T-norm based fuzzy logics preserving degrees of truth

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Abstract

T-norm based fuzzy logics are usually considered as *truth preserving*, that is, taking 1 as the only truth value to be preserved in inferences. In this paper we study t-norm based fuzzy logics *preserving degrees of truth*, that is, preserving the lower bounds of the truth degrees of the premises. These logics are axiomatizable by using a restricted form of Modus Ponens together with the rule of Adjunction for $\wedge$ and their properties turn out to be very diferent from the ones satisfied by the *truth preserving logics*. For instance, they are selfextensional (but not Fregean) and with few exceptions they are not algebraizable (not even protoalgebraic).

Keywords: many-valued logics, residuated lattices, protoalgebraic logics, substructural logics

1 Introduction

From the algebraic semantics point of view, most of the fuzzy (and substructural) logics studied in the literature given by distinguished subvarieties K of the variety RL of residuated lattices[1] (like those of BL, MV, MTL, Heyting algebras, etc.) are defined by taking 1 (the maximum of any algebra of a given variety) as the only truth value to be preserved by inference, in the sense of yielding valid consequences from valid premises. We will refer to them as *truth preserving logics* and will denote them by $\vdash_{\mathsf{K}}$. Truth preserving logics $\vdash_{\mathsf{K}}$ are algebraizable in the sense of Blok and Pigozzi [2], and thus there is a nice correspondence between varieties of residuated lattices and axiomatic extensions of Höhle's Monoidal Logic ML (or Ono's FL_{ew}), i.e., the truth preserving logic of residuated lattices.

The intended role played above by residuated lattices corresponds to truth-value structures, but indeed truth preserving logics do not take full advantage of being many-valued, as they only focus on the truth value 1 (*the truth*) and not on other intermediate truth values. A possible way to circumvent this inconvenience while keeping the truth preserving framework is to introduce truth-constants into the language. Actually, in this way the expanded language allows one to have formulas of the kind $\overline{r} \to \varphi$ which, when evaluated to 1, express that the truth value of φ is greater or equal than r. This methodology actually goes back to Pavelka [21] where he built a propositional many-valued logic which turned out to be equivalent to the expansion of Łukasiewicz logic by adding a truth constant $\overline{r}$ for each *real* $r \in [0, 1]$ into the language, together with a number of additional axioms. This logic was further developed by Nóvak [19] and Hájek [17], and more recently a similar approach has been applied in [8, 10] to study the expansions with truth constants of other fuzzy logics including Gödel, Product, Nilpotent Minimum logics as well as other continuous t-norm

[1] In this paper we assume a residuated lattice to be bounded, commutative and integral.

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based logics. All these expansions, like the truth preserving logics $\vdash_{\mathsf{K}}$, have been shown to be algebraizable.

In this paper, following the approach initiated in [13] for the particular case of Łukasiewicz logic, we aim at going beyond the truth preserving framework in order to exploit many-valuedness by focussing on the notion of inference $\models_{\mathsf{K}}^{\leqslant}$ which results from preserving lower bounds of truth values, and hence not only preserving the value 1. In this setting the language remains the same as in truth preserving logics $\vdash_{\mathsf{K}}$, but what changes is the inference relation. This kind of inference corresponds to the so-called *logics preserving degrees of truth*, discussed at length in [11, 13, 20], and follows a very general pattern which could be considered for any class of truth structures endowed with an ordering relation.[2]

Notation: Throughout the paper we fix the following propositional (algebraic) language $\mathcal{L} = \langle \wedge, \vee, \star, \rightarrow, 1, 0 \rangle$ with connectives of arity $\langle 2, 2, 2, 2, 0, 0 \rangle$. We will use the notation φ^n to denote $\varphi \star \overset{n}{\ldots} \star \varphi$. Operations interpreted in a specific algebra will be denoted with the same symbol. $\boldsymbol{Fm}$ will denote the formula algebra (of this similarity type), and Fm its universe. If $\boldsymbol{A}$ is an algebra, $\mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$ will denote the set of all homomorphisms from $\boldsymbol{Fm}$ to $\boldsymbol{A}$, in other words, the set of evaluations of formulas in the algebra $\boldsymbol{A}$, and $\mathrm{Co}\boldsymbol{A}$ will denote the set of all congruences of $\boldsymbol{A}$.

