# Relation between OWA operator and the SMARTER method 

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#### Abstract

This paper presents two methods to obtain the weights in multiattribute decision making, when the only relation between the attributes is the order. The first one is the SMARTER method. The second is the OWA operator and finally the relation between them. The same operation and examples are also given.


Keywords: multiattribute decision making, SMARTER, OWA operator

## 1 Introduction

The decision analysis field has often encountered difficulties in transforming theoretical ideas into practical decision support tools.

Multicriteria decision-making (MCDM) is an important part of decision science, deals with the problem of helping the decision-maker to choose the best option into a set of alternatives, and according to several criteria. In classical MCDM methods, the ratings and the weights of the criteria are known precisely.
In the real world, the uncertainty, constraints, and even the vague knowledge of the experts imply that decision-makers cannot provide exact numbers to express their opinions. In a lot of cases, the only knowledge is an order relation.

In the process of decision-making, the decisionmaker generally needs to provide his/her preferences over a set of $n$ decision alternatives.

Different ways of admitting approximate preferences have attracted much attention in the decision analysis literature Pöyhönen and Hämäläinen (2001).
In the TRADEOFF procedure (Keeney and Raiffa, 1976) the decision-maker compares two hypothetical alternatives that differ in two attributes only.
In PRIME (Preference Ratios In Multiattribute Evaluation) the decision maker evaluates ratios of value differences (Salo and Hämäläinen, 2001). The decision maker enters these ratios as intervals of numbers.

In AHP (Analytic Hierarchy Process) a decision maker can use verbal expressions to facilitate the preference elicitation. Verbal expressions are converted into numbers according to the nine point integer scale $\{1, \ldots ., 9\}$ of Saaty (1980) or their reciprocal $\{1 / 9, \ldots, 1\}$.
According to Edwards and Barron (1994) and Valiris el al (2005), the Simple Multi-Attribute Rating Technique (SMART), is "by far the most common method actually used in real, decisionguiding multi-attribute utility measurements". For the SMART technique, ratings of alternatives are assigned directly, in a natural scale of the criteria where available. The advantage of the SMART model is that it is independent of the alternatives.
In SMARTER (von Winterfeldt and Edwards, 1986), where one judges the performance of an alternative by choosing an appropriate value between a predetermined lower limit for the worst (real or imaginary) alternative and a predetermined upper limit for the best (real or ideal) alternative.

The OWA operator provides examples of aggregation operators in which the ordering of the arguments play a central role in the operation. The OWA operator provides a parameterized family of averaging operators, having as its arguments a finite set of values. A fundamental aspect of these operators is the reordering process that associates the arguments with the weights.

In this work, we provide a class of operators related with the ordering of the argument values.

The paper is organized as follows: the next section introduces the framework for the OWA operator. Section 3 deals with the evaluation of the SMARTER approach. Section 4 examines the relation between OWA and SMARTER, and an illustrative example is given. Finally, the most important conclusions are outlined.

## 2 OWA operator

In this section we review the basic concepts associated with the OWA operators [12].

### 2.1 Definition and Properties

An Ordered Weighted Averaging operator (OWA) of dimension $n$ is a function

$$
F: R^{n} \rightarrow R
$$

that has an associated $n$ vector

$$
W=\left[w_{1}, w_{2}, \ldots, w_{n}\right]^{T}
$$

such that
1.- $w_{i} \in[0,1]$
2.- $\sum_{i} w_{i}=1$

Furthermore, $F\left(a_{1}, a_{2}, . ., a_{n}\right)=\sum_{j} w_{j} b_{j}$
where $b_{j}$ is the $\mathrm{j}^{\text {th }}$ largest of the $a_{i}$
If we represent the ordered arguments, the $b_{j}$, by a vector $B$, called ordered argument vector. We can express the expression (1) as:

$$
\begin{equation*}
F_{W}\left(a_{1}, a_{2}, . ., a_{n}\right)=W . B \tag{2}
\end{equation*}
$$

In particular, a weight $w_{i}$ is not associated with a specific argument but with an ordered position of the aggregate. This ordering operation essentially provides a non-linear aspect to this aggregation operation.

A number of properties can be associated with these operators. It is first noted that the OWA aggregation is commutative, that is the aggregation is indifferent to the initial indexing of the argument. A second characteristic associated with these operators is monotonic. Thus, if $\hat{a}_{i} \geq a_{i}$ for all $i$, then

$$
F\left(\hat{a}_{1}, \hat{a}_{2}, \ldots, \hat{a}_{n}\right) \geq f\left(a_{1}, a_{2}, . ., a_{n}\right)
$$

Another characteristic associated with these operators is that of idempotency. In particular, if for all $i, a_{i}=a$ then $F\left(a_{1}, a_{2}, . ., a_{n}\right)=a$.

