On Algebras Based on Representable Uninorms

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Abstract

Representable uninorms form a special class of uninorms that have an additive generator over the whole real line bounded by $-\infty, +\infty$. Algebras based on left-continuous and conjunctive representable uninorms, called RU-algebras, form a subvariety of commutative bounded residuated lattices, and are strictly related to Abelian ℓ-groups.

In this work, we study this relation by showing that the category of Abelian ℓ-groups is equivalent to a full subcategory of RU-algebras. Moreover we prove that the variety of RU-algebras is generated by every single infinite RU-chain. Finally, we briefly study some simple model-theoretic properties of the class of RU-chains that are related to ordered divisible Abelian groups.

Keywords: Representable Uninorms, Residuated Lattices, Abelian ℓ-Groups, Ordered Divisible Abelian Groups, Quantifier Elimination.

1 Introduction

A *uninorm* $*$ is a binary, commutative, associative and monotone operation over $[0, 1]$, having a neutral element $e \in [0, 1]$ (see [14]). Note that each uninorm $*$ behaves like a t-norm over $[0, e]$, like a t-conorm over $[e, 1]$, and $\min(x, y) \leq x*y \leq \max(x, y)$ if $x \leq e \leq y$ or $y \leq e \leq x$ (see [2]). Moreover, a uninorm which is continuous necessarily is either a t-norm or a t-conorm (see [7]), hence, in a certain sense, there is no continuous uninorm.

Whenever $0*1 = 0$ we call $*$ a *conjunctive* uninorm. A uninorm $*$ admits a residual implication $\to$ iff it is conjunctive and left-continuous (see [3]). In this case

$$\langle [0, 1], *, \to, \min, \max, e, f, 0, 1\rangle,$$

with $f \in [0, 1]$, is a commutative bounded pointed residuated lattice (see below and [12]).

A remarkable class of uninorms is given by *representable uninorms*, i.e. uninorms that can be represented by means of a one-variable bijective function $h : [0, 1] \to \overline{\mathbb{R}}$, with $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, $h(0) = -\infty$, $h(e) = 0$, and $h(1) = +\infty$ such that:

$$x * y = h^{-1}(h(x) + h(y)).$$

These uninorms are also called *almost-continuous* being continuous on $(0, 1)$.

Theorem 1.1 ([2, 11])**.** *Given a uninorm $*$ with neutral element $e \in (0, 1)$, the following are equivalent:*

(i) $*$ is representable,

(ii) $*$ is strictly increasing and continuous on $(0, 1)$.

Any two conjunctive representable uninorms are order isomorphic, and in particular they are isomorphic to the Cross Ratio uninorm:

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$$x * y = \begin{cases} \frac{xy}{xy+(1-x)(1-y)} & (x,y) \in [0,1]^- \\ 0 & \text{otherwise} \end{cases},$$

where $[0,1]^-$ is $[0,1]^2 \backslash \{(0,1),(1,0)\}$.

A pointed bounded commutative residuated lattice ([12]) is a structure

$$\mathcal{A} = \langle A, *, \rightarrow, \sqcap, \sqcup, t, f, 0, 1 \rangle,$$

where $f \in A$ is an arbitrary element used to define a negation operator $\neg x$ as $x \rightarrow f$, and such that $\langle A, *, \rightarrow, \sqcap, \sqcup, 0, 1 \rangle$ is a bounded residuated lattice with top element 1 and bottom element 0, and $\langle A, *, t \rangle$ is a commutative monoid.

An RU-algebra is a pointed bounded commutative residuated lattice satisfying the following conditions:

- $t \leq ((x \rightarrow y) \sqcap t) \sqcup ((y \rightarrow x) \sqcap t)$,
- $t \leq (1 \rightarrow x) \sqcup (x \rightarrow 0) \sqcup (y \rightarrow (x * (x \rightarrow y)))$,
- $(x \rightarrow t) \rightarrow t = x$,
- $1 \rightarrow t = 0$,
- $f = t$.

It is easy to see that the class of RU-algebras forms a variety.