2 Definitions, basic properties and axiomatization

In this section we will consider for each variety K of residuated lattices, two logics: the truth preserving one (the most studied in fuzzy logic) and the truth degree preserving one (the one that the paper deals with). Following modern algebraic logic literature (see for instance [5, 15, 23]), in this paper we identify a ***logic*** L with its consequence relation, which can be denoted as $\vdash_L$ or similar symbols. This is a relation $\vdash_L \subseteq P(Fm) \times Fm$, and we use the common relational notation and write $\Gamma \vdash_L \varphi$ which is to be interpreted as "φ follows from Γ in the logic L".

Traditionally, the substructural (fuzzy) logic associated in the literature with each subvariety K of RL is the logic $\vdash_{\mathsf{K}}$ which will be called **the logic that preserves truth with respect to the class** K (where truth is represented by the constant 1). This logic has K as its algebraic counterpart; more precisely, $\vdash_{\mathsf{K}}$ is a (finitely and regularly) algebraizable logic having K as its equivalent algebraic semantics, with defining equation[3] $x \approx 1$ and equivalence formula $x \leftrightarrow y := (x \rightarrow y) \star (y \rightarrow x)$. From this it follows that $\vdash_{\mathsf{K}}$ is the (finitary) logic determined by the two clauses

$$
\begin{aligned}
&(i)\ \ \varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathsf{K}} \psi \iff \\
&\qquad \mathsf{K} \models \varphi_0 \approx 1\ \&\ \ldots\ \&\ \varphi_{n-1} \approx 1 \Rightarrow \psi \approx 1, \\
&(ii)\ \ \emptyset \vdash_{\mathsf{K}} \psi \iff \mathsf{K} \models \psi \approx 1,
\end{aligned}
$$

where the symbol $\models$, when written without any sub- or superscript, stands for first-order (or quasiequational) satisfaction, and & and $\Rightarrow$ are the symbols for first-order conjunction and implication; the first clause amounts to saying that

$$
\begin{aligned}
&\varphi_0, \ldots, \varphi_{n-1} \vdash_{\mathsf{K}} \psi \iff \\
&\quad \forall \boldsymbol{A} \in \mathsf{K}, \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}), \\
&\quad \text{if } v(\varphi_i) = 1 \text{ for all } i < n, \text{ then } v(\psi) = 1.
\end{aligned}
$$

Following the discussion in the introduction, we associate with each subvariety K of RL another logic, which will be our main object of study:

Definition 2.1. *The logic $\models_{\mathsf{K}}^{\leqslant}$, which we call **the logic that preserves degrees of truth with respect to** K, is defined as follows, for $\Gamma \cup \{\psi\} \subseteq Fm$:*

1. *For $\Gamma = \{\varphi_0, \ldots, \varphi_{n-1}\}$, $n \geqslant 1$,*
$$
\begin{aligned}
&\varphi_0, \ldots, \varphi_{n-1} \models_{\mathsf{K}}^{\leqslant} \psi \iff \\
&\forall \boldsymbol{A} \in \mathsf{K}, \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A}), \forall a \in A, \\
&\textit{if } v(\varphi_i) \geqslant a \textit{ for all } i < n \textit{ then } v(\psi) \geqslant a.
\end{aligned}
$$

[2] The proofs of the theorems stated in this paper will appear in a more extended paper under preparation [3].

[3] We will use the notation $x \approx y$ for equations and $x \preccurlyeq y$ will stand for $x \wedge y \approx x$.

2. $\emptyset \models^{\leqslant}_{\mathsf{K}} \varphi$ *when for all* $\boldsymbol{A} \in \mathsf{K}$*, for all* $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$*,* $v(\varphi) = 1$.

3. *For an infinite* $\Gamma \subseteq Fm$*,* $\Gamma \models^{\leqslant}_{\mathsf{K}} \psi$ *when there exist* $\varphi_0, \ldots, \varphi_{n-1} \in \Gamma$ *such that* $\varphi_0, \ldots, \varphi_{n-1} \models^{\leqslant}_{\mathsf{K}} \psi$

It is obvious that this definition actually yields a finitary logic, that is, a finitary consequence relation on the set of formulas. It also follows from clauses 3 and 1 that $\models^{\leqslant}_{\mathsf{K}}$ really *preserves degrees of truth with respect to* K, that is, it satisfies, for all $\Gamma \cup \{\psi\} \subseteq Fm$,

$$\begin{aligned}&\Gamma \models^{\leqslant}_{\mathsf{K}} \psi \implies \\ &\quad \forall \boldsymbol{A} \in \mathsf{K}\,, \forall v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})\,, \forall a \in A\,, \\ &\quad \text{if } \; v(\gamma) \geqslant a \text{ for all } \gamma \in \Gamma \; \text{ then } \; v(\psi) \geqslant a.\end{aligned}$$

When the set of premises is finite, it is easy to show that $\models^{\leqslant}_{\mathsf{K}}$ admits the following equivalent expressions.