The satisfaction of these three conditions, as noted by Dubois and Prade (1985), assures these operators of being in the class of operators called mean operators. It can also be shown; it follows from the fact that an OWA operator is a mean operator, that the Min and Max of the arguments bound the OWA aggregation. That for any OWA aggregation F,

$$
\begin{equation*}
\operatorname{Min}_{i}\left[a_{i}\right] \leq F\left(a_{1}, a_{2}, . ., a_{n}\right) \leq \operatorname{Max}_{i}\left[a_{i}\right] \tag{3}
\end{equation*}
$$

In Yager (1993) discusses a number of families of these operators. The various different mean operators are implemented by appropriate selection of the weights in the associated weighting vector.
In the following, we shall look at some of these operators.

The Max operator is recovered if

$$
\begin{equation*}
W=W^{*}=[1,0, . ., 0]^{T} \tag{4}
\end{equation*}
$$

the Max OWA operator emphasizes the largest element in the argument bag.
The Min operator is recovered if

$$
\begin{equation*}
W=W_{*}=[0,0, . ., 1]^{T} \tag{5}
\end{equation*}
$$

the Min operator emphasizes solely the smallest element in the bag of arguments.

An interesting class of operators is the $\mathrm{k}^{\text {th }}$ largest argument in the aggregation. We noted that the Max and the Min are cases of this when the $k$ is 1 and $n$, respectively
The ordinary simple average is recovered if

$$
\begin{equation*}
W=W_{A V G}=[1 / n, 1 / n, . ., 1 / n]^{T} \tag{6}
\end{equation*}
$$

In this case

$$
\begin{equation*}
F\left(a_{1}, a_{2}, . ., a_{n}\right)=\sum_{i} \frac{1}{n} b_{i}=\frac{1}{n} \sum_{i} a_{i} \tag{7}
\end{equation*}
$$

It is noted that in this case, no reordering is required.

### 2.2. Mathematical Programming Problem

O'Hagan (1988) suggested a methodology for obtaining the OWA weighting vector based upon the use of these characterizing measures. This approach, which only requires the specification of just the $\alpha$ value, generates a class of OWA weights, which are called MEOWA weights. The determination of these weights, $w_{1}, . ., w_{n}$, from $\alpha$ requires the solution of the following Mathematical Programming Problem (MPP).

## Maximice

$$
H(W)=-\sum w_{i} \ln w_{i}
$$

subject to
(1) $\alpha=\frac{1}{n-1} \sum_{i=1}^{n}\left((n-1) w_{i}\right)$
(2) $\quad \sum_{i} w_{i}=1$
(3) $w_{i} \in[0,1]$

In this MPP formulation, restriction (1) is the imposition of the condition that the desired $\alpha$ value is attained. Constraints (2) and (3) just assure us that the weights satisfy the basic requirements of the OWA weights.
The objective function used in this approach is one of trying to maximize the dispersion or entropy, that is calculate the weights to be the ones which use as much information as possible in the aggregation.

$$
\begin{align*}
\text { For } \alpha=0.75 & w_{3}=[0.62,0.27,0.11] \\
& w_{4}=[0.52,0.27,0.15,0.06]  \tag{8}\\
& w_{5}=[0.46,0.26,0.15,0.08,0.05]
\end{align*}
$$

It can easily be shown Yager (1988) that
1.- orness $\left(W^{*}\right)=1$
2.- orness $\left(W_{*}\right)=0$
3.- orness $\left(W_{\text {AVG }}\right)=0.5$

A measure of andness can be defined as

```
andness (W)=1-orness(W)
```


## 3. SMARTER method

Multiattribute value analysis derives an overall value for each alternative. This is composed of the values of the alternatives with respect to each attribute and of the weights of the attributes.

If the attributes are mutually preferentially independent, an additive value function can use to aggregate the component values (Keeney and Raiffa, 1976). The overall value of an alternative $x$ is then

$$
\begin{equation*}
v(x)=\sum_{i=1}^{n} w_{i} v_{i}\left(x_{i}\right) \tag{9}
\end{equation*}
$$

where $x_{i}$ is the consequence of an alternative $x$ for attribute $i, n$ is the number of attributes, and $w_{i} \geq 0$ is the weight of the attribute $i$. The sum up weights is normalized to one.
Edwards (1977) originally described SMART as the whole process of rating alternatives and weighting attributes. With SMART the weights are elicited in two steps (Edwards, 1977; von Winterfeldt and Edwards, 1986):

1. Rank the importance of the changes in the attributes from the worst attribute levels to the best levels.
2. Make ratio estimates of the relative importance of each attribute relative to the one ranked lowest in importance.
As we have already commented, Edwards and Barron (1994) also presented a new version, SMARTER, which only uses the ranking of attributes to derive the weights.