RU standard algebras are structures $\langle [0,1], *, \rightarrow, \min, \max, t, 0, 1 \rangle$, where $*$ is a representable left-continuous conjunctive uninorm, $\rightarrow$ its residuum, and t its neutral element. The prototypical standard RU-algebra is the RU-algebra based on the real unit interval where the monoidal operation corresponds to the Cross Ratio uninorm, its residuum is given by

$$x \rightarrow y = \begin{cases} \frac{(1-x)y}{x(1-y)+(1-x)y} & (x,y) \in [0,1]^{--} \\ 1 & \text{otherwise} \end{cases},$$

where $[0,1]^{--}$ is $[0,1]^2 \backslash \{(0,0),(1,1)\}$, and $t = \frac{1}{2}$. Every standard RU-algebra is isomorphic to the algebra based on the Cross Ratio uninorm, which can be also shown to generate the whole variety, as proven by Gabbay and Metcalfe in [3][1].

It is easy to see that any standard RU-algebra is isomorphic to the Abelian ℓ-group of reals bounded by $-\infty$ and $+\infty$, i.e.:

$$\mathcal{R} = \langle \overline{\mathbb{R}}, +, -, \min, \max, 0, -\infty, +\infty \rangle.$$

Indeed, any two RU-standard algebras are isomorphic to each other, and, in particular, to the RU-algebra based on the Cross Ratio uninorm, which, in turn, is isomorphic to $\mathcal{R}$ through the mapping $h : [0,1] \rightarrow \overline{\mathbb{R}}$ such that $x \mapsto \log(\frac{x}{1-x})$, whenever $x \in (0,1)$, while $0 \mapsto -\infty$ and $1 \mapsto +\infty$[2].

The above mentioned connection with Abelian ℓ-groups will be generalized in the next section. Indeed, we will show that there is an equivalence between the category of Abelian ℓ-groups and a full subcategory of RU-algebras satisfying certain conditions[3].

Furthermore, we will prove, in Section 3, that the variety of RU-algebras is generated by each single infinite RU-chain. This will be done by exploiting a translation into the universal theory of ordered Abelian groups. As an immediate consequence, we will obtain that both the RU-algebra related to the additive group of integers, the RU-algebra based on the rational unit interval $[0,1] \cap \mathbb{Q}$, and the standard RU-algebra over the reals generate the whole variety.

In Section 4, we will prove some easily derivable model-theoretic properties of the first-order theory of RU-chains related to ordered divisible Abelian groups.

We will end this work with some comments

[1] Actually, Gabbay and Metcalfe do not explicitly show that the algebra based on the Cross Ratio uninorm generates the whole variety, but this result is implicit in their work. In this paper, we give an alternative and more general proof of this fact.

[2] This allows to prove that the equational theory of RU-algebras is decidable and its satisfiability problem is in NP, as shown in [9]. The same result was also previously achieved by proof-theoretic means by Gabbay and Metcalfe in [3].

[3] While working on this paper we realized that some similar work was previously done by Gabbay, Metcalfe, and Olivetti in the forthcoming book [4], but in a different setting.

on our future work on this subject.

2 Categorical equivalence

Let $\mathcal{ALG}$ be the category of Abelian ℓ-groups and let $\mathcal{RU}^\star$ be the full subcategory of RU-algebras satisfying the following condition for all $0 < x < 1$:

$$(\star) \qquad 1 \to x = x \to 0 = 0.$$

For both categories, morphisms are homorphisms of their respective objects.

Let $\mathcal{G} = \langle G, +, -, \wedge, \vee, 0_G \rangle$ be an Abelian ℓ-group. Let $\perp$ and $\top$ be two elements not belonging to G, and define the set $\hat{G} = G \cup \{\perp, \top\}$ so that for every $x \in G$, $\perp < x < \top$. Let $\Upsilon(\mathcal{G})$ be the following structure

$$\Upsilon(\mathcal{G}) = \langle \hat{G}, +, \ominus, \wedge, \vee, 0_G, \perp, \top \rangle,$$

where $x \ominus y = y - x$, for all $x \in G$, $x + \perp = \perp$, $x + \top = \top$, $\top \ominus x = \perp$, $\perp \ominus x = \top$, $x \ominus \top = \top$, and $x \ominus \perp = \perp$. Moreover

$$\perp + \perp = \perp + \top = \top \ominus \perp = \perp,$$

and

$$\top + \top = \perp \ominus \top = \perp \ominus \perp = \top \ominus \top = \top.$$

Then we have:

Lemma 2.1. *Given an Abelian ℓ-group $\mathcal{G}$, the structure $\Upsilon(\mathcal{G})$ is an* RU-*algebra satisfying condition* $(\star)$.