Lemma 2.2. *If* $\{\varphi_0, \ldots, \varphi_{n-1}, \psi\} \subseteq Fm$ *then then relation* $\varphi_0, \ldots, \varphi_{n-1} \models^{\leqslant}_{\mathsf{K}} \psi$ *is equivalent to each one of the following statements:*

- $\varphi_0 \wedge \cdots \wedge \varphi_{n-1} \models^{\leqslant}_{\mathsf{K}} \psi$.
- $\mathsf{K} \models \varphi_0 \wedge \cdots \wedge \varphi_{n-1} \preccurlyeq \psi$.

The logics $\vdash_{\mathsf{K}}$ and $\models^{\leqslant}_{\mathsf{K}}$ have a quite different behaviour but, of course, they also have some tight relations, as described next. Indeed, as a consequence of Definition 2.1 and Lemma 2.2, if $\{\varphi_0, \ldots, \varphi_{n-1}, \psi\} \subseteq Fm$ then the following equivalences hold:

- $\emptyset \models^{\leqslant}_{\mathsf{K}} \varphi \iff \emptyset \vdash_{\mathsf{K}} \varphi$
- $\varphi_0, \ldots, \varphi_{n-1} \models^{\leqslant}_{\mathsf{K}} \psi \iff$
  $\emptyset \vdash_{\mathsf{K}} \varphi_0 \wedge \cdots \wedge \varphi_{n-1} \to \psi$.

Hence the two logics *have the same theorems* (or *tautologies*), and furthermore $\models^{\leqslant}_{\mathsf{K}}$ is determined by the theorems of $\vdash_{\mathsf{K}}$.

As an immediate consequence of this relationship we obtain the following relations between the two logics:

- $\varphi \dashv\models^{\leqslant}_{\mathsf{K}} \psi \iff \vdash_{\mathsf{K}} \varphi \leftrightarrow \psi$,
- the logic $\vdash_{\mathsf{K}}$ is an extension of the logic $\models^{\leqslant}_{\mathsf{K}}$.

This extension is in general strict as one can easily notice, e.g., $\varphi \vdash_{\mathsf{K}} \varphi \star \varphi$ trivally holds, while in general $\varphi \not\models^{\leqslant}_{\mathsf{K}} \varphi \star \varphi$.

Next we will obtain a Hilbert style axiomatization of the logic $\models^{\leqslant}_{\mathsf{K}}$. For each variety K of residuated lattices, we denote by $TAUT(\mathsf{K})$ the set of tautologies (theorems) of $\vdash_{\mathsf{K}}$, which is also the set of tautologies (theorems) of $\models^{\leqslant}_{\mathsf{K}}$. This set of formulas is semantically determined (by $\varphi \approx 1$ true in K) and does not depend on any particular axiomatization of $\vdash_{\mathsf{K}}$. Actually, it is known that $\vdash_{\mathsf{K}}$ can be axiomatized by taking all formulas in $TAUT(\mathsf{K})$ as axioms and the sole rule of Modus Ponens.

Formally, we take an *inference rule* to be any set of pairs $\langle \Gamma, \varphi \rangle$ where Γ is a finite set of formulas. In order to exhibit an axiomatization of the logic $\models^{\leqslant}_{\mathsf{K}}$ we consider the following two rules of inference:

$$\begin{aligned}&(\text{Adj-}\wedge) \quad \big\{\big\langle \{\varphi, \psi\}\,, \varphi \wedge \psi \big\rangle : \varphi, \psi \in Fm\big\}. \\ &(\text{MP-r}) \quad \big\{\big\langle \{\varphi, \varphi \to \psi\}\,, \psi \big\rangle : \varphi, \psi \in Fm \text{ and } \\ &\qquad\qquad \varphi \to \psi \in TAUT(\mathsf{K})\big\}.\end{aligned}$$

Notice that the rule (MP-r), a restricted form of Modus Ponens, can be applied only when the premise of the form $\varphi \to \psi$ belongs to $TAUT(\mathsf{K})$, that is, it is a tautology of $\vdash_{\mathsf{K}}$.

