Principal Fuzzy Type Theories for Fuzzy Logic in Broader Sense *

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Abstract

In this paper, we briefly discuss the concept of fuzzy logic in broader sense and its present stage of development. Furthermore, we introduce a special algebra called EQ-algebra in which the basic operation is that of fuzzy equality. Then we introduce axiomatics of a new core fuzzy type theory — IEQ-FTT — and demonstrate how three principal fuzzy type theories, namely IMTL-, Łukasiewicz, and BL- can be derived from it.

Keywords: Fuzzy type theory, EQ-algebra, residuated lattice, fuzzy logic.

1 Introduction

Fuzzy Logic in Broader Sense (FLb) is an extension of Fuzzy Logic in Narrow Sense (FLn), which aims at developing a *formal theory of human reasoning with the stress to utilization of vagueness contained in the meaning of special class of natural language expressions.* This program was initiated by V. Novák in 1995 in [6]. Note that it overlaps with two other paradigms proposed in the literature: *commonsense reasoning* (cf. [3] and the citations therein) and *precisiated natural language* (PNL; [12, 13]) but it is not equal to them. The main drawback of the up-to-date formalizations of commonsense reasoning is neglecting vagueness contained in the meaning of natural language expressions. On the other hand, the main drawback of PNL is lack of precise, mathematically justified tools. Thus, our concept of FLb is a glue between both these paradigms.

FLb consists (so far) of the following theories:

(a) Formal theory of evaluative linguistic expressions,

(b) formal theory of fuzzy IF-THEN rules,

(c) formal theory of perception-based logical deduction,

(d) formal theory of intermediate quantifiers.

Formal basis for all these theories is *fuzzy type theory* — a higher-order fuzzy logic (see [7, 9]) which continues the development of classical type theory (cf. [1, 2, 5]). Expressive power and experiences led us to separation of a few distinguished kinds of fuzzy logic: IMTL-logic, Łukasiewicz logic and Basic fuzzy logic (BL). All of them have propositional as well as predicate version and enjoy completeness property (their propositional versions enjoy even standard completeness). These logics have been extended also to higher order versions (fuzzy type theories) which enjoy the generalized completeness property (i.e. completeness w.r.t. generalized models).

Unlike other formal logical systems where the fundamental connective is implication, the *fundamental connective in FTT is fuzzy equality/equalivalence.* It is important to note that this connective emerges incessantly in many abstract thoughts. Therefore, fuzzy type theory becomes elegant and philosophically interesting. The principal question now raises,

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how fuzzy equality should be interpreted in the structure of truth values. In [9], we used the biresiduation operation in residuated lattice. This operation, however, is derived from implication and therefore we face a methodological discrepancy: the *basic connective* in syntax is interpreted by a *derived operation* in semantics. Therefore, in [**?**] we introduced a new special algebraic structure of truth values called EQ-algebra in which the basic operations is fuzzy equality (equivalence) and implication is derived from it. Since the product operation has been relaxed from the implication, this algebra generalizes residuated lattices. In this paper we overview EQ-algebras and develop a formal system of fuzzy type theory on the basis of one special case of them.

2 EQ-algebras

As mentioned, the basic connective in FTT is fuzzy equality. Hence, a natural question arises, whether we can introduce an algebra of truth values specific for FTT. The first attempt has been presented in [10] and also in [**?**] where the concept of EQ-algebra was introduced. In detail, this concept has been presented in [**?**].

From the point of view of logic, the main difference between residuated lattices and EQ-algebras lays in the way how implication operation is obtained. While in residuated lattice, it is obtained from (strong) conjunction, in EQ-algebra it is obtained from equivalence. Though properties of both kinds of algebras are similar, they differ in several essential points.