The idea is to use the centroid method of Solymosi and Dombi (1986) so that the weight of an attribute ranked to be ith is

$$
\begin{equation*}
v_{i}=\frac{1}{n} \sum_{k=i}^{n} \frac{1}{k} \tag{10}
\end{equation*}
$$

where $n$ is the number of attributes.
For $n=4$ these values are:
$v_{1}=\frac{1}{4}\left(\frac{1}{4}+\frac{1}{3}+\frac{1}{2}+1\right)=\frac{25}{48}$
$v_{2}=\frac{1}{4}\left(\frac{1}{4}+\frac{1}{3}+\frac{1}{2}\right)=\frac{13}{48}$
$v_{3}=\frac{1}{4}\left(\frac{1}{4}+\frac{1}{3}\right)=\frac{7}{48}$

$$
v_{4}=\frac{1}{4}\left(\frac{1}{4}\right)=\frac{1}{16}
$$

In this way we have obtained the vector of weights as

$$
[0.5208,0.2708,0.1458,0.0625]
$$

In the same form, it is possible to obtain the rest of the vectors for the different numbers of attributes. In the table 1 we present the weights for the values of $n$.
Table 1: Weights for indicated table of attributes

| $\mathbf{n}$ | $\mathbf{2}$ | $\mathbf{3}$ | $\mathbf{4}$ | $\mathbf{5}$ | $\mathbf{6}$ |
| :---: | :--- | :--- | :--- | :--- | :--- |
| $\mathbf{v}$ |  |  |  |  |  |
| $v_{1}$ | 0.7500 | 0.6111 | 0.5208 | 0.4567 | 0.4083 |
| $v_{2}$ | 0.2500 | 0.2778 | 0.2708 | 0.2567 | 0.2417 |
| $v_{3}$ |  | 0.1111 | 0.1458 | 0.1567 | 0.1583 |
| $v_{4}$ |  |  | 0.0625 | 0.0900 | 0.1028 |
| $v_{5}$ |  |  |  | 0.0400 | 0.0611 |
| $v_{6}$ |  |  |  |  | 0.0278 |

## 4. The Operator

If we view the OWA weights as a column vector (as in table 1). We can refer to the weights with the low indices as weights at the top, and those with higher indices as weights at the bottom, (Yager 1996).

Using this convention, we see that if most of the weights are at the top, then the aggregation is emphasizing the higher-valued arguments in the calculation of the median values. If most of the weights are at the bottom, then the aggregation is emphasizing the smaller-valued argument in the aggregation.
The Min aggregation is a strong example of this type of aggregation. If the weights are equally distributed between those above the middle and those below the middle, we can see that the
aggregation is not favouring the higher-valued element over the lower-valued element.

Definition 1: Let $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ be a numeric set, that has associated an aggregation function $F=\left\{f_{1}, f_{2}, \ldots, f_{n}\right\}$, which is decreasing, that is, $f_{i} \leq f_{i+1} \forall i=1,2, \ldots, n$, then

$$
\begin{align*}
& \text { Operator }=\sum_{i=1}^{n} w_{\boldsymbol{i}} \cdot \boldsymbol{x}_{\boldsymbol{i}}  \tag{11}\\
& w_{i}=w_{i-1}+g\left(f_{i}\right)
\end{align*}
$$

Where: $\quad w_{i} \in[0,1], \sum_{i=1}^{n} w_{i}=1$.
The representation of this operator is given in figure 1.


Figure 1: Operator representation

Definition 2: Let $X=\left\{x_{1}, x_{2}, \ldots, x_{n}\right\}$ be an alternative set, that have associated an order between then, that is, $x_{i} \leq x_{i+1} \quad \forall i=1,2, \ldots, n$, and suppose that the function considered for aggregation is $g\left(f_{i}\right)=1 / i n$, then

$$
\begin{equation*}
w_{i}=w_{i+1}+1 / i n \tag{12}
\end{equation*}
$$

and

$$
w_{n}=\frac{1}{n \cdot n}
$$

$$
\begin{aligned}
& w_{i} \in[0,1] \\
& \sum_{i=1}^{n} w_{i}=1
\end{aligned}
$$

1.- It is important to verify that $\sum_{i=1}^{n} w_{i}=1$ :

$$
\sum_{i=1}^{n} w_{i}=\frac{1}{n \cdot n}+\left(\frac{1}{n \cdot n}+\frac{1}{n \cdot(n-1)}\right)+\ldots+
$$

$$
\left(\frac{1}{n \cdot n}+\frac{1}{n \cdot(n-1)}+\ldots+\frac{1}{2 \cdot n}\right)+
$$

