Heuristics for Solving Investment Problems in a Fuzzy Environment

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Abstract

The paper presents models for the portfolio selection problem, which allow two kinds of uncertainty to be tackled: that arising from the data and that associated to the imprecise decision maker criteria. In the first case, the models are formulated in terms of LR-fuzzy numbers and are solved by means of genetic algorithms. In the second case, some algorithms designed by the authors for fuzzy location problems are adapted.

Keywords: Portfolio selection, Heuristics, Soft Computing.

1 Introduction

Among the financial products offered by investment companies, one of the most relevant for firms is the choice and management of the investment portfolio it is perhaps choosing and managing investment portfolios. Optimization is assumed to be useful in improving financial asset selection in a portfolio, taking into account the risk of every possible situation. Undoubtedly, choosing the best investment options for a portfolio is important to the investor in economic terms, and constitutes a complex problem because of the following aspects:

a) the large number of parameters involved,
b) the uncertainty in the data,
c) the fact that it is difficult to emulate investor desires in a model.

Both stochastic and fuzzy programming have been used in order to deal with uncertainty. Some authors have based their research on stochastic approaches (Kall and Wallace, 1994). However, sometimes the use of probability distributions is not justified. Using fuzzy models avoids unrealistic modelling and offers a chance to reduce information costs. In Inuiuchi and Ramik (2000), the portfolio selection problem exemplifies the advantages and disadvantages of different fuzzy mathematical programming approaches.

This paper presents a summary of some optimization schemes for managing portfolio selection problems based on soft-computing methods. The techniques used to solve these models include exact optimization methods and heuristics techniques, some of which are general and others are adapted from fuzzy location models.

2 Formulating the Portfolio Selection Problem

The standard formulation of Markowitz model (MV), is

\[
\begin{align*}
\text{Min} & \quad x^TQx \\
\text{s.t.} & \quad \sum_{i=1}^{n} E(R_i)x_i \geq \rho \\
& \quad \sum_{i=1}^{n} x_i = 1 \\
& \quad l_i \leq x_i \leq u_i, \quad 1 \leq i \leq n
\end{align*}
\]

(1)

where \(x_i\) is the proportion of a total investment fund chosen by the investor, devoted to asset \(i^{th}\) for \(n\) risky assets, \(R_i\) is the return on the \(i^{th}\) asset.
(it is assumed to be a random variable). Thus, the vector of returns \( R = (R_1, ..., R_n) \) can be summarized by the mean vector \( \text{E}(R) \) and the covariance matrix \( Q \). The parameter \( \rho \) can be assumed to represent the minimal rate of return required by the investor and \( u_i \) (respectively \( l_i \)) the maximum (minimum) amount of total funds that can be invested in asset \( i \).

In the model MV, the averages \( \text{E}(R_i) \) and the elements of the covariance matrix \( Q \) can be approximated using historical data.

Dissatisfaction with the traditional notion of variance as a measure of risk is due to the fact that no distinction is made between gains and losses. In fact, Markowitz (1959) provided two suggestions for measuring downside risk: a semi-variance, which is the sum of the squares of negative deviations from the mean, or a semi-variance computed from a target return.

Moreover, different elements can be fuzzified in the portfolio selection problem. Some authors use possibility distributions to model the uncertainty on returns. Tanaka and Guo focus on the achievements of possibility theory and its applications for operation research, in particular to portfolio selection models. Their approach permits expert knowledge to be incorporated by means of possibility grading, to reflect the degree of similarity between the future state of stock markets and their state in previous periods (see, for instance, Tanaka and Guo, 1999; Guo and Tanaka, 2003). Other authors study the portfolio selection problem using fuzzy formulations. Watada (1997) proposed a fuzzy portfolio selection model where fuzzy numbers were used to represent the decision makers’ aspiration levels for the expected rate of return and a certain degree of risk. Ortí et al (2002) propose the incorporation of fuzzy numbers to represent the uncertainty of the future return on assets and set out portfolio selection as a problem of nonlinear multi-objective programming with fuzzy parameters.

