Heuristics for Solving Investment Problems in a Fuzzy Environment

María José Canós Darós
Dep. Matemáticas para la Economía y la Empresa. Avda. Tarongers s/n. 46071-Valencia
e-mail: maria.j.canos@uv.es

Carlos Ivorra Castillo
Dep. Matemáticas para la Economía y la Empresa. Avda. Tarongers s/n. 46071-Valencia
e-mail: carlos.ivorra@uv.es

Vicente Liern Carrión
Dep. Matemáticas para la Economía y la Empresa. Avda. Tarongers s/n. 46071-Valencia
e-mail: vicente.liern@uv.es

Abstract

The paper presents models for the portfolio selection problem, which allow two kinds of uncertainty to be tackled: that arising from the data and that associated to the imprecise decision maker criteria. In the first case, the models are formulated in terms of LR-fuzzy numbers and are solved by means of genetic algorithms. In the second case, some algorithms designed by the authors for fuzzy location problems are adapted.

Keywords: Portfolio selection, Heuristics, Soft Computing.

1 Introduction

Among the financial products offered by investment companies, one of the most relevant for firms is the choice and management of the investment portfolio it is perhaps choosing and managing investment portfolios. Optimization is assumed to be useful in improving financial asset selection in a portfolio, taking into account the risk of every possible situation. Undoubtedly, choosing the best investment options for a portfolio is important to the investor in economic terms, and constitutes a complex problem because of the following aspects:

a) the large number of parameters involved,

b) the uncertainty in the data,

c) the fact that it is difficult to emulate investor desires in a model.

Both stochastic and fuzzy programming have been used in order to deal with uncertainty Some authors have based their research on stochastic approaches (Kall and Wallace, 1994). However, sometimes the use of probability distributions is not justified. Using fuzzy models avoids unrealistic modelling and offers a chance to reduce information costs. In Inuiguchi and Ramík (2000), the portfolio selection problem exemplifies the advantages and disadvantages of different fuzzy mathematical programming approaches.

This paper presents a summary of some optimization schemes for managing portfolio selection problems based on soft-computing methods. The techniques used to solve these models include exact optimization methods and heuristics techniques, some of which are general and others are adapted from fuzzy location models.

2 Formulating the Portfolio Selection Problem

The standard formulation of Markowitz model (MV), is

$$\begin{array}{ll} Min & x^t Q x \\ s.t. & \sum_{i=1}^{n} E(R_i) x_i \geq \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & l_i \leq x_i \leq u_i, \ 1 \leq i \leq n \end{array} \tag{1}$$

where x_i is the proportion of a total investment fund chosen by the investor, devoted to asset i^{th} for n risky assets,.R_i is the return on the i^{th} asset

L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, pp. 1254–1259
Torremolinos (Málaga), June 22–27, 2008

(it is assumed to be a random variable). Thus, the vector of returns R = (R_1, ..., R_n) can be summarized by the mean vector E(R) and the covariance matrix Q. The parameter ρ can be assumed to represent the minimal rate of return required by the investor and u_i (respectively l_i) the maximum (minimum) amount of total funds that can be invested in asset i.

In the model MV, the averages E(R_i) and the elements of the covariance matrix Q can be approximated using historical data.

Dissatisfaction with the traditional notion of variance as a measure of risk is due to the fact that no distinction is made between gains and loses. In fact, Markowitz (1959) provided two suggestions for measuring downside risk: a semi-variance, which is the sum of the squares of negative deviations from the mean, or a semi-variance computed from a target return.

Moreover, different elements can be fuzzified in the portfolio selection problem. Some authors use possibility distributions to model the uncertainty on returns. Tanaka and Guo focus on the achievements of possibility theory and its applications for operation research, in particular to portfolio selection models. Their approach permits expert knowledge to be incorporated by means of possibility grading, to reflect the degree of similarity between the future state of stock markets and their state in previous periods (see, for instance, Tanaka and Guo, 1999; Guo and Tanaka, 2003). Other authors study the portfolio selection problem using fuzzy formulations. Watada (1997) proposed a fuzzy portfolio selection model where fuzzy numbers were used to represent the decision makers' aspiration levels for the expected rate of return and a certain degree of risk. Ortí et al (2002) propose the incorporation of fuzzy numbers to represent the uncertainty of the future return on assets and set out portfolio selection as a problem of nonlinear multi-objective programming with fuzzy parameters.