Theorem 2.3 (Completeness)**.** *The logic* $\models^{\leqslant}_{\mathsf{K}}$ *is the logic defined by the axiomatic system having all the formulas in* $TAUT(\mathsf{K})$ *as axioms, and the rules* (Adj-$\wedge$) *and* (MP-r) *as rules of inference.*

Given an axiomatic presentation for $\vdash_{\mathsf{K}}$ that uses some axioms $AX(\mathsf{K})$ and only the ordinary rule of Modus Ponens $\big\{\big\langle \{\varphi, \varphi \to \psi\}\,, \psi \big\rangle : \varphi\,, \psi \in Fm\big\}$, the set $AX(\mathsf{K})$ can replace the set $TAUT(\mathsf{K})$ as the set of axioms of $\vdash^{\leqslant}_{\mathsf{K}}$. This is so because each application of Modus Ponens in a proof of a theorem of $\vdash_{\mathsf{K}}$ will in fact be an application of (MP-r), therefore the same proof will be a proof in $\vdash^{\leqslant}_{\mathsf{K}}$. This is the case for many varieties of t-norm-based logics such as BL, MTL, MV, Product, Gödel, WNM or the variety generated by any finite family of continuous t-norms.

As regards to filters, it is well known that the filters $\mathcal{F}i_{\vdash_{\mathsf{K}}}(\boldsymbol{A})$ of the truth preserving logic $\vdash_{\mathsf{K}}$ over an algebra $\boldsymbol{A} \in \mathsf{K}$ are the implicative filters of $\boldsymbol{A}$, i.e., those lattice filters closed under modus ponens. On the other hand, it can be shown that the filters $\mathcal{F}i_{\models^{\leqslant}_{\mathsf{K}}}(\boldsymbol{A})$ of the logic preserving degrees of truth $\models^{\leqslant}_{\mathsf{K}}$ over $\boldsymbol{A}$ are just the lattice filters of $\boldsymbol{A}$. Formally, this implies the following proposition such that $\Gamma \models_{\langle \boldsymbol{A},F\rangle} \varphi$ holds by definition iff for all $v \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, if $v(\gamma) \in F$ for all $\gamma \in \Gamma$, then $v(\varphi) \in F$.

Proposition 2.4. *The logic $\models^{\leqslant}_{\mathsf{K}}$ is the intersection of the logics $\models_{\langle \boldsymbol{A},F\rangle}$ where $\boldsymbol{A} \in \mathsf{K}$ and F is a lattice filter of $\boldsymbol{A}$.*

Let K be a variety of residuated lattices, $\boldsymbol{A} \in \mathsf{K}$, and let $F \subseteq A$ be a non-empty order filter (or increasing subset). Then recall that the following conditions are equivalent: **1.** F is an implicative filter. **2.** F is closed under $\star$. **3.** F is a lattice filter such that for all $a \in A$, if $a \in F$ then $a^2 = a \star a \in F$.

Since both logics are defined from the class K, they are characterized by their filters on the algebras in K. Therefore, using the above equivalent conditions, we have the following characterization of $\vdash_{\mathsf{K}}$ in terms of $\models^{\leqslant}_{\mathsf{K}}$.

Corollary 2.5. *$\vdash_{\mathsf{K}}$ is the extension of $\models^{\leqslant}_{\mathsf{K}}$ obtained by adding to it any one of the following rules:*

(MP) $\varphi, \varphi \to \psi \ \vdash \ \psi$.

(Adj$\star$) $\varphi, \psi \ \vdash \ \varphi \star \psi$.

(square-closing) $\varphi \ \vdash \ \varphi^2$.

After the axiomatic presentation found for $\models^{\leqslant}_{\mathsf{K}}$ in Theorem 2.3, this Corollary can also be (informally) read as saying that $\models^{\leqslant}_{\mathsf{K}}$ is obtained from $\vdash_{\mathsf{K}}$ by weakening the rules of Modus Ponens and Adjunction, namely, restricting the rule of Modus Ponens and replacing Adjunction for $\star$ by Adjunction for $\wedge$.