Definition 1
An EQ-algebra is an algebra

$$\mathcal{E} = \langle E, \wedge, \otimes, \sim, \mathbf{1}\rangle \qquad (1)$$

of type (2, 2, 2, 0) where the axioms are fulfilled for all $a, b, c \in E$*:*

(E1) $\langle E, \wedge, \mathbf{1}\rangle$ *is a commutative idempotent monoid (i.e.* $\wedge$*-semilattice with top element* $\mathbf{1}$*). We put* $a \le b$ *iff* $a \wedge b = a$*, as usual.*

(E2) $\langle E, \otimes, \mathbf{1}\rangle$ *is a commutative monoid and* $\otimes$ *is isotone w.r.t.* $\le$*.*

(E3) $a \sim a = \mathbf{1}$,

(E4) $((a \wedge b) \sim c) \otimes (d \sim a) \le c \sim (d \wedge b)$,

(E5) $(a \sim b) \otimes (c \sim d) \le (a \sim c) \sim (b \sim d)$,

(E6) $(a \wedge b \wedge c) \sim a \le (a \wedge b) \sim a$,

(E7) $(a \wedge b) \sim a \le (a \wedge b \wedge c) \sim (a \wedge c)$,

(E8) $a \otimes b \le a \sim b$.

The operation "$\wedge$*" is called meet (infimum), "*$\otimes$*" is called product and "*$\sim$*" is a fuzzy equality*

Unlike [10**?**], which were the first exposition of the idea, we have weakened axiom (E8) and added axiom (E5).

Clearly, $\le$ is the classical partial order. We will put

$$a \to b = (a \wedge b) \sim a, \qquad (2)$$

and

$$\tilde{a} = a \sim \mathbf{1} \qquad (3)$$

where $a, b \in E$. The derived operation (2) will be called *implication*. Hence, we may rewrite (E6),(E7) into

$$a \to (b \wedge c) \le a \to b, \qquad \text{(E6')}$$
$$a \to b \le (a \wedge c) \to b, \qquad \text{(E7')}$$

respectively.

Lemma 1
The following properties hold in EQ-algebras:

(a) $a \sim b = b \sim a$, *(symmetry)*

(b) $(a \sim b) \otimes (b \sim c) \le (a \sim c)$, *(transitivity)*

(c) $(a \to b) \otimes (b \to c) \le a \to c$, *(transitivity of implication)*

(d) $a \otimes (a \to b) \le \tilde{b}$.

(e) $(a \to b) \otimes (b \to a) \le a \sim b \le (a \to b) \wedge (b \to a)$

Let $\mathcal{E}$ contain also the bottom element $\mathbf{0}$. Then we put

$$\neg a = a \sim \mathbf{0}, \qquad a \in E. \tag{4}$$

Definition 2
Let $\mathcal{E}$ be EQ-algebra. We say that it is:

(i) semiseparated *if for all $a \in E$,*

(E9) $a \sim \mathbf{1} = \mathbf{1}$ *implies* $a = \mathbf{1}$.

(ii) separated *if for all $a, b \in E$,*

(E10) $a \sim b = \mathbf{1}$ *implies* $a = b$.

(iii) spanned *if*

(E11) $\tilde{\mathbf{0}} = \mathbf{0}$.

(iv) good *if for all $a \in E$,*

(E12) $a \sim \mathbf{1} = a$.

(v) residuated *if for all $a, b, c \in E$,*

(E13) $(a \otimes b) \wedge c = a \otimes b$ *iff* $a \wedge ((b \wedge c) \sim b) = a$.

(vi) involutive *(IEQ-algebra) if for all $a \in E$,*

(E14) $\neg\neg a = a$.

Let $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \Rightarrow, \mathbf{0}, \mathbf{1}\rangle$ be a residuated lattice. We may introduce two kinds of biresiduation operation:

$$a \Leftrightarrow b = (a \Rightarrow b) \wedge (b \Rightarrow a), \tag{5}$$
$$a \,\hat{\Leftrightarrow}\, b = (a \Rightarrow b) \otimes (b \Rightarrow a). \tag{6}$$

Both operations are natural interpretations of equivalence since they are reflexive, symmetric, and transitive in the following sense:

$$(a \Box b) \otimes (b \Box c) \leq a \Box c$$

for all $a, b, c \in L$ where $\Box \in \{\Leftrightarrow, \hat{\Leftrightarrow}\}$.

Example 1
Let $\mathcal{L} = \langle L, \wedge, \vee, \otimes, \Rightarrow, \mathbf{0}, \mathbf{1}\rangle$ be a residuated lattice. Then

(i) $\mathcal{E}_{\mathcal{L}} = \langle L, \wedge, \otimes, \Leftrightarrow, \mathbf{1}\rangle$ is a separated EQ-algebra.

