Stratification in the Type-Reduced Set and the Generalised Karnik-Mendel Iterative Procedure

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Abstract

Type-2 defuzzification is a two-stage process. Type-reduction, the first stage, gives rise to a type-1 fuzzy set known as the type-reduced set (TRS), which for the second stage is defuzzified to give a crisp number. This paper focusses on the TRS itself, which reveals a fascinating internal structure. We provide a description of and explanation for this structure, then go on to suggest how it may be exploited to enable the application of the Karnik-Mendel Iterative Procedure to generalised type-2 fuzzy sets.

Keywords: Type-Reduced Set, Stratification, Type-2, Type-Reduction, Defuzzification.

1 Background

The research presented here emerged as we were investigating why the sampling method of type-reduction (section 2.4) worked so efficiently with tiny sample sizes. During this piece of research (which is still ongoing), we focussed on the type-reduced set (TRS), which we came to realise was worthy of study in its own right, and furthermore had properties which could be exploited in a generalisation of the Karnik-Mendel Iterative Procedure [3].

The sampling method itself (subsection 2.4) was put forward as a strategy for making generalised type-2 fuzzy inferencing less computationally complex by alleviating the defuzzification bottleneck (subsection 2.4) in order to open the way for the practical development of generalised type-2 applications. Type-1 and interval type-2 fuzzy sets have dominated the field of fuzzy inferencing, but we contend that *speedy* generalised type-2 fuzzy inferencing is achievable and highly desirable.

For continuous generalised type-2 fuzzy sets, Coupland and John [1] have proposed a geometric approach to fuzzy inferencing. However this article concerns the *discretised* type-2 fuzzy set (T2FS). We first (section 2) look at the formation of the type-reduced set, moving on to consider its structure and properties (sections 3 and 4). Lastly (sections 5 and 6) we exploit the TRS structure in order to generalise the Karnik-Mendel Iterative Procedure.

2 Formation of the Type-Reduced Set

2.1 Fuzzy Inferencing Systems

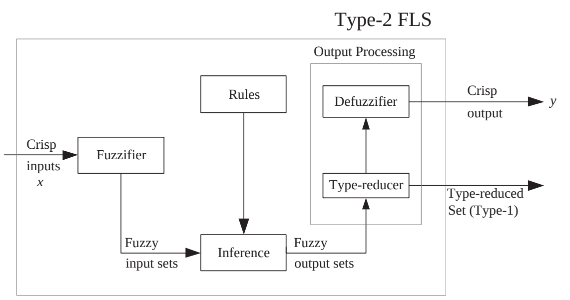

Figure 1: Type-2 FIS (from Mendel [5], page 288).

The type-reduced set is normally created as part of the processing of a type-2 fuzzy inferencing system (FIS). An FIS is a computerised aid

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to decision making, which uses fuzzy sets, and operates by applying fuzzy logic operators to common-sense linguistic rules. An FIS can be of any type; its type is determined by the highest type of the fuzzy sets it employs. Here we are concerned with a type-2 FIS in the Mamdani style. An FIS (of any type) usually starts with a crisp number, and works through the five stages of fuzzification, antecedent computation, implication, aggregation/combination of consequents, and defuzzification. In a type-2 FIS, the defuzzification stage consists of two parts, *type-reduction* and defuzzification proper, as shown in figure 1:

Type-Reduction The T2FS is reduced to a type-1 set, the type-reduced set by exploiting the embedded type-2 fuzzy sets (section 2.2).

Type-1 Defuzzification To arrive at a single number to represent the T2FS, this TRS is defuzzified using one of the many type-1 defuzzification approaches.

The TRS as the Centroid of the T2FS The TRS is the result of the type-reduction process. Mathematically, we interpret it as the centroid of a type-2 set as it represents, in a sense, the centre of the T2FS ([5], page 10).