3 Classification in the hierarchies of abstract algebraic logic

In this section we investigate the location of the logic $\models^{\leqslant}_{\mathsf{K}}$ inside the two general hierarchies of logics considered in abstract algebraic logic, the ***Frege hierarchy*** and the ***Leibniz hierarchy***; see [12, 15] for more details on the hierarchies. The Frege hierarchy is organized, so to speak, around the kind of replacement properties of the logic, while the Leibniz hierarchy is organized around properties of the so-called *Leibniz operator* on the filters of the logic.

The basic, largest class in the Frege hierarchy is that of ***selfextensional logics*** [23], which are the logics L whose interderivability relation $\dashv\vdash_L$ is a congruence of the formula algebra. This means that L satisfies the following *weak form of the replacement property*: for any $\alpha, \beta, \varphi(x, \vec{y}) \in Fm$,

$$\begin{array}{rl} \text{if} & \alpha \dashv\vdash_L \beta \\ \text{then} & \varphi(\alpha, \vec{y}) \dashv\vdash_L \varphi(\beta, \vec{y}). \end{array}$$

It turns out that all our logics belong to this class:

Proposition 3.1. *For each variety K of residuated lattices, the consequence relation $\models^{\leqslant}_{\mathsf{K}}$ is selfextensional.*

A smaller class in this hierarchy is that of Fregean logics. A logic L is ***Fregean*** when for each theory T of L, the interderivability relation of L modulo T is a congruence of $\boldsymbol{Fm}$. This corresponds to the following *strong form of the replacement property*: for any theory T of L, any $\alpha, \beta, \varphi(x, \vec{y}) \in Fm$,

$$\begin{array}{rl} \text{if} & T, \alpha \vdash_L \beta \ \text{ and } \ T, \beta \vdash_L \alpha \\ \text{then} & T, \varphi(\alpha, \vec{y}) \vdash_L \varphi(\beta, \vec{y}). \end{array}$$

Fregean logics were introduced in [14, 22], and have been extensively studied in the context of protoalgebraic logics in [6, 7], see also [5].

Proposition 3.2. *Let K be a variety of residuated lattices. Then the following conditions are equivalent:*

1. *The logic $\models^{\leqslant}_{\mathsf{K}}$ is Fregean.*

2. K *is a variety of Heyting algebras.*

Thus, as far as the Frege hierarchy is concerned, our class of (selfextensional) logics is divided into two groups: those generated by a class of Heyting algebras, which are Fregean, and the rest, which are not.

Now we focus our attention on the Leibniz hierarchy. This hierarchy is much richer and more complex than the Fregean one, and most of its classes can be defined or characterized in several ways, although the majority of them concern several properties of the ***Leibniz operator***. This operator corresponds to the mapping $F \longmapsto \boldsymbol{\Omega_A} F$ restricted to the set of all L-filters on a fixed algebra $\boldsymbol{A}$, where $\boldsymbol{\Omega_A} F$, called the ***Leibniz congruence***, is the largest of all the congruences of $\boldsymbol{A}$ that are ***compatible*** with F in the sense that they do not relate elements inside F with elements outside F, that is, $\boldsymbol{\Omega_A} F = \max\{\theta \in \mathrm{Co}\boldsymbol{A} : \text{ if } \langle a, b\rangle \in \theta \text{ and } a \in F \text{ then } b \in F\}$. It can be shown [2] that such congruence always exists and can be characterized syntactically.

The largest class of logics in the Leibniz hierarchy is the class of ***protoalgebraic logics***. They can be defined in a number of equivalent ways. We first consider them as the logics such that the monotonicity condition

$$\text{if } G \subseteq F \text{ then } \boldsymbol{\Omega_A} G \subseteq \boldsymbol{\Omega_A} F \qquad \text{(M)}$$

holds for every algebra $\boldsymbol{A}$ and every pair of filters F, G of the logic over $\boldsymbol{A}$.

It has been shown in [13, Theorem 3.11] that condition (M) fails in the variety of MV-algebras, and hence $\models^{\leqslant}_{\mathsf{MV}}$, the logic which preserves degrees of truth associated to Łukasiewicz's infinite-valued logic, is non-protoalgebraic. Another argument shows that (M) fails in the product algebra associated with the negative cone of the linearly lexicographically ordered abelian group $\mathbb{Z} \times_l \mathbb{Z}$ (see [4]). This last fact proves that $\models^{\leqslant}_{\Pi}$, the logic which preserves degrees of truth associated to Product logic, is non-protoalgebraic as well.