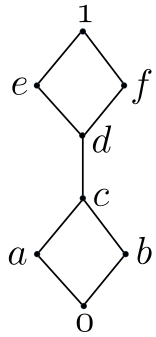

Figure 1: Eight elements IEQ-algebra.

(ii) If $\mathcal{L}$ is linearly ordered then also $\hat{\mathcal{E}}_{\mathcal{L}} = \langle L, \wedge, \otimes, \hat{\Leftrightarrow}, \mathbf{1}\rangle$ is a separated EQ-algebra (since both $\Leftrightarrow$ and $\hat{\Leftrightarrow}$ concide).

(iii) Let $*$ be a monoidal operation on L such that $* \leq \otimes$. Then $\mathcal{E} = \langle L, \wedge, *, \Leftrightarrow, \mathbf{1}\rangle$ is a separated EQ-algebra. If $* < \otimes$ then $\mathcal{E}$ is not residuated.

Example 2
Example of finite non-trivial non-residuated IEQ-algebra is the following: its (semi)lattice structure is in Figure 1. Product and fuzzy equality are defined as follows:

$\otimes$	$\mathbf{0}$	a	b	c	d	e	f	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
a	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	a
b	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	b
c	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	c
d	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	d	d	d	d
e	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	d	e	d	e
f	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	d	d	d	f
$\mathbf{1}$	$\mathbf{0}$	a	b	c	d	d	f	$\mathbf{1}$

$\sim$	$\mathbf{0}$	a	b	c	d	e	f	$\mathbf{1}$
$\mathbf{0}$	$\mathbf{1}$	e	f	d	c	a	b	$\mathbf{0}$
a	e	$\mathbf{1}$	d	f	c	a	c	a
b	f	d	$\mathbf{1}$	e	c	c	b	b
c	d	f	e	$\mathbf{1}$	c	c	c	c
d	c	c	c	c	$\mathbf{1}$	f	e	d
e	a	a	c	c	f	$\mathbf{1}$	d	e
f	b	c	b	c	e	d	$\mathbf{1}$	f
$\mathbf{1}$	$\mathbf{0}$	a	b	c	d	e	f	$\mathbf{1}$

There are also examples of non-trivial linearly ordered IEQ-algebras and many other exam-

ples of non-trivial finite EQ-algebras including linearly ordered ones. Therefore, in general neither non-linear nor linear EQ-algebras coincide with residuated lattices.

Lemma 2
Let $\mathcal{E}$ be a good EQ-algebra. Then the following holds for all $a, b \in E$:

(a) It is spanned and separated.

(b) Axiom (E8) is provable from the other axioms and can be omitted.

(c) $a \otimes (a \to b) \leq b$,

(d) $a \leq b$ *iff* $a \to b = \mathbf{1}$,

(e) $a \leq (a \sim b) \sim b$.

Lemma 3
An EQ-algebra $\mathcal{E}$ is residuated iff

$$(a \otimes b) \to c = a \to (b \to c)$$

holds for all $a, b, c \in E$.

Theorem 1
Each IEQ-algebra $\mathcal{E}$ is good, spanned and separated.

An EQ-algebra $\mathcal{E}$ is *complete* if it is a complete $\wedge$-semilattice. A lattice ordered EQ-algebra is an EQ-algebra that is a lattice. The EQ-algebra $\mathcal{E}$ is a *lattice EQ-algebra* (ℓEQ-algebra) if it is lattice ordered and, moreover, the following additional substitution axiom holds for all $a, b, c, d \in E$:

(E15) $((a \vee b) \sim c) \otimes (d \sim a) \leq ((d \vee b) \sim c)$.

A complete EQ-algebra is a complete lattice ordered EQ-algebra. Every finite EQ-algebra is latice ordered. A complete residuated EQ-algebra is a complete residuated lattice.

In IEQ-algebra $\mathcal{E}$, it is possible to define

$$a \vee b = \neg(\neg a \wedge \neg b). \quad (7)$$

It is easy to prove that each IEQ-algebra is lattice ordered.