Essentially, two kinds of uncertainties can be distinguished in the portfolio selection model:

a) that arising from historical data, which are used to estimate the volatility and expected returns of the assets.

b) that depending on investor criteria.

We devote one section to each type of uncertainty.

3 Modelling Uncertainty in the Data

It is a well known fact that the portfolio selection problem has fuzzy characteristics (see León et al., 2002). In this context, where investors seek the minimum risk under a given expected return, unfeasibility is caused by the conflict between the desired return and diversification requirements. Hence, in the investor’s opinion, the viability of the instance depends on how severe perturbations must be in order to make it feasible. The fuzzy linear programming techniques used to repair infeasible instances with respect to the original will of the investor as much as possible. The uncertainty of the returns on assets is represented as an expected interval of a fuzzy number (Dubois and Prade, 1987) and the risk of the portfolio is defined as the fuzzy measurement of the mean total semi-deviation (León et al, 2004). Moreover, decision maker criteria are also added to the model.

3.1 Portfolio Selection Using LR-fuzzy Numbers

Let us assume that \( \tilde{R}_i \) is an LR-fuzzy number with trapezoidal possibility distribution. The return of the \( i^{th} \) asset is therefore denoted by

\[
\tilde{R}_i = (a_{i1}, a_{i2}, c_i, d_i)_{LR},
\]

then, the mean of the fuzzy return of the portfolio,

\[
\tilde{c} = \sum_{i=1}^{n} R_i x_i,
\]

has the following expected interval as a mean

\[
E(\tilde{c}) = \left[ \sum_{i=1}^{n} \frac{a_{i1} x_i}{2}, \sum_{i=1}^{n} \frac{a_{i2} x_i}{2}, \sum_{i=1}^{n} \frac{c_i x_i}{2}, \sum_{i=1}^{n} \frac{d_i x_i}{2} \right],
\]

and the risk is:

\[
g(\tilde{c}) = \left[ 0, \sum_{i=1}^{n} \frac{1}{2} \left( a_{i2} + a_{i1} \right) + \frac{1}{4} (c_i - d_i) \right].
\]
In order to determine the intervals of the mean returns and risks, following the proposal from León et al. (2004), the upper bound of the interval can be minimized and a representative of the expected interval can be chosen. The mathematical program is:

\[
\begin{align*}
\text{Min} & \quad \sum_{i=1}^{n} \left( a_{wi} - a_{li} + \frac{1}{2}(c_{i} + d_{i}) \right) x_{i} \\
\text{s.t.} & \quad \sum_{i=1}^{n} \left( \frac{1}{2}(a_{ui} - a_{li}) + \frac{1}{4}(c_{i} + d_{i}) \right) x_{i} \geq \rho \\
& \quad \sum_{i=1}^{n} x_{i} = 1 \\
& \quad l_{i} \leq x_{i} \leq u_{i}, \ 1 \leq i \leq n
\end{align*}
\]

(2)

### 3.2 Portfolio Selection Using Genetic Algorithms

Formulating (MV) implies a problem with the number of assets that can belong to the optimal portfolio. For this reason some authors suggest the inclusion of a maximum bound for the number of assets to be included in a portfolio in the formulation of the problem. Mixed binary quadratic programming can be used to model this with the following results:

\[
\begin{align*}
\text{Min} & \quad x^{\prime}Qx \\
\text{s.t.} & \quad \sum_{i=1}^{n} E(R_{i}) x_{i} \geq \rho \\
& \quad \sum_{i=1}^{n} x_{i} = 1 \\
& \quad \sum_{i=1}^{n} y_{i} \leq N_{u} \\
& \quad y_{i} \in \{0,1\} \\
& \quad l_{i} \leq x_{i} \leq y_{i} u_{i}, \ 1 \leq i \leq n
\end{align*}
\]

