$$

*That is, $U$ must be given by $U(x, y) =$*

$$\begin{cases} \max(x,y) & \textit{if } y > g(x) \textit{ or} \\ & \big(y = g(x) \textit{ and } x > g^2(x)\big) \\ \min(x,y) & \textit{otherwise.} \end{cases} \quad (7)$$

In particular, all left-continuous idempotent uninorms satisfy the condition in the proposition above since for them $U(x, g(x)) = \min(x, g(x))$ for all $x \in [0, 1]$. However, note again that there are non-left-continuous conjunctive uninorms satisfying the condition in the proposition above. It suffices to take an idempotent uninorm in $\mathcal{U}_{\min}$ for which the associated function $g$ is given by

$$g(x) = \begin{cases} 1 & \text{if } x < e \\ e & \text{if } x \geq e. \end{cases}$$

With respect to the general case we have the following results.

**Proposition 2** *Let $U = \langle g, e\rangle_{\text{ide}}$ be an idempotent uninorm and $I_U$ its residual implication. If $I_U$ satisfies the law of importation with a conjunctive uninorm $U_c$ then $U_c$ must be idempotent, say $U_c = \langle g_c, e_c\rangle_{\text{ide}}$ and $e_c = e$ and $g_c = g$.*

**Theorem 7** *Let $U = \langle g, e\rangle_{\text{ide}}$ be an idempotent uninorm and $I_U$ its residual implication.*

*Then $I_U$ satisfies (2) with a conjunctive uninorm $U_c$ if and only if $U_c$ is also idempotent, say $U_c = \langle g_c, e_c\rangle_{\text{ide}}$ with $e_c = e$, $g_c = g$, and $U_c$ is given by equation (7).*

Note that from the theorem above, given a conjunctive idempotent uninorm $U_c = \langle g_c, e_c\rangle_{\text{ide}}$ with $g(0) = 1$, all idempotent uninorms $U$ with the same neutral element $e_c$ and the same associated function $g_c$ have the same residual implication $I_U$, and this implication satisfies the law of importation with $U_c$. Note that when $U$ is left-continuous, necessarily $g^2(x) \geq x$ for all $x \in [0,1]$ (see [3]) and consequently the only $U_c$ such that $I_U$ satisfies the law of importation with $U_c$ is the left-continuous one, that is, the proper uninorm $U$.

#### 3.2.3 $U$ is a uninorm in $\mathcal{U}_{\min}$

It is well known that a uninorm $U$ in $\mathcal{U}_{\min}$ is totally determined by its underlying t-norm and t-conorm. In fact, it is given by $U(x,y) = \min(x,y)$ for all $x, y \in [0,1]$ such that $\min(x,y) < e < \max(x,y)$ (see [7] for details on this kind of uninorms). We will denote a uninorm $U$ in $\mathcal{U}_{\min}$ by $U = \langle T, e, S\rangle_{\min}$. Note that in this case all considered uninorms are in fact non-left-continuous but we will obtain again solutions of the law of importation.

In what follows we will suppose that the underlying t-norm $T$ and t-conorm $S$ of the uninorm $U$ in $\mathcal{U}_{\min}$ are left-continuous.

**Proposition 3** *Let $U$ be a uninorm in $\mathcal{U}_{\min}$ and $I_U$ its residual implication. Then $I_U$ always satisfies the law of importation with the same $U$.*

In fact there are no other solutions when $U$ is in $\mathcal{U}_{\min}$ but the proof of this fact is not trivial and it needs a previous result.

**Proposition 4** *Let $U$ be a uninorm in $\mathcal{U}_{\min}$ and $I_U$ its residual implication. If $I_U$ satisfies the law of importation with a conjunctive uninorm $U_c$ then $U_c$ must be also in $\mathcal{U}_{\min}$, say $U_c = \langle T_c, e_c, S_c\rangle_{\min}$, and moreover, $e_c = e$.*

Now, it can be proved that, when $I_U$ satisfies the law of importation with $U_c$, not only $U_c$ must be in $\mathcal{U}_{\min}$ with $e_c = e$, but also the underlying t-norm and t-conorm of $U_c$ must coincide with those of $U$. From these results the final characterization can be easily derived.

**Theorem 8** *Let $U$ be a uninorm in $\mathcal{U}_{\min}$ with continuous underlying t-conorm and $I_U$ its residual implication. Then $I_U$ satisfies the law of importation with a conjunctive uninorm $U_c$ if and only if $U_c = U$.*

#### 3.2.4 $U$ is a uninorm continuous at $]0,1[^2$

This kind of uninorms can be divided in two groups that we will denote by $\mathcal{U}_{\cos,\min}$ and $\mathcal{U}_{\cos,\max}$ and whose structures can be viewed in figures 1 and 2, respectively (more details on these uninorms can be found in [9]). From these structures, uninorms $U$ in $\mathcal{U}_{\cos,\min}$ will be denoted by $U = \langle T_1, \alpha, T_2, \beta, (R,e)\rangle_{\cos,\min}$ and those in $\mathcal{U}_{\cos,\max}$ by $U = \langle (R,e), \gamma, S_1, \delta, S_2\rangle_{\cos,\max}$.