Lemma 4
Let $\mathcal{E}$ be an IEQ-algebra. Then the following holds for all $a, b \in E$:

(a) $a \sim b = \neg a \sim \neg b$,

(b) $\mathcal{E}$ is ℓIEQ-algebra.

Let $\mathcal{E}$ be a spanned EQ-algebra. A *delta operation* in $\mathcal{E}$ is an operation $\Delta : E \longrightarrow E$ fulfilling the following axioms:

(i) $\Delta\mathbf{1} = \mathbf{1}$,

(ii) $\Delta a \leq a$,

(iii) $\Delta a \leq \Delta\Delta a$,

(iv) $\Delta(a \sim b) \leq \Delta\tilde{a} \sim \Delta\tilde{b}$,

(v) $\Delta(a \wedge b) = \Delta a \wedge \Delta b$.

If $\mathcal{E}$ is, moreover, lattice ordered then Δ must also fulfil the following:

(v) $\Delta(a \vee b) \leq \Delta a \vee \Delta b$,

(vi) $\Delta a \vee \neg\Delta a = \mathbf{1}$.

Lemma 5
(a) If $a \leq b$ then $\Delta a \leq \Delta b$.

(b) $\Delta(a \to b) \leq \Delta\tilde{a} \to \Delta\tilde{b}$.

(c) If $\mathcal{E}$ is good then $\Delta(a \to b) \leq \Delta a \to \Delta b$.

If the algebra is linearly ordered then we can define Δ-operation by $\Delta(\mathbf{1}) = \mathbf{1}$ and $\Delta(x) = \mathbf{0}$ otherwise. There is no nontrivial Δ-operation (i.e., different from identity) in the IEQ-algebra from Example 2.

3 Core FTT

From now on, the FTT presented in [9] will be referred to as IMTL-FTT. In this section, we introduce a new *core fuzzy type theory*. It will be denoted by IEQ-FTT because its structure of truth values is formed by an IEQ$_\Delta$-algebra

$$\mathcal{E}_\Delta = \langle E, \wedge, \otimes, \sim, \mathbf{0}, \mathbf{1}, \Delta \rangle \quad (8)$$

where Δ is a delta operation introduced above. Recall that $\mathcal{E}_\Delta$ is, in fact, the ℓIEQ-algebra and so we can consider also the join operation $\vee$ in it.

We will develop the IEQ-FTT in correspondence with IMTL-FTT introduced in detail in [9]. The definition of types and formulas remains unchanged. The special constants are: $\mathbf{E}_{(o\alpha)\alpha}$ (fuzzy equality) for every $\alpha \in$ *Types*, $\mathbf{C}_{(oo)o}$ (conjunction), $\mathbf{S}_{(oo)o}$ (strong conjunction), $\mathbf{D}_{oo}$ (delta connective), and $\iota_{\epsilon(o\epsilon)}, \iota_{o(oo)}$ (description operators).

The definitions of truth $\top$, falsity $\bot$, negation $\neg$ and implication $\Rightarrow$ remain unchanged.

(a) Special connectives:

$$\vee := \lambda x_o \lambda y_o \neg(\neg x_o \wedge \neg y_o) \quad (\textit{disjunction})$$
$$\& := \lambda x_o \lambda y_o (\mathbf{S}_{(oo)o}\, y_o) x_o. \quad (\textit{strong conjunction}) \tag{9}$$

(b) Quantifiers: Let $A_o \in Form_o$ and x_α be a variable of type α. Then we put:

$$(\forall x_\alpha) A_o := (\lambda x_\alpha\, A_o \equiv \lambda x_\alpha\, \top), \quad (\textit{general quantifier})$$
$$(\exists x_\alpha) A_o := \neg(\forall x_\alpha) \neg A_o. \quad (\textit{existential quantifier})$$

The definition of n-fold strong conjunction is as usual.

The following formulas of type o are logical axioms of fuzzy type theory. The types α, β are arbitrary types $\alpha, \beta \in$ *Types* unless specified otherwise.