2.2 Embedded Sets

Type-2 type-reduction is dependent on the concept of an *embedded set*. The concept of a type-2 embedded set follows directly from Mendel and John's representation theorem [7]. Figure 2 shows two embedded sets of a T2FS discretised into slices with a separation of 0.1 units. The number of sets in a T2FS is dependent upon the discretisation technique, the spacing of the discretising slices, and the shape of the FOU[1].

2.3 Algorithm for Type-Reduction

Algorithm 1 is the type-reduction algorithm originally described by Mendel [5]. The TRS produced by this algorithm, as it is a type-1 set, may be easily defuzzified by calculating its centroid.

[1]FOU stands for *Footprint Of Uncertainty*, the projection of the T2FS on the $x - y$ plane

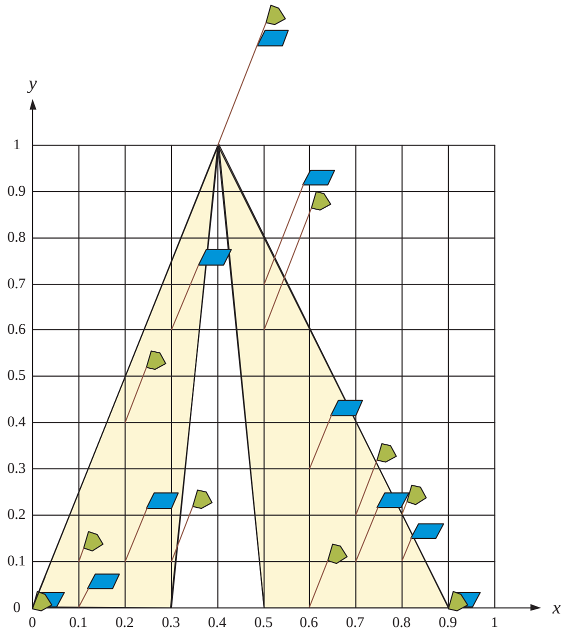

Figure 2: Two embedded sets, indicated by different flag styles. The flag height reflects the secondary membership grade. Both the domain and co-domain are discretised into 11 slices. The shaded region is the FOU.

2.4 The Sampling Method of Defuzzification

The type-reduction algorithm as presented in section 2.3 involves the processing of *all* the embedded sets within the original discretised type-2 set. These sets are very numerous. For instance, when a prototype type-2 FIS performed an inference using sets which had been discretised into 51 slices across both the x and y-axes, the number of embedded sets in the aggregated set was calculated to be in the order of 2.9×10^{63}. Though individually easily processed, embedded sets in their totality give rise to a processing bottleneck simply by virtue of their high cardinality.

The sampling method of defuzzification [2] is an efficient, cut-down alternative to dealing with all the embedded sets. In this technique, only a relatively small random sample of the totality of embedded sets is processed. The resultant defuzzified value, though surprisingly accurate, is nonetheless an approximation.

Algorithm 1 Type-Reduction of a T2FS to a Type-1 Fuzzy Set

for all embedded sets **do**
    find the minimum secondary membership grade
    calculate the domain value of the type-1 centroid of the type-2 embedded set
    pair the secondary grade with the domain value to produce a set of ordered pairs (x,z) $\{$some values of x may correspond to more than 1 value of $z\}$
end for
for all domain values **do**
    select the maximum secondary grade $\{$each x corresponds to a unique co-domain value$\}$
end for

3 Structure of the TRS

This section describes the structure and properties of the TRS. The membership function of the TRS of a discretised T2FS may be thought of as a set of tuples. Figures 3 to 5 show typical TRSs derived from different sized samples of randomly generated embedded sets (section 2.4), originating from the same discretised T2FS.