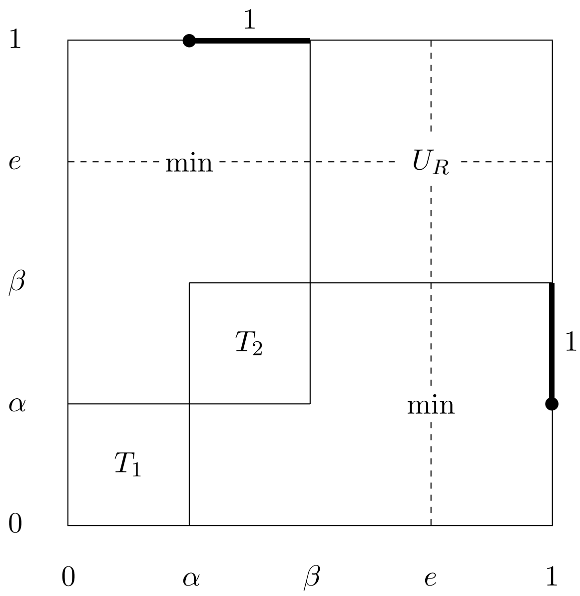


Figure 1: A uninorm $U = \langle T_1, \alpha, T_2, \beta, (R,e)\rangle_{\cos,\min}$ in $\mathcal{U}_{\cos,\min}$

An R-implication can be derived from any uninorm in $\mathcal{U}_{\cos,\min}$ and with techniques similar to those used for representable uninorms and uninorms in $\mathcal{U}_{\min}$ the following result can be obtained.

**Theorem 9** *Let* $U =$

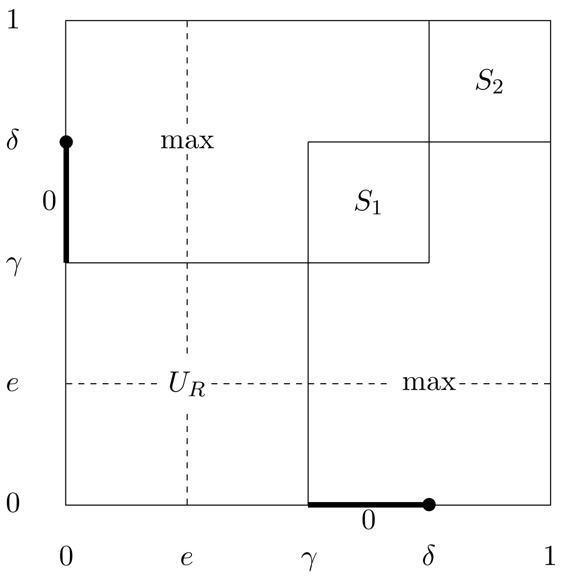


Figure 2: A uninorm $U = \langle (R,e), \gamma, S_1, \delta, S_2 \rangle_{\cos,\max}$ in $\mathcal{U}_{\cos,\max}$

$\langle T_1, \alpha, T_2, \beta, (R,e) \rangle_{\cos,\min}$ *be a uninorm in* $\mathcal{U}_{\cos,\min}$ *and* $I_U$ *its residual implication. Then* $I_U$ *satisfies the law of importation with a conjunctive uninorm* $U_c$ *if and only if* $U_c(\alpha, 1) = U_c(1, \alpha) = \alpha$ *and* $U_c = U$ *except maybe in these points.*

With respect to uninorms in $\mathcal{U}_{\cos,\max}$, R-implications can be derived from them if and only if $\delta = 1$ (see [17]). We will denote a uninorm in this subclass by $U = \langle (R,e), \gamma, S, 1 \rangle_{\cos,\max}$. In this case there are no new solutions of the law of importation as it follows from the following theorem.

**Theorem 10** *Let* $U$ *be a uninorm in* $\mathcal{U}_{\cos,\max}$ *with* $\delta = 1$, *say* $U = \langle (R,e), \gamma, S, 1 \rangle_{\cos,\max}$. *Let* $I_U$ *be its residual implication, then* $I_U$ *satisfies the law of importation with a conjunctive uninorm* $U_c$ *if and only if* $\gamma = 1$ *(that is,* $U$ *is representable, say* $\langle h, e \rangle_{\mathrm{rep}}$*) and* $U_c = \langle h, e, \wedge \rangle_{\mathrm{rep}}$.

### 3.3 QL-implications

In this section we deal with those binary operations obtained by

$$I_{QL}(x,y) = U_d(N(x), U_c(x,y)) \qquad (8)$$

for all $x, y \in [0,1]$, where $U_d$ is a disjunctive uninorm, $N$ is a strong negation and $U_c$ is a conjunctive uninorm, that will be called QL-operators (from uninorms). It is proved in [14] the following necessary condition for $I_{QL}$ to be an implication: $U_d$ must be a t-conorm, say $S$, such that

$$S(x, N(x)) = 1 \quad \text{for all} \quad x \in [0,1]. \qquad (9)$$

Although this condition is not sufficient (see [14]), it is enough to easily derive the following result with respect to the law of importation.