Essentially, two kinds of uncertainties can be distinguished in the portfolio selection model:

a) that arising from historical data, which are used to estimate the volatility and expected returns of the assets.

b) that depending on investor criteria.

We devote one section to each type of uncertainty.

3 Modelling Uncertainty in the Data

It is a well known fact that the portfolio selection problem has fuzzy characteristics (see León et al., 2002). In this context, where investors seek the minimum risk under a given expected return, unfeasibility is caused by the conflict between the desired return and diversification requirements. Hence, in the investor's opinion, the viability of the instance depends on how severe perturbations must be in order to make it feasible. The fuzzy linear programming techniques used to repair infeasible instances with respect to the original will of the investor as much as possible. The uncertainty of the returns on assets is represented as an expected interval of a fuzzy number (Dubois and Prade, 1987) and the risk of the portfolio is defined as the fuzzy measurement of the mean total semi-deviation (León et al, 2004). Moreover, decision maker criteria are also added to the model.

3.1 Portfolio Selection Using LR-fuzzy Numbers

Let us assume that $\tilde{R}_i$ is an LR-fuzzy number with trapezoidal possibility distribution. The return of the ith asset is therefore denoted by

$$\tilde{R}_i = \left(a_{li}, a_{ui}, c_i, d_i\right)_{LR},$$

then, the mean of the fuzzy return of the portfolio,

$$\tilde{c} = \sum_{i=1}^{n} R_i x_i,$$

has the following expected interval as a mean

$$E(\tilde{c}) = \left[\sum_{i=1}^{n} a_{li} x_i - \frac{1}{2}\sum_{i=1}^{n} c_i x_i,\ \sum_{i=1}^{n} a_{ui} x_i + \frac{1}{2}\sum_{i=1}^{n} d_i x_i\right],$$

and the risk is:

$$g(\tilde{c}) = \left[0,\ \sum_{i=1}^{n}\left(\frac{1}{2}\left(a_{ui} + a_{li}\right) + \frac{1}{4}\left(c_i - d_i\right)\right) x_i\right].$$

In order to determine the intervals of the mean returns and risks, following the proposal from León et al (2004), the upper bound of the interval can be minimized and a representative of the expected interval can be chosen. The mathematical program is:

$$\begin{array}{ll} Min & \sum_{i=1}^{n}\left(a_{ui}-a_{li}+\frac{1}{2}\left(c_i+d_i\right)\right)x_i \\ s.t. & \sum_{i=1}^{n}\left(\frac{1}{2}\left(a_{ui}-a_{li}\right)+\frac{1}{4}\left(c_i+d_i\right)\right)x_i \geq \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & l_i \leq x_i \leq u_i,\ 1 \leq i \leq n \end{array} \tag{2}$$

3.2 Portfolio Selection Using Genetic Algorithms

Formulating (MV) implies a problem with the number of assets that can belong to the optimal portfolio. For this reason some authors suggest the inclusion of a maximum bound for the number of assets to be included in a portfolio in the formulation of the problem. Mixed binary quadratic programming can be used to model this with the following results:

$$\begin{array}{ll} Min & x^tQx \\ s.t. & \sum_{i=1}^{n} E(R_i)x_i \geq \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & \sum_{i=1}^{n} y_i \leq N_u \\ & y_i \in \{0,1\} \\ & y_i l_i \leq x_i \leq y_i u_i,\ 1 \leq i \leq n \end{array} \tag{3}$$

where N_u is the maximum number of assets that can belong to a portfolio

By limiting the number of assets in the model, a small variation in the parameters fixed by the decision-maker (invested capital or required return) can force a change in the composition of the optimal portfolio which can alter risk significantly. If z* is the optimal risk of the model (3), the associated fuzzy model can be represented by this non linear optimization problem:

$$\begin{array}{ll} Min & \lambda \\ s.t. & x^tQx \lesssim z^* \\ & \sum_{i=1}^{n} E(R_i)x_i \gtrsim \rho \\ & \sum_{i=1}^{n} x_i = 1 \\ & \sum_{i=1}^{n} y_i \leq N_u \\ & y_i \in \{0,1\} \\ & y_i l_i \leq x_i \leq y_i u_i,\ 1 \leq i \leq n \end{array} \tag{4}$$

where λ is the level of global satisfaction of a portfolio, defined as

$$\lambda = \min\{\mu_r(x,y), \mu_f(x,y)\}$$

Moreover, μ_r and μ_f represent, respectively, the membership functions to the objective set and the opportunity set, that is, the level of improvement in the risk and the degree of feasibility of the portfolio.