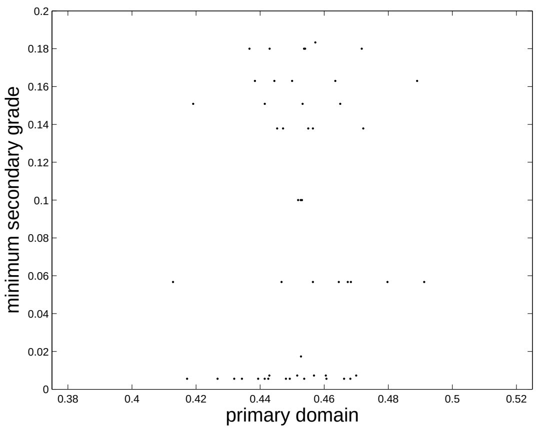

Figure 3: The TRS strata. Sample size = 50.

After producing numerous TRS graphs (all from T2FSs with triangular secondary membership functions) one observation stood out: *The TRS is stratified*. The explanation for the appearance of strata is that they derive directly from the original T2FS, and are artifacts produced by the combination of the processes of discretisation and fuzzy inferencing. Since during type-reduction the minimum secondary grade of each embedded set is selected, unsurprisingly the same minimum values appear repeatedly, for different domain values.

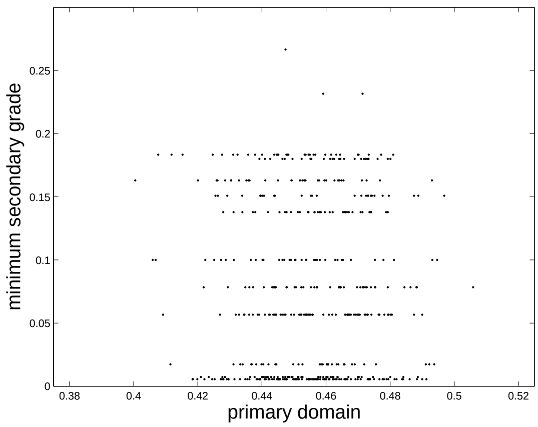

Figure 4: The TRS strata. Sample size = 500.

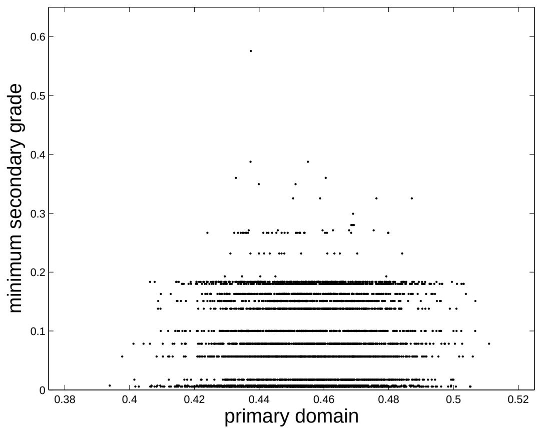

Figure 5: The TRS strata. Sample size = 5000.

Further characteristics of the stratified structure are:

- Its general shape is trapezoidal i.e. progressing upwards the strata tend to narrow. However it cannot be assumed that any given stratum is narrower than the one directly below it.
- It is non-symmetrical (though it often appears almost symmetrical).
- The widest stratum is low in height, though it is not necessarily the lowest stratum.

- The heights of the lowest strata tend to be very small; how small depends on the fineness of the discretisation.

- On the whole the lower strata are denser.

Future research will ascertain whether these observations are invariably true. The impact, if any, of using secondary membership functions other than triangular is to be explored. The rest of this paper does not rely on these comments.

4 Mendel's Probability Analysis of the Stratified Structure

In relation to the sampling method of defuzzification (section 2.4), Mendel [6] has observed that as the domain discretisation becomes finer, the probability of a randomly selected embedded set containing at least one of the minimum secondary grades (from one of the secondary membership functions) approaches 1. We reproduce Mendel's argument:

> Assumptions:
>
> 1. Primary variable x is discretized into N values $x_1, x_2, \ldots, x_N$.
> 2. We are free to choose N, e.g. make it as small as we choose.
> 3. All primary memberships are discretized into the same number of levels, M. (Even if you do not do this, the analysis below is interesting, and can be modified to the case of non-equal discretization.)
> 4. The smallest secondary grade for each of the N secondary MFs occurs exactly one time in each of the secondary MFs (this is controversial, but it could be changed with a more complicated analysis).
>
> My first goal is to compute the probability of choosing an embedded T2 FS that contains at least one of the minimum secondary grades.
>
> 1. The total number of embedded T2 FSs is M^N.
> 2. The total number of embedded T2 FSs that do not contain at least one of the minimum secondary values is $(M-1)^N$.
> 3. The total number of embedded T2 FSs that contain at least one of the minimum secondary values is $M^N - (M-1)^N$.
> 4. The probability of choosing an embedded T2 FS that contains at least one of the minimum secondary values is $p(1 \text{ or more}|M,N)$, where
>
> $$p(1 \text{ or more}|M,N) = \frac{M^N - (M-1)^N}{M^N}. \tag{1}$$
>
> Next, I want to study $p(1 \text{ or more}|M,N)$, especially as N increases. Note from (1) that
>
> $$p(1 \text{ or more}|M,N) = 1 - \left(\frac{M-1}{M}\right)^N. \tag{2}$$
>
> Because $(M-1)/M < 1$, it is true that:
>
> **Fact:** As N increases
>
> $$p(1 \text{ or more}|M,N) \rightarrow 1 \quad \blacksquare \tag{3}$$

Mendel's argument shows convincingly that as the number of vertical slices increases, the probability of a randomly selected embedded set containing at least one of the minimum secondary grades of a secondary membership function tends to 1.

Table 1 shows the relationship between M and N and $p(1 \text{ or more}|M,N)$. It is readily apparent that with increasing N, $p(1 \text{ or more}|M,N)$ increases, tending towards 1. Even with an N of only 50, the probability is within 1% of 1. It is also evident that as M increases, $p(1 \text{ or more}|M,N)$ decreases. The effect of increasing M is opposed to that of increasing N. However their effects are not equally balanced; the higher N becomes, the lower the impact of M.

5 Mathematical Formulation of the Stratified Structure

5.1 Assumptions and Definitions

The TRS as a Crisp Set Any non-continuous type-1 fuzzy set A may be thought of as a

Table 1: Table showing how $p(1 \text{ or more}|M,N)$ varies with M and N.

M╲N	**10**	**20**	**50**	**100**	**200**
10	0.6513	0.8784	0.9948	0.9999	0.9999
20	0.4013	0.6415	0.9231	0.9941	0.9999
50	0.1829	0.3324	0.6358	0.8674	0.9824
100	0.0956	0.1821	0.3950	0.6340	0.8660
200	0.0489	0.0954	0.2217	0.3942	0.6330

crisp, finite, set of co-ordinate points, i.e. $A \equiv \{(x, \mu_A(x))\}$. The TRS of a discretised T2FS may therefore be regarded as a *crisp* set of tuples.

Definition 1 (Cardinality)**.** *For type-1 fuzzy set A, $\mid A \mid$, the cardinality of A, is the number of tuples in A.*

Scalar Cardinality The concept of *scalar cardinality* is frequently encountered in the following analysis. The following definition of scalar cardinality for type-1 fuzzy sets is adapted from Klir and Folger ([4], p17):

Definition 2 (Scalar Cardinality)**.** *"The* scalar cardinality *of a fuzzy set A defined on a finite universal set X is the summation of the membership grades of all the elements of X in A. Thus,*

$$|| A || = \sum_{x \in X} \mu_A(x). \text{ ''}$$

We note that the symbol '$\sum$' as used here represents 'sum'.

We now formally define a stratum.

Definition 3 (Stratum)**.** *A* stratum *is a subset*[2] *of TRS T for which every element has the same μ co-ordinate. Let S_ω be a stratum. Then*

$$\begin{aligned} S_\omega &= \{\{x, \mu_T(x)\} \in T \mid \mu_T(x) = \omega\} \\ &\quad \textit{for some } \omega \in [0,1]. \end{aligned}$$

It follows that a type-1 fuzzy set is the union of all its strata, i.e.

$$T = \bigcup_{i=1}^{j} S_{\omega_i}.$$

[2]In the commonly accepted 'crisp' sense of the word. Klir and Folger [4], p19, give a different definition of 'subset' in the type-1 context.