Proof. It is clear that $\langle \hat{G}, \wedge, \vee, \perp, \top \rangle$ is a bounded lattice, and that $\langle \hat{G}, +, 0_G \rangle$ is a commutative monoid. Moreover, an easy inspection easily shows that the following conditions hold

- $x + y \le z$ iff $x \le y \ominus z$,
- $0_G \le ((x \ominus y) \wedge 0_G) \vee ((y \ominus x) \wedge 0_G)$,
- $0_G \le (\top \ominus x) \vee (x \ominus \perp) \vee ((x + (y - x)) - y)$,
- $(x \ominus 0_G) \ominus 0_G = x$,
- $\top \ominus 0_G = \perp$,
- $\top \ominus x = x \ominus \perp = \perp$, for all $x \in G$.

Therefore $\Upsilon(\mathcal{G})$ is an RU-algebra satisfying $(\star)$. □

Now, let $\mathcal{A} = \langle A, *, \to, \sqcap, \sqcup, t, 0, 1 \rangle$ be a non-trivial RU-algebra satisfying $(\star)$. Let $\Upsilon^{-1}(\mathcal{A})$ be the following structure:

$$\Upsilon^{-1}(\mathcal{A}) = \langle A^-, +, -, \wedge, \vee, u \rangle,$$

where A^- is $A \backslash \{0, 1\}$, $x + y$ is $x * y$, $-x$ is $x \to t$, u is t, and $\wedge$ and $\vee$ correspond to $\sqcap$ and $\sqcup$, respectively.

Lemma 2.2. *$\Upsilon^{-1}(\mathcal{A})$ is an Abelian ℓ-group such that $\Upsilon(\Upsilon^{-1}(\mathcal{A})) \cong \mathcal{A}$.*

Proof. It is easy to see that $\Upsilon^{-1}(\mathcal{A})$ satisfies the properties of Abelian ℓ-groups. We just show that $x + (-x) = u$. Notice that trivially $x * (x \to t) \le t$, and from the condition

$$t \le (1 \to x) \vee (x \to 0) \vee (y \to (x * (x \to y))),$$

we have that $t = t * t \le x * (x \to t)$, since $1 \to x = x \to 0 = 0$ for all $x \in A^-$.

Moreover, by construction, it immediately follows that $\Upsilon(\Upsilon^{-1}(\mathcal{A})) \cong \mathcal{A}$. □

Now, for every morphism $h : \mathcal{G} \to \mathcal{H}$ in $\mathcal{ALG}$, define $\Upsilon(h) : \Upsilon(\mathcal{G}) \to \Upsilon(\mathcal{H})$ as $\Upsilon(h)(x) = h(x)$, for all $x \in \mathcal{G}$, and $\Upsilon(h)(\top) = \top$, $\Upsilon(h)(\perp) = \perp$. For every morphism $\phi : \mathcal{A} \to \mathcal{B}$ in $\mathcal{RU}^\star$, define $\Upsilon^{-1}(\phi) : \Upsilon^{-1}(\mathcal{A}) \to \Upsilon^{-1}(\mathcal{B})$ as the restriction of ϕ on A^-.

Theorem 2.3. *The pair of functors $\Upsilon : \mathcal{ALG} \to \mathcal{RU}^\star$, and $\Upsilon^{-1} : \mathcal{RU}^\star \to \mathcal{ALG}$ constitutes an equivalence of the categories $\mathcal{ALG}$ and $\mathcal{RU}^\star$*

Proof. It is easy to see that Υ is a faithful and full functor, hence injective and surjective w.r.t. morphisms. The rest of the proof is easily derivable from Lemma 2.1 and Lemma 2.2. This proves the equivalence of categories (see [8]). □

As an immediate consequence we have:

Corollary 2.4. *For every non-trivial linearly ordered* RU-*algebra $\mathcal{A}$ there exists an ordered Abelian group $\mathcal{G}$ such that $\mathcal{A} \cong \Upsilon(\mathcal{G})$.*

3 Infinite Chains

It is easy to see that the smallest non-trivial RU-algebra is the three-element chain $\{0, t, 1\}$, which also is isomorphic to the RU-algebra obtained from the trivial Abelian ℓ-group composed by one element. For all the remaining chains we can prove the following:

Proposition 3.1. *The algebra $\Upsilon(\mathbb{Z})$ can be embedded into any RU-chain with more than three elements.*