It is obvious that non-protoalgebraic logics must always yield instances of algebras and filters where (M) fails. However, in our opinion this is not the best strategy to prove non-protoalgebraicity. The difficulty is that there are varieties (e.g. MV and Π) where the Leibniz operator is monotonic over lattice filters in a generator of the variety (e.g., when the generator is a simple algebra) while there are other algebras in the variety where the monotonicity fails. Next we provide an equational characterization of protoalgebraicity that has the advantatge that it is enough to check it in the algebras generating the variety. The proof relies on another general method of characterizing the protoalgebraicity of $\models^{\leqslant}_{\mathsf{K}}$: the existence of a set of formulas $\Delta(x, y)$ in two variables such that

$$\emptyset \models^{\leqslant}_{\mathsf{K}} \Delta(x, x) \quad \text{and} \quad x, \Delta(x, y) \models^{\leqslant}_{\mathsf{K}} y. \quad \text{(P)}$$

Since $\models^{\leqslant}_{\mathsf{K}}$ is finitary and conjunctive (in the sense of Lemma 2.2), the set $\Delta(x, y)$ can actually be reduced to just one formula $\delta(x, y)$. We are going to see that we can always take $(x \to y)^n \star (y \to x)^n$, for some $n \in \omega$, as the formula $\delta(x, y)$. Notice that in general there is no uniqueness, up to equivalence in $\models^{\leqslant}_{\mathsf{K}}$, of the formula $\delta(x, y)$ satisfying (P); for instance, for classical propositional logic we can consider either the formula $x \to y$ or the formula $x \leftrightarrow y$, which are not equivalent in this logic. In the next result and the following ones, note that for terms φ, ψ, the expression $\varphi \preccurlyeq \psi$ is, on a lattice, actually equivalent to an equation.

Consider the variety $\mathsf{Prot_n} \subseteq \mathsf{RL}$ defined by the equation:

$$x \wedge \big((x \to y)^n \star (y \to x)^n\big) \preccurlyeq y \qquad (\mathrm{Prot}_n)$$

Next we state some equivalent presentations.

Theorem 3.3. *If* $n \geqslant 1$*, any of the following equations can replace* (Prot_n) *in the definition of the variety* $\mathsf{Prot_n}$ *inside* RL*:*

$$x \wedge y^n \approx x \star y^n \qquad (\mathrm{SC}_n)$$

$$x \wedge (x \to y)^n \preccurlyeq y \qquad (\mathrm{MP}_n)$$

$$x \wedge (x \to y)^n \wedge (y \to x)^n \preccurlyeq y \qquad (\mathrm{SMP}_n)$$

The significance of the family of classes $\mathsf{Prot_n}$ lies in the following important result:

Theorem 3.4. *Let K be a variety of residuated lattices. Then the following conditions are equivalent:*

1. *The logic $\models_{\mathsf{K}}^{\leqslant}$ is protoalgebraic.*
2. *There is an $n \in \omega$ such that K satisfies the equation $x \wedge \big((x \rightarrow y)^n \star (y \rightarrow x)^n\big) \preccurlyeq y$, that is, such that $\mathsf{K} \subseteq \mathsf{Prot_n}$.*

Let us consider the following equations

$$x \vee (\neg x^n) \approx 1 \qquad (\mathrm{EM}_n)$$

$$x^n \wedge y^n \approx x^n \star y^n \qquad (\mathrm{IMC}_n)$$

$$x^n \approx x^{n+1} \qquad (\mathrm{E}_n)$$

The varieties of residuated lattices given by these equations will be denoted, respectively, by $\mathsf{EM_n}$, $\mathsf{IMC_n}$ and $\mathsf{E_n}$. The varieties $\mathsf{EM_n}$ and $\mathsf{E_n}$ have been widely considered in the literature, see [16, Chapter 11], as the variety generated by the simple n-contractive residuated lattices and as the variety of the n-contractive residuated lattices. The other names have been chosen so as to follow the associations SC: "Strong Contraction", MP: "Modus Ponens", SMP: "Symmetric Modus Ponens", Prot: "Protoalgebraic", and IMC: "Idempotent Meet Contraction" (cf. Lemma 3.5).