Moreover the strata *partition* the type-1 fuzzy set.

The concept of a stratum is related to that of an α-cut. Klir and Folger ([4], p16) define an α-cut as follows:

Definition 4 (**α**-Cut)**.** *"An α-cut of a fuzzy set A is a crisp set A_α that contains all the elements of the universal set X that have a membership grade in A greater than or equal to the specified value of α. This definition can be written as*

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}. \text{ ''}$$

Thus an α-cut is a set of domain values. On similar lines we propose the following definition of an α-*support*:

Definition 5 (**α**-Support)**.** *An* α-support *of a type-1 fuzzy set T is a crisp set $\underline{\underline{T}}_\alpha$ that contains all the elements of the universal set X that have a membership grade in T equal to the specified value of α. This definition can be written as*

$$\underline{\underline{T}}_\alpha = \{x \in X \mid \mu_T(x) = \alpha\}.$$

Each stratum corresponds to an α-support, and vice versa. The difference between the two concepts is that the stratum is a set of co-ordinates, but the α-support is a set of domain values, the associated co-domain value being specified by the subscript of the α-support, e.g. $\underline{\underline{T}}_{0.2}$.

5.2 Strata Representation Theorem

Theorem 1 (Strata Representation Theorem)**.** *A type-1 fuzzy set may be represented as the union of its strata.*

Proof. This is a consequence of the fact that any type-1 fuzzy set can be partitioned into its strata. A finite type-1 fuzzy set T is specified if and only if all ordered pairs $(x, \mu_T(x))$ are specified. Consider the level set of T, $\Lambda_T = \{\alpha_1, \alpha_2, \ldots, \alpha_h\}$, where α_1 is the lowest α-support and α_h the highest. $T_{\alpha_i} = \{x \in X \mid \mu_T(x) = \alpha_i\}$, $\forall i = 1 \ldots h$. By listing all the co-ordinates (x_1, α_1) to (x_n, α_1), then (x_1, α_2) to (x_n, α_2), and ultimately (x_1, α_h) to (x_n, α_h) all points $(x, \mu_T(x))$ are specified stratum by stratum. □

5.3 Centroid Formula Exhibiting Strata

The TRS, like any type-1 fuzzy set, is often defuzzified by finding the x co-ordinate of its centroid (section 2.3), using the formula:

$$x = \frac{\sum \mu_i x_i}{\sum \mu_i},$$

where μ_i is the height of the i^{th} TRS point, x_i is its x co-ordinate, and x the defuzzified value. Let T be a TRS which has h strata of heights $\alpha_1, \alpha_2, \ldots, \alpha_h$ in ascending order. Then re-ordering the products in the numerator according to the stratum upon which they lie, we obtain

$$x = \frac{\sum_1^{n_1} \alpha_1 x_{1_i} + \cdots + \sum_1^{n_h} \alpha_h x_{h_i}}{\|T\|},$$

which may be rewritten

$$x = \frac{\alpha_1 \sum_1^{n_1} x_{1_i} + \cdots + \alpha_h \sum_1^{n_h} x_{h_i}}{\|T\|}.$$

6 The Generalised Karnik-Mendel Iterative Procedure

The most widely adopted method for type-reducing an interval type-2 fuzzy set is the Karnik-Mendel Iterative Procedure (KMIP) [3]. The result of type-reduction of an interval type-2 fuzzy set is an interval type-1 set where the centroid lies between the two endpoints. The iterative procedure is an efficient method for finding these endpoints. The centroid of the type-1 set (i.e. the defuzzified value of the T2FS) is taken to be the centre of this interval.

In section 5.3 we showed how the defuzzified value may be calculated using a stratum by stratum approach. We now take the idea of defuzzification by strata a stage further with the simple observation that each stratum is like an interval set, the only difference being the stratum height (figure **??**). So why not defuzzify each stratum as if it were an interval set, and combine the results appropriately to give the defuzzified value of the T2FS?