Proof. We know that every non-trivial RU-chain $\mathcal{A}$ is isomorphic to the RU-algebra $\Upsilon(\Upsilon^{-1}(\mathcal{A}))$ of the Abelian ℓ-group $\Upsilon^{-1}(\mathcal{A})$ obtained from $\mathcal{A}$, as shown in Lemma 2.2. Abelian ℓ-groups are torsion-free, and so the Abelian ℓ-group over the integers $\mathbb{Z}$ is embeddable in any non-trivial Abelian ℓ-group [5]. Then we have that $\Upsilon(\mathbb{Z})$ is embeddable into any $\Upsilon(\Upsilon^{-1}(\mathcal{A}))$. □

Now, our aim will be to show that every single infinite RU-chain generates the whole variety. To prove this, we will exploit the connection with ordered Abelian groups, and with their universal theory, in the following language $\langle +, -, 0, < \rangle$ (i.e. the language of ordered groups). We will need the following result proven by Gurevich and Kokorin (see also [13]).

Theorem 3.2 ([6])**.** *Any two non-trivial ordered Abelian groups satisfy the same universal sentences.*

Let $\mathcal{A}$ be an RU-chain, we say that $a \in A$ is *extremal* if either $a = 0$ or $a = 1$. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be tuples of elements from any two RU-chains $\mathcal{A}$ and $\mathcal{B}$ (not necessarily different). We say that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are *coherent* if, for each j, either $a_j = 0$ and $b_j = 0$, or $a_j = 1$ and $b_j = 1$, or $a_j \in A^-$ and $b \in B^-$. If a term $\epsilon(x_1, \ldots, x_n)$ does not contain the bottom $\bot$ nor the top $\top$ as a subterm, then we call such a term *bound-free.*

It is easy to prove the following:

Lemma 3.3. *Let $\epsilon(x_1, \ldots, x_n)$ be any term in the language of RU-algebras. We have:*

(i) If $\epsilon(x_1, \ldots, x_n)$ is not bound-free, then either $\epsilon(x_1, \ldots, x_n)$ is extremal for every $a_1, \ldots, a_n$, or there exists a bound-free term $\epsilon'(x'_1, \ldots, x'_m)$, where $\{x'_1, \ldots, x'_m\} \subseteq \{x_1, \ldots, x_n\}$, such that $\epsilon(x_1, \ldots, x_n)$ and $\epsilon'(x'_1, \ldots, x'_m)$ are equivalent.

(ii) For any RU-chain $\mathcal{A}$ and any $a_1, \ldots, a_n$, with each a_j being non-extremal, and $\epsilon(x_1, \ldots, x_n)$ being bound-free, $\epsilon(a_1, \ldots, a_n)$ is non-extremal.

(iii) For any two RU-chains $\mathcal{A}$ and $\mathcal{B}$ (not necessarily different), if for $a_1, \ldots, a_n \in A$, $\epsilon(a_1, \ldots, a_n)$ is extremal (non-extremal, resp.), then for any $b_1, \ldots, b_n \in B$ coherent with $a_1, \ldots, a_n$, $\epsilon(a_1, \ldots, a_n)$ is extremal (non-extremal resp.) as well.

Now, we proceed with another lemma that makes direct use of the connection with the universal theory of ordered Abelian groups.

Lemma 3.4. *Let $\mathcal{A}$ be any RU-chain and $\Upsilon^{-1}(\mathcal{A})$ its associated ordered Abelian group. Let $\epsilon(x_1, \ldots, x_n) = \tau(x_1, \ldots, x_n)$ be any equation in the language of RU-algebras where both members are bound-free. There exists a universal formula φ in the language of ordered groups, such that φ is true in $\Upsilon^{-1}(\mathcal{A})$, iff $\epsilon(a_1, \ldots, a_n) = \tau(a_1, \ldots, a_n)$ holds for all $a_1, \ldots, a_n \in A^-$.*

Proof. For each $\diamond \in \{*, \to, \sqcap, \sqcup\}$, let $\psi_\diamond(x, y, z)$ be the definition of $\diamond$ in the language of ordered groups.

Let $\epsilon = \tau$ be any equation in $\mathcal{A}$ (in the language of RU-algebras). Let

$$\Gamma^\epsilon = \{\gamma_1^\epsilon, \ldots, \gamma_m^\epsilon\} \quad \text{and} \quad \Gamma^\tau = \{\gamma_1^\tau, \ldots, \gamma_s^\tau\}$$

be the sets of subterms of ϵ and τ, respectively (with γ_m^ϵ corresponding to ϵ, and γ_s^τ corresponding to τ).