It is straightforward to check that for every $\mathsf{K} \in \{\mathsf{EM}, \mathsf{Prot}, \mathsf{IMC}, \mathsf{E}\}$, if $n \leqslant m$ then $\mathsf{K_n} \subseteq \mathsf{K_m}$. Moreover, for $n = 0$ all previous classes coincide with the variety that has only the trivial algebra, except in the case of SC, where $\mathsf{SC_0} = \mathsf{RL}$. For $n = 1$ it is not difficult to show that $\mathsf{EM_1}$ is the class of Boolean algebras and that $\mathsf{Prot_1} = \mathsf{IMC_1} = \mathsf{E_1}$ is the class of Heyting algebras.

It is well known that in every residuated lattice the set of idempotent elements of $\boldsymbol{A}$ is closed under join, and that in general this is not the case for meet, as Example 4 in [9] proves. However, it is easy to show:

Lemma 3.5. *If $\boldsymbol{A} \in \mathsf{E_n}$, then the following conditions are equivalent:*

1. $\boldsymbol{A} \models x^n \wedge y^n \approx x^n \star y^n$.
2. $\boldsymbol{A} \models x^n \wedge y^n \approx (x \wedge y)^n$.
3. *The set of idempotent elements of $\boldsymbol{A}$ is closed under meet.*

Therefore the variety $\mathsf{IMC_n}$ could also be introduced using (E_n) together with any of the conditions stated in the previous lemma. It is obvious that in all MTL algebras the set of idempotent elements is closed under meet, but notice that the class of residuated lattices with this property is strictly bigger than MTL, as witnessed for instance by Example 5 in [9].

Theorem 3.3 allows to show that $\mathsf{Prot_n} \subseteq \mathsf{E_n}$. However we can do better:

Lemma 3.6. *Let $n \geqslant 1$. Then, $\mathsf{EM_n} \subseteq \mathsf{Prot_n} \subseteq \mathsf{IMC_n} \subseteq \mathsf{E_n}$.*

In the next result we establish, among several things, that the inclusions are proper. There, the term "ordinal sum" is used in the sense of linearly ordered semihoops; see [1, 18]. By MTL and BL we denote the classes of MTL algebras and of BL algebras, respectively.

Theorem 3.7. *Let $n \geqslant 2$.*

1. *The equations* $(\mathrm{Prot}_n), (\mathrm{IMC}_n)$ *and* (E_n)*, and their equivalents, are preserved under the operation of ordinal sums, while* (EM_n) *is not.*
2. $\mathsf{EM_n} \subsetneq \mathsf{Prot_n} \subsetneq \mathsf{IMC_n} \subsetneq \mathsf{E_n}$.
3. $\mathsf{MTL} \cap \mathsf{EM_n} \subsetneq \mathsf{MTL} \cap \mathsf{Prot_n} \subsetneq \mathsf{MTL} \cap \mathsf{IMC_n} = \mathsf{MTL} \cap \mathsf{E_n}$.
4. $\mathsf{BL} \cap \mathsf{EM_n} \subsetneq \mathsf{BL} \cap \mathsf{Prot_n} = \mathsf{BL} \cap \mathsf{E_n}$.
5. *If $\boldsymbol{A}$ is an MTL chain, then $\boldsymbol{A} \in \mathsf{Prot_n}$ if and only if $\boldsymbol{A}$ is an ordinal sum of simple n-contractive MTL chains.*

We note that the restriction to chains in Theorem 3.7 is unavoidable. This necessity is witnessed for instance by the algebra given in [9, Example 5], since it is in $\mathsf{Prot_2}$ but it is not an ordinal sum of simple n-contractive ones.

As a consequence of Theorem 3.7 we have that

1. If K is a variety of residuated lattices such that $\models_{\mathsf{K}}^{\leqslant}$ is protoalgebraic then there exists an $n \in \omega$ such that $\mathsf{K} \subseteq \mathsf{E_n}$. In particular, it follows that if K is any of the

varieties $\mathsf{MV}, \mathsf{\Pi}, \mathsf{BL}, \mathsf{MTL}$ or RL then $\models^{\leqslant}_{\mathsf{K}}$ is non-protoalgebraic.

2. If K is a variety of residuated lattices such that $\models^{\leqslant}_{\mathsf{K}}$ is protoalgebraic then in all algebras of K the set of idempotent elements is closed under meet.

3. If $\mathsf{K} \subseteq \mathsf{MTL}$, then the logic $\models^{\leqslant}_{\mathsf{K}}$ is protoalgebraic if and only if there exists a natural n such that all chains in K are ordinal sums (as semihoops) of simple n-contractive MTL chains (i.e., such that $\mathsf{K} \subseteq \boldsymbol{\Omega}(\mathsf{MTL} \cap \mathsf{EM_n})$ where $\boldsymbol{\Omega}$ is the operator used in [18]). Notice that not all finite MTL chains define a protoalgebraic logic; for instance, the nilpotent minimum chain $\boldsymbol{N_4}$ with four elements does not.