6.1 Assumptions and Definitions

Lowest and Highest Domain Values

Definition 6 (Lowest Domain Value)**.** *We define the* lowest domain value *of a stratum to be the least domain value of all the points in the stratum. Symbolically,*

$$LDV = \min \underset{=\alpha}{A} = \min\{x \in X \mid \mu_A(x) = \alpha\}.$$

where $\underset{=\alpha}{A}$ *is the* α*-support of type-1 fuzzy set A.*

Definition 7 (Highest Domain Value)**.** *We define the* highest domain value *of a stratum to be the greatest domain value of all the points in the stratum. Symbolically,*

$$HDV = \max \underset{=\alpha}{A} = \max\{x \in X \mid \mu_A(x) = \alpha\}.$$

where $\underset{=\alpha}{A}$ *is the* α*-support of type-1 fuzzy set A.*

6.2 No Secondary Grade Appearing in More Than One Secondary Membership Function

We begin (algorithms 2 and 3) by considering the simplest case of a T2FS whereby there are no instances of the same secondary membership grade appearing in more than one secondary membership function of the T2FS. Let $S_1, \ldots, S_k$ be the strata, ordered by height, with S_1 being the lowest, and S_k the highest. Let $\alpha_1, \ldots, \alpha_k$ respectively be the heights of strata $S_1, \ldots, S_k$, and let N_m be the number of TRS points on stratum S_m. Let M_m be the mean of the LDV (L_m) and HDV $(H_m)$of stratum S_m. (M_m is equivalent to the centre of the interval in the standard KMIP, and may be thought of as the defuzzified value of stratum S_m considered in isolation.)

The values $c_1, \ldots, c_k$ are constructs representing the contribution of each stratum to the final defuzzified value of the T2FS. C_m, the contribution of S_m is the product of the mean M_m, the number of points on the stratum N_m, and the stratum height α_m. Symbolically, $C_m = M_m N_m \alpha_m$. Let $L_1, \ldots, L_k$ respectively be the LDVs of strata $S_1, \ldots, S_k$, and $H_1, \ldots, H_k$ respectively be their HDVs.

Algorithm 2, in conjunction with algorithm 3, extends the KMIP to generalised T2FSs in the case where no secondary membership grade is found in more than one secondary membership function. We now consider the situation whereby the T2FS has instances of a secondary membership grade appearing in more than one secondary membership function.

Algorithm 2 KMIP Generalisation with No Secondary Grade Appearing in More Than One Secondary Membership Function

1: **for all** strata S_m of TRS T **do**
2: calculate L_m and H_m {see Algorithm 3}
3: calculate N_m
4: calculate M_m
5: calculate $C_m = M_m N_m \alpha_m$
6: **end for**
7: calculate $||T||$ {scalar cardinality of T}
8: calculate $d = \frac{\Sigma C_i}{||T||}$ {defuzzified value}

Algorithm 3 Calculating the LDV and HDV of a Stratum S_m (line 2 of Algorithm 2)

- change all secondary grades $< \alpha_m$ to 0 {no grades lower than $< \alpha_m$}
- change all secondary grades of the secondary membership function which includes α_m other than α_m to 0 {α_m always selected as minimum secondary grade}
- change all non-zero secondary grades to 1 {transform to an interval set}
- apply the KMIP to find L_m and H_m

6.3 Repetitions of Secondary Grades in Different Secondary Membership Functions of the T2FS

If the same secondary membership grade g appears in more than one secondary membership function, then all the defuzzified embedded sets of which g is the minimum grade will appear in the same stratum. To distinguish between points on the same stratum that derive from different instances of a secondary membership grade, we introduce the notion of a *substratum*. It is possible for a stratum to have only one substratum.