$$
\left(\frac{1}{n \cdot n}+\frac{1}{n \cdot(n-1)}+\ldots+\frac{1}{2 . n}+\frac{1}{n}\right)
$$

$$
\sum_{i=1}^{n} w_{i}=n \frac{1}{n \cdot n}+(n-1) \frac{1}{n(n-1)}+\ldots+2 \frac{1}{2 \cdot n}+\frac{1}{n}
$$

$$
\text { then } \sum_{i=1}^{n} w_{i}=1 \text {. }
$$

Example 1. We consider a Multiattribute Decision Evaluation, with four attributes and that the only information about the criteria is that $C_{1} \succ C_{2} \succ C_{3} \succ C_{4}$.
In this context the associated weights are:
$w_{4}=\frac{1}{4.4}=\frac{1}{16}=\frac{3}{48}$
$w_{3}=\left(\frac{1}{16}+\frac{1}{4.3}\right)=\frac{7}{48}$
$w_{2}=\left(\frac{7}{48}+\frac{1}{4.2}\right)=\frac{13}{48}$
$w_{1}=\left(\frac{13}{48}+\frac{1}{4}\right)=\frac{25}{48}$
with
$\alpha=\frac{1}{n-1} \sum_{i=1}^{n}\left((n-i) w_{i}\right)=\frac{1}{3}\left(3 \frac{25}{48}+2 \frac{13}{48}+\frac{7}{48}\right)=$ $\frac{3}{4}=0.75$

It is possible to see that this value coincide with that given in (8).

Theorem 1. Let be $O_{1}=\left[w_{1}, w_{2}, \ldots, w_{n}\right]$ and $O_{1}^{\prime}=\left\lfloor w_{1}^{\prime}, w_{2}^{\prime}, \ldots, w_{n}^{\prime}\right\rfloor$ two vectors of weights with associated orness $\alpha_{1}$ and $\alpha_{2}$, respectively. In this situation it is possible to see that,

1. The semi sum of $O_{1}$ and $O_{1}^{\prime}$ is a new OWA operator with weights the semi sum of the corresponding component of the two vectors.
$O=\frac{O_{1}+O_{1}^{\prime}}{2}=\left[\frac{w_{1}+w_{1}^{\prime}}{2}+\ldots+\frac{w_{n}+w_{n}^{\prime}}{2}\right]$
being
$\sum_{i=1}^{n} \frac{w_{i}+w_{i}^{\prime}}{2}=$
$\sum_{i=1}^{n} \frac{w_{i}}{2}+\sum_{i=1}^{n} \frac{w_{1}^{\prime}}{2}=\frac{1}{2} \sum_{i=1}^{n} w_{i}+\frac{1}{2} \sum_{i=1}^{n} w_{i}^{\prime}=\frac{1}{2}+\frac{1}{2}=1$
2. The orness associated with this new OWA operator $O$ is , $\alpha=\frac{\alpha_{1}+\alpha_{2}}{2}$
$\alpha=\binom{(n-1)\left(\frac{w_{1}+w_{1}^{\prime}}{2}\right)+(n-2)\left(\frac{w_{2}+w_{2}^{\prime}}{2}\right)+\ldots+}{2\left(\frac{w_{n-2}+w_{n-2}^{\prime}}{2}\right)+\frac{w_{n-1}+w_{n-1}^{\prime}}{2}}$
$=\frac{1}{2}(n-1) \sum_{i=1}^{n-1}(n-i) w_{i}+\frac{1}{2}(n-1) \sum_{i=1}^{n-1}(n-i) w_{i}$
$=\frac{1}{2} \alpha_{1}+\frac{1}{2} \alpha_{2}$

Corollary: If $\alpha_{2}=1-\alpha_{1}$, then $\alpha=\frac{1}{2}$

Example 2: Let be $O_{1}=[0.5208,0.2708$, $0.1458,0.0625$ ] and $\alpha_{1}=0.75$
$O_{1}^{\prime}=[0.2500,0.2500,0.2500,0.2500]$
and $\alpha_{2}=1 / 2$.
Then
$O=[0.3804,0.2604,0.1979,0.1563]$
being

$$
\begin{aligned}
& \alpha=1 / 3(3.0 .3854+2.0 .2604+0.1979)= \\
& 0.625
\end{aligned}
$$

and $\alpha=(0.75+0.50) / 2=0.625$

## 5. Conclusion.

We have shown that the weights obtained using the SMARTER method and obtained by an OWA operator, taking into account the O'Hagan approach, are the same when the orness $\alpha=0.75$.

Furthermore, we prove that the composite of two OWA operators is a new OWA operator with orness equal to the sum of the orness of the operator divided by 2.

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