Fundamental axioms

(CFT1) $\mathbf{\Delta}(x_\alpha \equiv y_\alpha) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv f_{\beta\alpha}\, y_\alpha)$

(CFT2$_1$) $(\forall x_\alpha)(f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha) \Rightarrow (f_{\beta\alpha} \equiv g_{\beta\alpha})$

(CFT2$_2$) $(f_{\beta\alpha} \equiv g_{\beta\alpha}) \Rightarrow (f_{\beta\alpha}\, x_\alpha \equiv g_{\beta\alpha}\, x_\alpha)$

(CFT3) $(\lambda x_\alpha B_\beta) A_\alpha \equiv C_\beta$
where C_β is obtained from B_β by replacing all free occurrences of x_α in it by A_α, provided that A_α is substitutable to B_β for x_α (*lambda conversion*).

(CFT4) $A_\alpha \equiv A_\alpha$

(CFT5) $(A_\alpha \equiv B_\alpha) \,\&\, (B_\alpha \equiv C_\alpha) \Rightarrow (A_\alpha \equiv C_\alpha)$
for all types $\alpha \neq o$.

Axioms of truth values

(CFT6) $(A_o \bigcirc B_o) \equiv (B_o \bigcirc A_o)$

(CFT7) $(A_o \bigcirc \top) \equiv A_o$

(CFT8) $(A_o \bigcirc B_o) \bigcirc C_o \equiv A_o \bigcirc (B_o \bigcirc C_o)$

where $\bigcirc \in \{\wedge, \&\}$

(CFT9) $(A_o \wedge A_o) \equiv A_o$

(CFT10) $\neg\neg A_o \equiv A_o$

(CFT11) $((A_o \wedge B_o) \equiv C_o) \,\&\, (D_o \equiv A_o) \Rightarrow ((D_o \wedge B_o) \equiv C_o)$

(CFT12) $(A_o \equiv B_o) \,\&\, (C_o \equiv D_o) \Rightarrow (A_o \equiv C_o) \equiv (B_o \equiv D_o)$

(CFT13) $((A_o \wedge B_o \wedge C_o) \equiv A_o) \Rightarrow (A_o \wedge B_o) \equiv A_o)$

(CFT14) $((A_o \wedge B_o) \equiv A_o) \Rightarrow (A_o \wedge B_o \wedge C_o) \equiv (A_o \wedge C_o)$

(CFT15) $(A_o \,\&\, B_o) \Rightarrow (A_o \equiv B_o)$

(CFT16) $((A_o \wedge B_o) \equiv A_o) \vee ((A_o \wedge B_o) \equiv B_o)$

Axioms of delta

(CFT17) $(g_{oo}(\mathbf{\Delta} x_o) \wedge g_{oo}(\neg \mathbf{\Delta} x_o)) \equiv (\forall y_o) g_{oo}(\mathbf{\Delta} y_o)$

(CFT18) $\mathbf{\Delta}(A_o \wedge B_o) \equiv \mathbf{\Delta} A_o \wedge \mathbf{\Delta} B_o$

(CFT19) $\mathbf{\Delta}(A_o \vee B_o) \Rightarrow \mathbf{\Delta} A_o \vee \mathbf{\Delta} B_o$

Axioms of quantifiers

(CFT20) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha) B_o)$
where x_α is not free in A_o

Axioms of descriptions

(CFT21) $\iota_{\alpha(o\alpha)}(\mathbf{E}_{(o\alpha)\alpha}\, y_\alpha) \equiv y_\alpha, \qquad \alpha = o, \epsilon$

Let us remark that this set of axioms is not optimized and so, it may contain redundancies. The problem of minimization of the set of axioms of IEQ-FTT is postponed to some of the future papers.

The inference rules (R) and (N) remain unchanged. The core FTT defined above will be referred to as IEQ-FTT.

Semantics: A *frame* for J is a tuple $\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{E}_\Delta \rangle$ where $\mathcal{E}_\Delta$ is a complete IEQ$_\Delta$-algebra of truth values, $=_\alpha$ is a fuzzy equality on M_α (see [9] and elsewhere). Recall that if $\beta\alpha$ is a type then the corresponding set $M_{\beta\alpha}$ contains (not necessarily all) functions $f : M_\alpha \longrightarrow M_\beta$. We put $M_o = E$ and assume that each set $M_{oo} \cup M_{(oo)o}$ contains all the operations from $\mathcal{E}_\Delta$. Let p be an assignment of elements from $\mathcal{M}$ to variables. Interpretation $\mathcal{I}^\mathcal{M}$ is a function that assigns every formula A_α, $\alpha \in Types$ and every assignment p a corresponding element, that is, a function of the type α.