We define a substratum:

Definition 8 (Substratum)**.** *A substratum is a subset of a stratum consisting of those points which derive from (the same instance of) the same secondary membership grade of the originating T2FS.*

The idea behind algorithm 4 is that every substratum within a stratum has to be processed. If an embedded set includes more than one instance of the same secondary grade, its associated TRS point will lie on more than one substratum. So that the same point is not considered more than once, we have to allow for repetitions. Let R_m be the number of repetitions of points on stratum S_m.

Algorithm 4 KMIP Generalisation with Repetitions of Secondary Grades in Different Secondary Membership Functions

```
for all strata S_m of TRS T do
    for all substrata S_{m_k} of S_m do
        apply KMIP to find L_{m_k} and H_{m_k} {LDV and HDV of substratum S_{m_k}}
        note L_m, the least of the L_{m_k} so far {lowest LDV}
        note H_m, the least of the H_{m_k} so far {greatest HDV}
        calculate M_m {mean of L_m and H_m}
        for all substrata S_{m_k} of S_m do
            calculate N_{m_k} {no. of points on substratum S_{m_k}}
        end for
        calculate R_m {no. of repetitions}
        calculate N_m = ΣN_{m_k} − R_m {no. of points on stratum S_m}
        calculate C_m = M_m N_m α_m {contribution of stratum S_m}
    end for
end for
calculate ||T|| {scalar cardinality of T}
calculate d = ΣC_i / ||T|| {defuzzified value}
```

7 Conclusion and Further Work

Type-reduction is an important part of the defuzzification process in type-2 fuzzy inferencing systems but presents many challenges. In this paper we have, for the first time, explored the stratified structure of the type-reduced set. The structure of the type-reduced set is both interesting and useful and we have exploited this structure in developing a new generalised version of the Karnik Mendel Iterative Procedure. The algorithm is presented in detail.

Further work will include:

The Tiered Type-2 Fuzzy Set We propose a new variant of the T2FS — *the tiered type-2 fuzzy set.* This hybrid of the interval and generalised sets has pre-discretised secondary grades, i.e. the secondary grades take values from a pre-determined set of numbers such as $0, 0.1, 0.2, 0.3, \ldots, 0.9, 1$. The obvious advantage of the tiered set is that it would readily lend itself to defuzzification using the Generalised KMIP presented here.

Continuous Generalised T2FSs It is hoped that greater understanding of how the TRS structure varies as the discretisation becomes finer might shed some light on *continuous* generalised T2FSs.

Overall TRS Shape Whether the secondary membership function has any bearing on the overall shape of the TRS has yet to be explored. All the TRSs we have looked at so far come from T2FSs with a triangular secondary membership function.

Efficiency of the Sampling Method We hope that the properties of the TRS will shed light on the reason for the extraordinary efficiency of the sampling method of defuzzification [2] when used with small sample sizes.

Structure of an Individual Stratum We would like to investigate the pattern formed by the TRS points as they are laid out on an individual stratum.

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References

[1] Simon Coupland and Robert I. John. Geometric Type-1 and Type-2 Fuzzy Logic Systems. *IEEE Transactions on Fuzzy Systems*, 15(1):3–15, February 2007.

[2] Sarah Greenfield, Robert I. John, and Simon Coupland. A Novel Sampling Method for Type-2 Defuzzification. In *Proc. UKCI 2005*, pages 120–127, London, September 2005.

[3] Nilesh N. Karnik and Jerry M. Mendel. Centroid of a Type-2 Fuzzy Set. *Information Sciences*, 132:195–220, 2001.

[4] George J. Klir and Tina A. Folger. *Fuzzy Sets, Uncertainty, and Information*. Prentice-Hall International, 1992.

[5] Jerry M. Mendel. *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*. Prentice-Hall PTR, 2001.

[6] Jerry M. Mendel. A Possible Explanation, 2005. e-mail correspondence with author.

[7] Jerry M. Mendel and Robert I. John. Type-2 Fuzzy Sets Made Simple. *IEEE Transactions on Fuzzy Systems*, 10(2):117–127, 2002.