Now, to each γ_j^ϵ and γ_k^τ associate a variable v_j^ϵ and v_k^τ, respectively (different variables for different subterms). For each $\diamond$, let

$$\Sigma_\diamond^\epsilon = \{(v_\sigma^\epsilon, v_{\sigma_1}^\epsilon, v_{\sigma_2}^\epsilon) \mid v_\sigma^\epsilon = (v_{\sigma_1}^\epsilon \diamond v_{\sigma_2}^\epsilon)\},$$

and

$$\Sigma_\diamond^\tau = \{(v_\sigma^\tau, v_{\sigma_1}^\tau, v_{\sigma_2}^\tau) \mid v_\sigma^\tau = (v_{\sigma_1}^\tau \diamond v_{\sigma_2}^\tau)\},$$

where each $v_{\sigma_j}^\epsilon$ and $v_{\sigma_k}^\tau$ is a variable associated to a subterm.

Now, for each $(v_\sigma^\epsilon, v_{\sigma_1}^\epsilon, v_{\sigma_2}^\epsilon) \in \Sigma_\diamond^\epsilon$ and $(v_\sigma^\tau, v_{\sigma_1}^\tau, v_{\sigma_2}^\tau) \in \Sigma_\diamond^\tau$, introduce the formulas $\psi_\diamond(v_\sigma^\epsilon, v_{\sigma_1}^\epsilon, v_{\sigma_2}^\epsilon)$ and $\psi_\diamond(v_\sigma^\tau, v_{\sigma_1}^\tau, v_{\sigma_2}^\tau)$, for each operation $\diamond$.

For each $\diamond$, denote by $\Theta_\diamond^\epsilon$ and $\Theta_\diamond^\tau$ the conjunction of all the above formulas. Let $\varphi^{\epsilon=\tau}$ be the following universal formula:

$$\forall v_1^\epsilon \dots v_m^\epsilon v_1^\tau \dots v_s^\tau \left((\bigwedge \Theta_\diamond^\epsilon) \wedge (\bigwedge \Theta_\diamond^\tau)\right) \Rightarrow (v_m^\epsilon = v_s^\tau),$$

where v_m^ϵ and v_s^τ are the variables corresponding to ϵ and τ, respectively (taken as subterms), and $\Rightarrow$ is the classical implication.

Now, $\varphi^{\epsilon=\tau}$ is a universal formula in the language of ordered groups, and $\epsilon = \tau$ holds in $\mathcal{A}$ for every $a_1, \dots, a_n \in A^-$ iff $\varphi^{\epsilon=\tau}$ is true in $\Upsilon^{-1}(\mathcal{A})$. □

Finally, we can prove the following:

Theorem 3.5. *Every infinite* RU*-chain generates the whole variety* $\mathbb{RU}$.

Proof. To prove the theorem, it suffices to show that every infinite RU-chain generates the same variety.

Let $\epsilon(x_1, \dots, x_n) = \tau(x_1, \dots, x_n)$ be an equation in the language of RU-algebras, and let $\mathcal{A}$ and $\mathcal{B}$ be any two different infinite RU-chains. Let us assume that the above equation does not hold in $\mathcal{A}$: i.e. there are some $a_1, \dots, a_n \in A$ such that $\epsilon(a_1, \dots, a_n) \neq \tau(a_1, \dots, a_n)$. We show that there are some $b_1, \dots, b_n \in B$ such that $\epsilon(b_1, \dots, b_n) \neq \tau(b_1, \dots, b_n)$. By Lemma 3.3(i), we can easily see that we can simply restrict ourselves to the case where both members of the equation are bound-free.

Suppose that at least one among $\epsilon(a_1, \dots, a_n)$ and $\tau(a_1, \dots, a_n)$ is extremal. Then, by Lemma 3.3(ii), some a_j are extremal, and consequently, by Lemma 3.3(iii), it suffices to take $b_1, \dots, b_n \in B$ coherent with $a_1, \dots, a_n \in A$.

Suppose that neither $\epsilon(a_1, \dots, a_n)$ nor $\tau(a_1, \dots, a_n)$ is extremal. If no a_j is extremal, then the fact that there must be some $b_1, \dots, b_n \in B^-$ such that $\epsilon(b_1, \dots, b_n) \neq \tau(b_1, \dots, b_n)$ is guaranteed by Lemma 3.4.