The model (4) makes it possible to fix the number of assets to invest in and also to obtain a portfolio by using genetic algorithms. This is useful when a quick solution is required or when the problem being tackled is large. This paper develops the approach appearing in Bermúdez et al. (2005, 2007), based on the basic rules of genetic algorithms, to obtain a simple algorithm structure.

4 Modelling Investor Preferences

The portfolio selection problem has two data concerning decision maker preferences, namely the capital to be invested and the risk to be assumed. The investor can be assumed to know with certainty the capital that he or she would like to consider, and in fact, in the model (1) this quantity has been normalized to the unit. However, determining the risk to be assumed can be more flexible. As a result, it is worth incorporating this flexibility in a fuzzy model.

In order to do so, it is preferable to work with the commonly called dual model of (1):

$$\begin{aligned} Max \quad & \sum_{i=1}^{n} E(R_i)x_i \\ s.t. \quad & x^t Qx \leq R \\ & \sum_{i=1}^{n} x_i = 1 \\ & l_i \leq x_i \leq u_i, \quad 1 \leq i \leq n \end{aligned} \tag{5}$$

Here the aim is to maximize the return subject to a maximum risk R.

We are going to show that the techniques used in the analysis of the p-median problem in Canós *et al*. (1999, 2001) can be generalized to deal with the risk fuzziness in the portfolio selection problem. The main idea is to consider partially feasible solutions involving slightly greater risk than that fixed by the decision maker, and study the possibilities that they offer in order to improve the expected return.

When compared with the p-median case, this problem happens to be more complicated, due to the p-median problem being linear and the risk constraint in the portfolio model being quadratic. Moreover, in the p-median case, a small reduction in covered demand affected optimal cost in a simple linear way, whereas the way in which the maximum expected return depends on the accepted risk is rather more complicated.

A fuzzy set $\tilde{S}$ of partially feasible solutions is defined so that portfolio selection belongs to $\tilde{S}$ with a degree of membership that depend on how much it exceeds the risk R fixed by the investor. On the other hand, a second fuzzy set $\tilde{G}$ is defined whose membership function reflects the improvement of the return provided by a partially feasible solution with respect to the optimal crisp return z*. In practice, we consider piecewise linear membership functions

$$\mu_{\tilde{S}}(x) = \begin{cases} 1 & \text{if } r \leq R \\ 1 - \dfrac{r - R}{p_f} & \text{if } R < r < R + p_f \\ 0 & \text{if } r \geq R + p_f \end{cases}$$

$$\mu_{\tilde{G}}(x) = \begin{cases} 0 & \text{if } z \leq z^* \\ \dfrac{z - z^*}{p_g} & \text{if } z^* < z < z^* + p_g \\ 1 & \text{if } z \geq z^* + p_g \end{cases}$$

where r and z are the risk and the return provided by the portfolio x (which is assumed to satisfy the constraints of (5) except in the first case), the parameter p_f is the maximum increment in risk that the decision maker can accept, and p_g is the increment of the return that the decision maker would consider completely satisfactory. From this, we can define a global degree of satisfaction

$$\lambda(x) = \min\left\{\mu_{\tilde{G}}(x), \mu_{\tilde{S}}(x)\right\},$$

which is the membership degree to the fuzzy intersection of $\tilde{S} \cap \tilde{G}$.

The fuzzy portfolio model becomes

$$\begin{aligned} Max \quad & \lambda(x) \\ s.t. \quad & x \in \tilde{S} \end{aligned} \tag{6}$$

In order to solve (6), the optimal solution of (5) will be calculated for each risk level R. For this, we solve explicitly the Kuhn-Tucker conditions for the problem. To carry out the computations in a generic framework, we start making a change of variables to diagonalize the risk matrix.