4. If $\mathsf{K} \subseteq \mathsf{BL}$, then the logic $\models^{\leqslant}_{\mathsf{K}}$ is protoalgebraic if and only if there exists a natural n such that $\mathsf{K} \subseteq \mathsf{E_n}$. In particular any finite BL chain defines a protoalgebraic logic.

5. If K is the variety generated by a family of continous t-norms, then K defines a protoalgebraic logic if and only if K is the variety of Gödel algebras (which is, in fact algebraizable).

6. The equation (Prot_n) (and its equivalent forms) gives a characterization of ordinal sums (as semihoops) of simple n-contractive MTL chains that is alternative (possibly simpler) than the one stated in [18, Prop. 4.24]. In this paper these ordinal sums were characterized using (both) equations $x^n \approx x^{n+1}$ and $(y^n \to x) \vee \big(x \to (x \star y)\big) \approx 1$.

Now we investigate when the logics preserving degrees of truth are equivalential. By definition $\models^{\leqslant}_{\mathsf{K}}$ is ***equivalential*** when there is a set of formulas $E(x, y)$ in two variables satisfying the condition (P) and the condition

$$E(x, y) \cup E(z, w) \models^{\leqslant}_{\mathsf{K}} E(x \circ z, y \circ w) \quad \text{(E)}$$

where $\circ \in \{\star, \wedge, \vee, \to\}$. Unlike in the case of protoalgebraic logics, in equivalential logics the set $E(x, y)$ is unique up to equivalence. In our case, since $\models^{\leqslant}_{\mathsf{K}}$ is finitary and conjunctive, if $E(x, y)$ is finite we can assume it to be just one formula $\varepsilon(x, y)$.

Theorem 3.8. *Let K be a variety of residuated lattices and let $n \geqslant 1$. Then the following conditions are equivalent:*

1. $\mathsf{K} \subseteq \mathsf{Prot_n}$.

2. *For every algebra $\boldsymbol{A}$ and every $F \in \mathcal{F}i_{\models^{\leqslant}_{\mathsf{K}}}(\boldsymbol{A})$, $\boldsymbol{\Omega}_{\boldsymbol{A}}F = \big\{\langle a, b\rangle \in A \times A : (a \leftrightarrow b)^n \in F\big\}$.*

3. *The logic $\models^{\leqslant}_{\mathsf{K}}$ is equivalential, with $\varepsilon(x, y) := (x \leftrightarrow y)^n$.*

Taking into account that $\mathsf{Prot_n} \subseteq \mathsf{E_n}$, we have:

Corollary 3.9. *Let K be a variety of residuated lattices. Then the following conditions are equivalent:*

1. *The logic $\models^{\leqslant}_{\mathsf{K}}$ is protoalgebraic.*

2. *For every algebra $\boldsymbol{A}$ and $F \in \mathcal{F}i_{\models^{\leqslant}_{\mathsf{K}}}(\boldsymbol{A})$ it holds that $\boldsymbol{\Omega}_{\boldsymbol{A}}F = \{\langle a, b\rangle \in A \times A : (a \leftrightarrow b)^n \in F \text{ for every } n \in \omega\}$.*

3. *The logic $\models^{\leqslant}_{\mathsf{K}}$ is equivalential.*

Finally, it is interesting to notice that all logics $\models^{\leqslant}_{\mathsf{K}}$ preserving degrees of truth are selfextensional, and all logics $\vdash_{\mathsf{K}}$ preserving truth are algebraizable. In the next result we also see that these properties "separate" the two groups: a logic in one group cannot have the characteristic property of the other group unless it actually belongs to it.

Theorem 3.10. *Let K be a variety of residuated lattices. Then the following conditions are equivalent:*

1. *The logic $\models^{\leqslant}_{\mathsf{K}}$ is algebraizable.*

2. $\models^{\leqslant}_{\mathsf{K}} = \vdash_{\mathsf{K}}$.

3. *The logic $\vdash_{\mathsf{K}}$ is selfextensional.*

4. K *is a variety of Heyting algebras, i.e.,* $\mathsf{K} \subseteq \mathsf{Prot_1}$.

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