Suppose that some a_j's are extremal. An easy adaptation of Lemma 3.3(i) shows that for any $\epsilon(x_1, \dots, x_n)$, if some x_j's take extremal values, then either $\epsilon(x_1, \dots, x_n)$ is extremal itself, or there exists $\epsilon'(x'_1, \dots, x'_m)$, with $\{x'_1, \dots, x'_m\} \subseteq \{x_1, \dots, x_n\}$, such that $\epsilon(x_1, \dots, x_n) = \epsilon'(x'_1, \dots, x'_m)$ for all $a_1, \dots, a_m \in A^-$ (and the extremal values of some x_j's are fixed). In that case, once again we can safely apply Lemma 3.4.

This proves that any two different infinite RU-chains generate the same variety. Hence the proof of the Theorem is complete. □

As a consequence, we obtain:

Corollary 3.6. *The only proper subvariety of* $\mathbb{RU}$ *is the variety generated by the three-element chain.*

4 Some Model-Theoretic Properties

In this section, we investigate RU-algebras from a model-theoretic point of view.

Recall that a first-order theory admits quantifier-elimination in a given language iff every formula is equivalent to a quantifier-free formula. Furthermore, recall that a first-order theory T is *model-complete* if for all $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \subseteq \mathcal{N}$ implies that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$. It is well-known that quantifier-elimination implies model-completeness. We immediately obtain:

Proposition 4.1. *The theory of* RU*-algebras does not admit quantifier elimination in the language of* RU*-algebras.*

Proof. It is easy to see that $\Upsilon(\mathbb{Z})$ is not an elementary substructure of $\Upsilon(\mathbb{G})$. Then, the theory of RU-algebras is not model-complete and consequently it does not enjoy quantifier elimination. □

Now, we focus on the subclass of RU-chains that are related to ordered divisible Abelian groups. Let $\mathcal{A}$ be a RU-chain such that $\Upsilon^{-1}(\mathcal{A})$ is an ordered divisible Abelian group. We call each such a chain a *divisible* RU-chain. It is a well-known fact that the theory of ordered divisible Abelian groups admits quantifier elimination in the language of ordered groups [10]. This suggests that also the theory of divisible RU-chains must enjoy quantifier elimination. In order to prove this fact, we rely on the following:

Lemma 4.2 (Corollary 3.1.6, [10])**.** *Let T be a theory in a given language $\mathcal{L}$. Suppose that for all quantifier-free formulas $\phi(\overline{v}, w)$, if $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{A}$ is a common substructure of $\mathcal{M}$ and $\mathcal{N}$, $\overline{a} \in A$, and there is $b \in M$ such that $\mathcal{M} \models \phi(\overline{a}, b)$, then there is $c \in N$ such that $\mathcal{N} \models \phi(\overline{a}, c)$. Then T has quantifier elimination.*

Then we have:

Theorem 4.3. *The first-order theory of divisible* RU*-chains admits quantifier elimination in the language of* RU*-algebras.*

Proof. Let $\mathcal{A}$ be any divisible RU-chain, and let $\Upsilon^{-1}(\mathcal{A})$ be its related divisible ordered Abelian group. Let h be a mapping from $\Upsilon^{-1}(\mathcal{A})$ into $A \cap ((\bot, -1] \cup 0 \cup [1, \top))$ defined as follows:

$$h(x) = \begin{cases} -1 & x = \bot \\ x - 1 & x \in (\bot, 0) \\ 0 & x = 0 \\ x + 1 & x \in (0, \top) \\ 1 & x = \top \end{cases} .$$

Let $\mathcal{A}'$ be an isomorphic copy of $\mathcal{A}$ over $A \cap ((\bot, -1] \cup 0 \cup [1, \top))$ defined through the mapping h. It is easy to see that the operations of $\mathcal{A}'$ can be defined by open formulas in the language $\langle +, -, 0, 1, < \rangle$ over $\Upsilon^{-1}(\mathcal{A})$.

Now, let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be divisible RU-chains with $\mathcal{A}$ being a subalgebra of both $\mathcal{B}$ and $\mathcal{C}$. Let $\mathcal{A}'$, $\mathcal{B}'$, and $\mathcal{C}'$ be isomorphic copies of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ defined as above through the mapping h. Let $\Upsilon^{-1}(\mathcal{A}), \Upsilon^{-1}(\mathcal{B})$ and $\Upsilon^{-1}(\mathcal{C})$ be the divisible ordered Abelian groups associated to $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively.