(3)

where \( N_{u} \) is the maximum number of assets that can belong to a portfolio

By limiting the number of assets in the model, a small variation in the parameters fixed by the decision-maker (invested capital or required return) can force a change in the composition of the optimal portfolio which can alter risk significantly. If \( z^{*} \) is the optimal risk of the model (3), the associated fuzzy model can be represented by this non linear optimization problem:

\[
\begin{align*}
\text{Min} & \quad \lambda \\
\text{s.t.} & \quad x^{\prime}Qx \geq z^{*} \\
& \quad \sum_{i=1}^{n} E(R_{i}) x_{i} \geq \rho \\
& \quad \sum_{i=1}^{n} x_{i} = 1 \\
& \quad \sum_{i=1}^{n} y_{i} \leq N_{u} \\
& \quad y_{i} \in \{0,1\} \\
& \quad y_{i} l_{i} \leq x_{i} \leq y_{i} u_{i}, \ 1 \leq i \leq n
\end{align*}
\]

(4)

Moreover, \( \mu_{u} \) and \( \mu_{l} \) represent, respectively, the membership functions to the objective set and the opportunity set, that is, the level of improvement in the risk and the degree of feasibility of the portfolio.

The model (4) makes it possible to fix the number of assets to invest in and also to obtain a portfolio by using genetic algorithms. This is useful when a quick solution is required or when the problem being tackled is large. This paper develops the approach appearing in Bermúdez et al. (2005, 2007), based on the basic rules of genetic algorithms, to obtain a simple algorithm structure.

### 4 Modelling Investor Preferences

The portfolio selection problem has two data concerning decision maker preferences, namely the capital to be invested and the risk to be assumed. The investor can be assumed to know with certainty the capital that he or she would like to consider, and in fact, in the model (1) this quantity has been normalized to the unit. However, determining the risk to be assumed can be more flexible. As a result, it is worth incorporating this flexibility in a fuzzy model.

In order to do so, it is preferable to work with the commonly called dual model of (1):
Max \( \sum_{i=1}^{n} E(R_{i})x_{i} \)
\[ s.t. \quad x^{t}Qx \leq R \]
\[ \sum_{i=1}^{n} x_{i} = 1 \]
\[ l_{i} \leq x_{i} \leq u_{i}, \quad 1 \leq i \leq n \]

Here the aim is to maximize the return subject to a maximum risk \( R \).

We are going to show that the techniques used in the analysis of the p-median problem in Canós et al. (1999, 2001) can be generalized to deal with the risk fuzziness in the portfolio selection problem. The main idea is to consider partially feasible solutions involving slightly greater risk than that fixed by the decision maker, and study the possibilities that they offer in order to improve the expected return.

When compared with the p-median case, this problem happens to be more complicated, due to the p-median problem being linear and the risk constraint in the portfolio model being quadratic. Moreover, in the p-median case, a small reduction in covered demand affected optimal cost in a simple linear way, whereas the way in which the maximum expected return depends on the accepted risk is rather more complicated.

A fuzzy set \( \tilde{S} \) of partially feasible solutions is defined so that portfolio selection belongs to \( \tilde{S} \) with a degree of membership that depend on how much it exceeds the risk \( R \) fixed by the investor. On the other hand, a second fuzzy set \( \tilde{G} \) is defined whose membership function reflects the improvement of the return provided by a partially feasible solution with respect to the optimal crisp return \( z^{*} \). In practice, we consider piecewise linear membership functions

\[
\mu_{\tilde{G}}(x) = \begin{cases} 
1 & \text{if } r \leq R \\
1 - \frac{r - R}{p_{f}} & \text{if } R < r < R + p_{f} \\
0 & \text{if } r \geq R + p_{f}
\end{cases}
\]

\[
\mu_{\tilde{S}}(x) = \begin{cases} 
0 & \text{if } z \leq z^{*} \\
\frac{z - z^{*}}{p_{g}} & \text{if } z^{*} < z < z^{*} + p_{g} \\
1 & \text{if } z \geq z^{*} + p_{g}
\end{cases}
\]