More specifically:

(i) $\mathcal{I}^\mathcal{M}(\mathbf{E}_{(oo)o}) = \sim$,

(ii) $\mathcal{I}^\mathcal{M}(\mathbf{E}_{(o\alpha)\alpha}) = =_\alpha$ for all $\alpha \in Types - \{o\}$,

(iii) $\mathcal{I}^\mathcal{M}(\mathbf{C}_{(oo)o}) = \wedge$,

(iv) $\mathcal{I}^\mathcal{M}(\mathbf{S}_{(oo)o}) = \otimes$,

(v) $\mathcal{I}^\mathcal{M}(\mathbf{D}_{oo}) = \Delta$.

(vi) Semantics of $\iota_{\alpha(o\alpha)}$ is an element obtained by defuzzification of a fuzzy set in M_α which chooses some element from its kernel.

A general model is a frame $\mathcal{M}$ such that $\mathcal{I}_p^\mathcal{M}(A_\alpha) \in M_\alpha$ holds true for all $\alpha \in Types$. This means that each set M_α from the frame $\mathcal{M}$ has enough elements so that the interpretation of each formula $A_\alpha \in Form$ is always defined in $\mathcal{M}$. If T is a theory, then a general model $\mathcal{M}$ is a model of T if all its special axioms are true in the degree 1 in $\mathcal{M}$.

Various properties common with IMTL-FTT are provable also in IEQ-FTT.

Theorem 2 (Rule of Two Cases)
If $T \vdash A_{o,x_\alpha}[\top]$ and $T \vdash A_{o,x_\alpha}[\bot]$ then $T \vdash A_{o,x_\alpha}[\mathbf{\Delta} y_o]$.

This theorem enables us to prove important formula (c) in the following lemma:

Lemma 6
(a) $\vdash \top$,

(b) $\vdash (A \equiv \top) \equiv A$,

(c) $\vdash (\mathbf{\Delta} A_o \Rightarrow (B_o \Rightarrow C_o)) \Rightarrow ((\mathbf{\Delta} A_o \Rightarrow B_o) \Rightarrow (\mathbf{\Delta} A_o \Rightarrow C_o))$,

(d) If $T \vdash A_o$ *then* $T \vdash \neg\mathbf{\Delta}\neg\mathbf{\Delta} A_o$,

(e) $\vdash A_\alpha \equiv B_\alpha$ *implies* $\vdash B_\alpha \equiv A_\alpha$.

Theorem 3 (Deduction theorem)
Let T be a theory, $A_o \in Form_o$ a formula. Then

$$T \cup \{A_o\} \vdash B_o \quad \textit{iff} \quad T \vdash \mathbf{\Delta} A_o \Rightarrow B_o$$

holds for every formula $B_o \in Form_o$.

Let T be a theory. We say that:

(i) T is *contradictory* if

$$T \vdash \bot.$$

Otherwise it is *consistent*.

(ii) T is *maximal consistent* if each its extension T', $T' \supset T$ is inconsistent.

(iii) T is *complete* if for every two formulas A_o, B_o

$$T \vdash A_o \Rightarrow B_o \quad \text{or} \quad T \vdash B_o \Rightarrow A_o.$$

(iv) T is *extensionally complete* if for every closed formula of the form $A_{\beta\alpha} \equiv B_{\beta\alpha}$, $T \not\vdash A_{\beta\alpha} \equiv B_{\beta\alpha}$ it follows that there is a closed formula C_α such that $T \not\vdash A_{\beta\alpha} C_\alpha \equiv B_{\beta\alpha} C_\alpha$.

Theorem 4
A theory T is contradictory iff each formula $A_o \in Form_o$ is provable in it.

Theorem 5
To every consistent theory T there is an extensionally complete consistent theory $\overline{T}$ which is extension of T.

We now have to consider the concept of *safe general model*, that is a model in which all the necessary suprema and infima exist (cf. [4]). The reason is that till now it is not known whether we can complete the IEQ-algebra of truth values.