As we are interested in smsall variations of R, the variables x_i that are zero in the optimal crisp portfolio can be removed and hence assume that non-negative conditions are not active. Standard linear algebra theory ensures us that we can decompose

$$S = A^t DA,$$

where the matrix D is a diagonal and A is regular. Then the change of variables y = Ax transforms the problem to

$$\begin{aligned} Max \quad & r' y \\ s.t. \quad & by = 1 \\ & y^T Dy = R \\ & l \leq A^{-1} y \leq u \end{aligned} \tag{7}$$

where $r' = r^t A^{-1}$ and $b = (1,1,...,1)A^{-1}$.

As the last inequalities are not active the Kuhn-Tucker conditions for (7) are simply the Lagrange conditions of the classical problem resulting from removing them. These are:

$$\sum_i b_i y_i = 1,$$

$$\sum_i d_i y_i^2 = R,$$

$$r'_i - b_i\lambda - 2d_i y_i \mu = 0, \quad 1 \leq i \leq n.$$

where λ, μ are the Kuhn-Tucker multipliers of the capital and the risk constraint respectively. We also have the sign Kuhn-Tucker condition

$$\mu \geq 0.$$

Solving these equations, y_i can be expressed as functions $y_i(R)$ which makes it possible to:

a) Determine the interval in which R can oscillate in order for the solution $y(R)$ to satisfy the constraints

$$l \leq A^{-1} y \leq u.$$

If values of R exceeding this interval need to be considered, we should deal with the solution of the problem analogue to (6) for a different choice of the variables.

b) Computing the return as a function $F(R)$ from which the degree of feasibility $\mu_f(R)$ and the degree of improvement of the goal $\mu_g(R)$ can be calculated as functions of R.

The expression obtained is

$$F(R) = \pm \frac{2LR-1}{2\sqrt{\Delta(R)}} \sum_{i=1}^{n} \left(\frac{r'^2_i}{d_i} - \frac{r'_i b_i}{d_i} \frac{2KLR - K \pm \sqrt{\Delta(R)}}{2L^2 R - L} \right),$$

where

$$K = \sum_{i=1}^{n} \frac{b_i r'_i}{2d_i},$$

$$L = \sum_{i=1}^{n} \frac{b_i^2}{2d_i},$$

$$M = \sum_{i=1}^{n} \frac{r'^2_i}{4d_i},$$

$$\Delta(R) = K^2 - 2ML + (4ML^2 - 2K^2 L)R.$$

These expressions can be used to determine the risk R^* such that

$$\mu_f(R^*) = \mu_g(R^*)$$

which is easily shown to be the risk maximizing λ. Hence, the portfolio $x(R^*)$ corresponding to $y(R^*)$ by the change of variables is the optimal solution of (6).

4.1 A Numerical Example

Below is a simple example of the fuzzy portfolio selection problem. Five assets from the historical data introduced by Markowitz (Markowitz, 1959) are used.

Table 1: Returns of Five Securities

Year	AmT	ATT	USS	GM	ATS
1937	-0,305	-0,173	-0,318	-0,477	-0,457
1938	0,513	0,098	0,285	0,714	0,107
1940	0,055	0,2	-0,047	0,165	-0,424
1941	-0,126	0,03	0,104	-0,043	-0,189
1942	-0,003	0,067	-0,039	0,476	0,865
1943	0,428	0,3	0,149	0,225	0,313
1944	0,192	0,103	0,26	0,29	0,637
1945	0,446	0,216	0,419	0,216	0,373
1946	-0,088	-0,046	-0,078	-0,272	-0,037

We have fixed a risk level R=0.03. The optimal crisp portfolio is formed by assets AmT, ATT, GM and ATS providing an optimal return z* = 0.102386.

For the fuzzy model, we have fixed p_f=0.02, p_g=0.02 and solved it by using Mathematica®.

The optimal return for a given risk R, happens to be

$$F(R) = \frac{-0.019 + 1.059R + 0.081\sqrt{-0.529 + 28.231R}}{\sqrt{-0.5299 + 28.2313R}}$$

Computations are valid for risks in the interval [0.0237814, 0.0876355].

Figure 1 shows the functions $\mu_f(R)$ and $\mu_g(R)$. They intersect at R*= 0.0412373 corresponding to λ= 0.438133. The return of the fuzzy portfolio is 0.111149.