where \( r \) and \( z \) are the risk and the return provided by the portfolio \( x \) (which is assumed to satisfy the constraints of (5) except in the first case), the parameter \( p_{f} \) is the maximum increment in risk that the decision maker can accept, and \( p_{g} \) is the increment of the return that the decision maker would consider completely satisfactory. From this, we can define a global degree of satisfaction

\[
\lambda(x) = \min\{\mu_{\tilde{G}}(x), \mu_{\tilde{S}}(x)\}
\]

which is the membership degree to the fuzzy intersection of \( \tilde{S} \cap \tilde{G} \).

The fuzzy portfolio model becomes

\[
\begin{align*}
\text{Max} & \quad \lambda(x) \\
\text{s.t.} & \quad x \in \tilde{S}
\end{align*}
\]

In order to solve (6), the optimal solution of (5) will be calculated for each risk level \( R \). For this, we solve explicitly the Kuhn-Tucker conditions for the problem. To carry out the computations in a generic framework, we start making a change of variables to diagonalize the risk matrix.

As we are interested in small variations of \( R \), the variables \( x_{i} \) that are zero in the optimal crisp portfolio can be removed and hence assume that non-negative conditions are not active. Standard linear algebra theory ensures us that we can decompose

\[
S = A' DA,
\]

where the matrix \( D \) is a diagonal and \( A \) is regular. Then the change of variables \( y = Ax \) transforms the problem to

\[
\begin{align*}
\text{Max} & \quad r'y \\
\text{s.t.} & \quad by = 1 \\
& \quad y^{T} Dy = R \\
& \quad l \leq A^{-1} y \leq u
\end{align*}
\]

where \( r' = r'A^{-1} \) and \( b = (1,1,...,1)A^{-1} \).

As the last inequalities are not active the Kuhn-Tucker conditions for (7) are simply the Lagrange conditions of the classical problem resulting from removing them. These are:

\[
\begin{align*}
\sum_{i} b_{i} y_{i} &= 1, \\
\sum_{i} d_{i} y_{i}^{2} &= R, \\
(r'_{i} - b y_{i}) - 2 d_{i} y_{i} \mu &= 0, \quad 1 \leq i \leq n.
\end{align*}
\]
where \( \lambda, \mu \) are the Kuhn-Tucker multipliers of the capital and the risk constraint respectively. We also have the sign Kuhn-Tucker condition \( \mu \geq 0 \).

Solving these equations, \( y_i \) can be expressed as functions \( y(R) \) which makes it possible to:

a) Determine the interval in which \( R \) can oscillate in order for the solution \( y(R) \) to satisfy the constraints

\[
I \leq A^{-1} y \leq u.
\]

If values of \( R \) exceeding this interval need to be considered, we should deal with the solution of the problem analogue to (6) for a different choice of the variables.

b) Computing the return as a function \( F(R) \) from which the degree of feasibility \( \mu_f(R) \) and the degree of improvement of the goal \( \mu_g(R) \) can be calculated as functions of \( R \).

The expression obtained is

\[
F(R) = \frac{2LR - 1}{2\sqrt{\Delta(R)}} \sum_{i=1}^{n} \left( \frac{r^2_i - r_i b_i}{d_i} \right) \frac{2KLR - K \pm \sqrt{\Delta(R)}}{2L^2R - L},
\]

where

\[
K = \sum_{i=1}^{n} \frac{b_i r_i^2}{2d_i},
\]

\[
L = \sum_{i=1}^{n} \frac{b_i^2}{2d_i},
\]

\[
M = \sum_{i=1}^{n} \frac{r_i^2}{4d_i},
\]

\[
\Delta(R) = K^2 - 2ML + (4ML^2 - 2K^2L)R.
\]

These expressions can be used to determine the risk \( R^* \) such that

\[
\mu_f(R^*) = \mu_g(R^*)
\]

which is easily shown to be the risk maximizing \( \lambda \). Hence, the portfolio \( \lambda(R^*) \) corresponding to \( y(R^*) \) by the change of variables is the optimal solution of (6).