Theorem 6
A theory T of IEQ-FTT is consistent iff it has a safe general model $\mathcal{M}$.

4 Principal FTT's

We have discussed in [11] that most important fuzzy type theories for further development of FLb are those based on IMTL-algebra, MV-algebra (standard Łukasiewicz algebra), BL-algebra, and in a sense also LΠ-algebra. All these algebras are residuated lattices. It is difficult to guess in the present stage of development whether we can replace them by IEQ-FTT.

To obtain residuated lattice-based FTT, we need to add the following axiom to IEQ-FTT:

(RFT1) $((A_o \,\&\, B_o) \Rightarrow C_o) \equiv (A_o \Rightarrow (B_o \Rightarrow C_o))$

The resulting fuzzy type theory is IMTL-FTT.

4.1 Łukasiewicz-fuzzy type theory

This is a leading kind of fuzzy type theory that is (up to now) fundamental for the development of FLb. It differs from IEQ-FTT by the following definitions:

$$\vee := \lambda x_o(\lambda y_o(x_o \Rightarrow y_o) \Rightarrow y_o), \qquad \text{(disjunction)}$$

$$\& := \lambda x_o(\lambda y_o(\neg(x_o \Rightarrow \neg y_o))). \qquad (\textit{strong conjunction})$$

Logical axioms of Ł-FTT are (CFT1)–(CFT21), (RFT1) and also the axiom

(LFT1) $(A_o \vee B_o) \equiv (B_o \vee A_o)$.

There is also simpler alternative which uses Rose-Rosser implication axioms for characterization of the structure of truth values. Then, axioms (CFT6)–(CFT16) should be replaced by the following axioms:

Implication axioms

(LFT'1) $A_o \Rightarrow (B_o \Rightarrow A_o)$

(LFT'2) $(A_o \Rightarrow B_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o))$

(LFT'3) $(\neg B_o \Rightarrow \neg A_o) \equiv (A_o \Rightarrow B_o)$

(LFT'4) $(A_o \vee B_o) \equiv (B_o \vee A_o)$

Theorem 7
A theory T of Ł-FTT is consistent iff it has a general model $\mathcal{M}$.

4.2 BL-fuzzy type theory

Recall that BL stands for *basic fuzzy logic* developed by P. Hájek in [4]. We may introduce also BL-fuzzy type theory (BL-FTT).

Axioms of BL-FTT are (CFT1)–(CFT9), (CFT11)–(CFT21), (RFT1) and also the following:

(BL-FT1) $(A_o \wedge B_o) \equiv A \,\&(A_o \Rightarrow B_o)$

(BL-FT2) $\vdash B_{o,x_\alpha}[A_\alpha] \Rightarrow (\exists x_\alpha)B_o$.

(BL-FT3) $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow ((\exists x_\alpha)A_o \Rightarrow B_o)$

(BL-FT4) $(\forall x_\alpha)(A_o \vee B_o) \Rightarrow ((\forall x_\alpha)A_o \vee B_o)$

Theorem 8
A theory T of BL-FTT is consistent iff it has a safe general model $\mathcal{M}$.

Let us remark that in [8], also axioms for ŁΠ fuzzy type theory have been formulated. It is discussible whether such a complicated theory is a proper formal system to be used in FLb.

5 Conclusion

In this paper, we have introduced axiomatics of a new core fuzzy type theory — IEQ-FTT — and demonstrated how it can be modified to obtain three principal fuzzy type theories that are IMTL-, Łukasiewicz and BL-FTT. The motivation for introducing new axiomatics follows from the requirement to establish FTT on the basis of an algebra of truth values that is more natural than the residuated lattice because the basic operation (connective) in the latter is implication while the basic connective in FTT is fuzzy equality/equivalence. Therefore, we have developed a special algebra of truth values called EQ-algebra and introduced it briefly in Section 2. This serves as the background for introducing IEQ-FTT in Section 4 which is a core fuzzy type theory based on EQ-algebra with double negation. When modifying the list of its axioms and some of the definitions of special formulas, we obtain IMTL-, Łukasiewicz and BL-FTT. The reasons why we take them as principal fuzzy type theories follow from the initial requirements of fuzzy logic in broader sense.

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