Let $\varphi(x, \overline{y})$ be a quantifier-free formula in the language of RU-algebras. Following a translation similar to the one carried out in the previous section, it is easy to see that there exists a quantifier-free formula $\varphi'(x, \overline{y})$ in the language of ordered groups (possibly with integer coefficients) such that for any divisible RU-chain $\mathcal{A}$, and any $b, \overline{a} \in A$, $\varphi(b, \overline{a})$ holds in $\mathcal{A}$ iff $\varphi'(h(b), h(\overline{a}))$ holds in $\Upsilon^{-1}(\mathcal{A})$. Let $\overline{a} \in A$ and $b \in B$ such that $\mathcal{B} \models \varphi(b, \overline{a})$. Note that we also have $h(\overline{a}) \in \Upsilon^{-1}(\mathcal{A})$ and $h(b) \in \Upsilon^{-1}(\mathcal{B})$, hence $\Upsilon^{-1}(\mathcal{A}) \models \varphi'(h(b), h(\overline{a}))$. Ordered divisible Abelian groups are elementarily equivalent to each other, and so, there must be some $h(c) \in \Upsilon^{-1}(\mathcal{C})$ such that $\Upsilon^{-1}(\mathcal{C}) \models \varphi'(h(c), h(\overline{a}))$. Hence it easily follows, by applying the above lemma, that the theory of divisible RU-chains has quantifier elimination. □

We briefly recall now, some notions from Model Theory.

An amalgam is a tuple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, f, g)$ such that $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are structures of the same signature, and $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{A} \to \mathcal{C}$ are embeddings. A class $\mathbf{K}$ of structures is said to have the *amalgamation property* if for every amalgam with $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}$ and $A \neq \emptyset$ there exist a structure $\mathcal{D} \in \mathbf{K}$ and embeddings $f' : \mathcal{B} \to \mathcal{D}$, $g' : \mathcal{C} \to \mathcal{D}$ such that $f' \circ f = g' \circ g$. A class $\mathbf{K}$ of structures is said to have the *strong amalgamation property*, if it has the amalgamation property and, moreover, $f'[\mathcal{B}] \cap g'[\mathcal{C}] = (f' \circ f)[\mathcal{A}] = (g' \circ g)[\mathcal{A}]$, where for any set X and function h on X, $h[X] = \{h(x) \mid x \in X\}$.

An ordered structure $\langle A, <, \dots \rangle$ is *o-minimal* if every definable $X \subseteq A$ is a finite union of points and intervals.

Let $\mathrm{Th}(\mathcal{RU})$ denote the first-order theory of divisible RU-chains, and let $\mathrm{Th}_\forall(\mathcal{RU})$ denote the the universal theory of divisible RU-chains. From the above quantifier-elimination result, we easily obtain the following consequences, easily derivable from general results in Model Theory (see [1, 10]):

Corollary 4.4.

(*i*) $\mathrm{Th}(\mathcal{RU})$ *is model-complete.*

(*ii*) $\mathrm{Th}(\mathcal{RU})$ *is complete.*

(*iii*) $\mathrm{Th}(\mathcal{RU})$ *is equivalent to a* $\forall\exists$*-theory.*

(*iv*) *The class of divisible* RU*-chains enjoys the strong amalgamation property.*

(*v*) *The class of models of* $\mathrm{Th}_\forall(\mathcal{RU})$ *enjoys the amalgamation property.*

(*vi*) *Each divisible* RU*-chain is o-minimal.*

5 Final Remarks

In this work we have briefly studied some algebraic and model-theoretic properties of RU-algebras by exploiting their strong connection with Abelian ℓ-groups. In particular we have shown that the category of Abelian ℓ-groups is equivalent to the full subcategory of RU-algebras satisfying condition $(\star)$; we have shown that every single infinite RU-chain generates the whole variety; and, finally we have dealt with some simple model theoretic properties related to quantifier elimination for those RU-chains related to ordered divisible Abelian groups.

RU-algebras certainly form an interesting class of structures, and seem to be easy to investigate due to their relation to Abelian ℓ-groups. For these reasons, we aim at developing and extending the work presented in this paper, and studying other properties of RU-algebras. In particular, we will study the free objects in the variety both from an algebraic and from a geometric point of view.

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