### 4.1 A Numerical Example

Below is a simple example of the fuzzy portfolio selection problem. Five assets from the historical data introduced by Markowitz (Markowitz, 1959) are used.

<table>
<thead>
<tr>
<th>Year</th>
<th>AmT</th>
<th>ATT</th>
<th>USS</th>
<th>GM</th>
<th>ATS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1937</td>
<td>-0.305</td>
<td>-0.173</td>
<td>-0.318</td>
<td>-0.477</td>
<td>-0.457</td>
</tr>
<tr>
<td>1938</td>
<td>0.513</td>
<td>0.098</td>
<td>0.285</td>
<td>0.714</td>
<td>0.107</td>
</tr>
<tr>
<td>1940</td>
<td>0.055</td>
<td>0.2</td>
<td>-0.047</td>
<td>0.165</td>
<td>-0.424</td>
</tr>
<tr>
<td>1941</td>
<td>-0.126</td>
<td>0.03</td>
<td>0.104</td>
<td>-0.043</td>
<td>-0.189</td>
</tr>
<tr>
<td>1942</td>
<td>-0.003</td>
<td>0.067</td>
<td>-0.039</td>
<td>0.476</td>
<td>0.865</td>
</tr>
<tr>
<td>1943</td>
<td>0.428</td>
<td>0.3</td>
<td>0.149</td>
<td>0.225</td>
<td>0.313</td>
</tr>
<tr>
<td>1944</td>
<td>0.192</td>
<td>0.103</td>
<td>0.26</td>
<td>0.29</td>
<td>0.637</td>
</tr>
<tr>
<td>1945</td>
<td>0.446</td>
<td>0.216</td>
<td>0.419</td>
<td>0.216</td>
<td>0.373</td>
</tr>
<tr>
<td>1946</td>
<td>-0.088</td>
<td>-0.046</td>
<td>-0.078</td>
<td>-0.272</td>
<td>-0.037</td>
</tr>
</tbody>
</table>

We have fixed a risk level \( R=0.03 \). The optimal crisp portfolio is formed by assets AmT, ATT, GM and ATS providing an optimal return \( z^* = 0.102386 \).

For the fuzzy model, we have fixed \( p^f=0.02 \), \( p^g=0.02 \) and solved it by using Mathematica®.

The optimal return for a given risk \( R \), happens to be

\[
F(R) = \frac{-0.019 + 1.059R + 0.081\sqrt{-0.529 + 28.231R}}{\sqrt{-0.529 + 28.231R}}
\]

Computations are valid for risks in the interval [0.0237814, 0.0876355].

Figure 1 shows the functions \( \mu_f(R) \) and \( \mu_g(R) \). They intersect at \( R^* = 0.0412373 \) corresponding to \( \lambda = 0.438133 \). The return of the fuzzy portfolio is 0.111149.

We observe that the global satisfaction degree is quite low. This means that risk has to be increased a great deal in order to obtain a not
very significant increase in the return on the asset. In other words, the crisp solution seems to be quite stable. However, it is still to be studied how solutions depend on the chosen membership functions by means of a suitable sensitivity analysis, analogue to that developed in Canós et al. (2008).

5 Conclusions

The models proposed for dealing with data uncertainty, are a further development of those presented by authors in previous editions of IPMU. They improve classical portfolio selection models in the sense that they lead to solutions with better returns and they adjust in a more realistic way to stock market trends (León et al. 2004). The models concerning fuzziness in investor criteria, have been less studied by the literature, and should be developed in greater detail before combining them with the previous ones. The task of combining both kinds of uncertainty and developing suitable intelligent systems to deal with them is still pending.

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References