**Proposition 5** *Let* $S$ *be a t-conorm satisfying (9),* $U$ *a conjunctive uninorm and* $N$ *a strong negation. If the corresponding QL-operator satisfies the law of importation with a conjunctive uninorm* $U_c$*, then* $U_c$ *and* $U$ *are both t-norms.*

Thus, from the proposition above we obtain that in this case there are no new solutions of the law of importation different from those already known for QL-implications derived from t-norms and t-conorms (see [2] or also Theorem 3).

### 3.4 D-implications

In this section we want to study the case of D-implications, that is, those given by equation (6). We do this in two steps, one devoted to D-implications derived from t-norms and t-conorms, and the other devoted to D-implications derived from uninorms. Before we recall some general results.

Operations obtained by

$$I_D(x,y) = U_d(U(N(x), N(y)), y) \qquad (10)$$

for all $x, y \in [0,1]$, where $U_d$ is a disjunctive uninorm, $N$ is a strong negation and $U$ is a conjunctive uninorm, will be called D-operators (from uninorms). In order for $I_D$ to be an implication, it is proved in [14] that necessarily $U_d$ must be a t-conorm $S$ satisfying (9). In the case that $S$ is continuous then it must exist an increasing bijection $\varphi : [0,1] \to [0,1]$ such that $S = W_\varphi^*$ and

$N \geq N_\varphi$. Again this condition is not sufficient and the situation is different depending on the fact that $U$ is a t-norm or a conjunctive uninorm. However, as it is done for QL-implications, note that for D-implications we will deal only with the special case when $N = N_\varphi$, as much for t-norms as for uninorms.

### 3.4.1 D-implications from t-norms

In this case we have operations of the form

$$I_D(x, y) = W^*_\varphi(T(N_\varphi(x), N_\varphi(y)), y)$$

for all $x, y \in [0, 1]$, where $T$ is a t-norm. This is equivalent to say (see [13]) that

$$I_{\varphi,T}(x, y) = \varphi^{-1}(\varphi(T(N_\varphi(x), N_\varphi(y)))+\varphi(y)). \quad (11)$$

Now some partial results can be derived as in the case of QL-implications. In fact, when $T$ is the minimum or the $\varphi$-conjugate of the product or of the Łukasiewicz t-norm, it is easy to see, from equation (11), that D-implications derived from $T$ coincide with the corresponding QL-implications derived from $T$. Consequently the same Theorem 3 applies for D-implications.

### 3.4.2 D-implications from uninorms

We can begin with a general result for D-operators.

**Proposition 6** *Let $S$ be a t-conorm satisfying (9), $U$ a conjunctive uninorm and $N$ a strong negation. If the corresponding D-operator satisfies the law of importation with a conjunctive uninorm $U_c$, then $U_c$ must be a t-norm.*

Now, we will study the case when $U_d = W^*_\varphi$ and $N = N_\varphi$ for an an increasing bijection $\varphi : [0, 1] \to [0, 1]$. That is, D-operators given by $I_{\varphi,U}(x, y) =$

$$\varphi^{-1}(\min(\varphi(U(N_\varphi(x), N_\varphi(y))) + \varphi(y)), 1). \quad (12)$$

where $U$ is a conjunctive uninorm. It is proved in [14] that, among the known classes of uninorms, only a special kind of uninorms in $\mathcal{U}_{\min}$ satisfy that the corresponding $U_{\varphi,U}$ is a D-implication. Moreover, in this case these D-implications are given by $I_{\varphi,U}(x, y) =$

$$\begin{cases} 1 & \text{if } x \leq N_\varphi(e) \\ \varphi^{-1}(1 - \varphi(x) + \varphi(y)) & \text{if } y \leq N_\varphi(e) < x \\ B(x, y) & \text{if } x, y \geq N_\varphi(e) \end{cases} \quad (13)$$

where $B(x, y) =$

$$\varphi^{-1}\left(\varphi\left(eT_U\left(\frac{N_\varphi(x)}{e}, \frac{N_\varphi(y)}{e}\right)\right) + \varphi(y)\right)$$

where $T_U$ is the underlying t-norm of $U$.

For this kind of operations we have the following result.

**Proposition 7** *Let $\varphi : [0, 1] \to [0, 1]$ be an increasing bijection and $U$ a uninorm in $\mathcal{U}_{\min}$ such that the corresponding D-operator $I_{\varphi,U}$ given by (13) is a D-implication. If $I_{\varphi,U}$ satisfies the law of importation with a conjunctive uninorm $U_c$ then both $U_c$ and $U$ are t-norms.*

That is, for the case of D-implications derived from uninorms, no new solutions appear different from those already known for D-implications derived from t-norms.

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