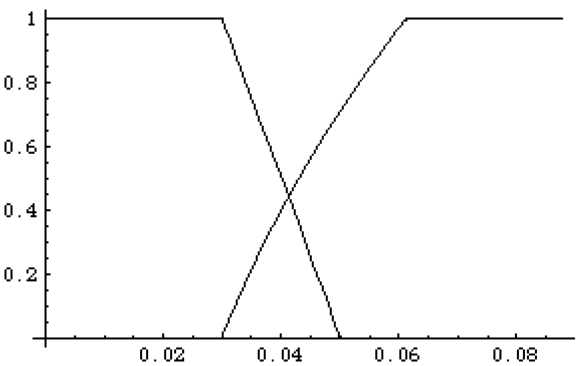

Figure 1: membership functions.

We observe that the global satisfaction degree is quite low. This means that risk has to be increased a great deal in order to obtain a not very significant increase in the return on the asset. In other words, the crisp solution seems to be quite stable. However, it is still to be studied how solutions depend on the chosen membership functions by means of a suitable sensitivity analysis, analogue to that developed in Canós et al. (2008).

5 Conclusions

The models proposed for dealing with data uncertainty, are a further development of those presented by authors in previous editions of IPMU. They improve classical portfolio selection models in the sense that they lead to solutions with better returns and they adjust in a more realistic way to stock market trends (León *et al.* 2004).

The models concerning fuzziness in investor criteria, have been less studied by the literature, and should be developed in greater detail before combining them with the previous ones.

The task of combining both kinds of uncertainty and developing suitable intelligent systems to deal with them is still pending.

Acknowledgements

This research has been partially supported by the Ministerio de Educación y Ciencia, TIN-2005-08404-C04-04.

References

[1] Bermúdez, J.D., Ivorra, C., Liern, V., Picó, F.J., Segura, J.V., Vercher, E. (2007): Selección de carteras de cardinalidad restringida mediante algoritmos genéticos y técnicas basdas en soft computing. Proceedings MAEB-2007, 223-229.

[2] Bermúdez, J.D., Liern, V., Segura, J.V., Torres, D., Vercher, E. (2005): Fuzzy Portfolio Selection: a comparative study. Proceedings Fourth Conference EUSFLAT 2005, 709-713.

[3] M. J. Canós, C. Ivorra, V. Liern (1999): An exact algorithm for the fuzzy p-median problem. European Journal of Operational Research, 116, pp. 80-86

[4] M. J. Canós, C. Ivorra, V. Liern (2001): The fuzzy p-median problem: A global analysis of the solutions. European Journal of Operational Research, 130, 430-436.

[5] M. J. Canós, C. Ivorra, V. Liern (2008): Marginal analysis for the fuzzy p-median problem, European Journal of Operational Research, Available online.

[6] Dubois, D., Prade, H. (1987): The mean value of a fuzzy number, Fuzzy Sets and Systems 24, pp. 279-300.

[7] Guo P.; Tanaka H. (2003): Decision analysis based on fused double exponential possibility distributions. European Journal of Operational Research, 148, pp. 467-479.

[8] Inuiguchi M, Ramík J. (2000): Possibilistic linear programming: a brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem. Fuzzy Sets and Systems, 111, 3-28.

[9] Kall, P., Wallace, S. T: (1994): Stochastic Programming. John Wiley, Chichester.

[10] León, T., Liern, V., Marco, P., Segura, J.V., Vercher, E. (2004): A downside risk approach for the portfolio selection problem with fuzzy returns, Fuzzy Economic Review 9, 61-77.

[11] León, T., Liern, V., Vercher, E. (2002): Viability of infeasible portfolio selection problems: A fuzzy approach, European Journal of Operational Research 139, 178-189.

[12] Markowitz, H. M. (1959): Portfolio selection: Efficient Diversification of Investments, John Wiley, New York.

[13] Ortí, F.J., Sáez, J., Terceño, A. (2002): On the treatment of uncertainty in portfolio selection, Fuzzy Economic Review 7,. 59-80.

[14] Tanaka, T.; Guo, P. (1999): Possibilistic data analysis and its application to portfolio selection problems, Fuzzy Economic Review 2, 2-23.

[15] Watada, J. (1997): Fuzzy portfolio selection and its applications to decision making, Tatra Mountains Math. Publ. 13, 219-248.