Contraction and Dilation Operators in a Semilinear Space over Residuated Lattice

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Abstract

The notion of a semilinear space over residuated lattice is introduced. Two problems of solvability of systems of linear-like equations with $\sup -*$ or $\inf \to$ compositions are considered in a finite semilinear space. We prove that each system of equations is solvable if and only if its right-hand side is a fixed point of the respective contraction or dilation operator. Sets of fixed points are characterized as subsemimodules over respective reducts of the residuated l-monoid.

Keywords: Semilinear space, Residuated lattice, Solvability of a system of equations, Contraction operator, Dilation operator, Fixed point.

1 Introduction

The concept of *integral, residuated, commutative l-monoid* (the other name is residuated lattice) has been introduced by U. Höhle in [7] to create a basic structure for fuzzy algebras. In fact, many fuzzy logic algebras (MTL-algebra, BL-algebra, MV-algebra, etc.) are particular cases of that structure. Traditionally, residuated lattices and fuzzy logic algebras are considered as structures of truth values of the respective fuzzy logics. Such "usage" determines the prevailing way of investigation of these structures which is influenced by logic.

On the other hand, the residuated lattice operations are used in modeling of fuzzy systems or systems of fuzzy IF-THEN rules. In this respect, residuated lattice demonstrates its linear behavior [4]. This comes through when we demand correctness of a model of the systems of fuzzy IF-THEN rules (in the sense of [11]). In this contribution, we show that a system of fuzzy relation equations can be considered as a system of linear-like equations in a semilinear space over a residuated lattice. We will focus on systems with $\sup -*$ or $\inf \to$ compositions because they are the most popular in practical applications.

The novelty of this contribution consists in demonstration of the similarity between problems of solvability of a system of fuzzy relation equations and a system of linear equations in linear algebra which puts an emphasis on a matrix of coefficients. We change an angle under which the problem of solvability is usually considered (see, e.g., [3, 5, 13, 14]) and concentrate on characterizations of possible right-hand sides that make the respective systems solvable. We prove that a right-hand side vector must be a fixed point of a special operator (contraction or dilation). Moreover, systems of fuzzy relation equations with different compositions are considered at a time.

2 Residuated Lattice

The concept of *integral, residuated, commutative l-monoid* has been introduced by U. Höhle in [7] to create a basic structure for fuzzy algebras. In fact, many fuzzy logic algebras (MTL-algebra, BL-algebra, MV-algebra,

L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, pp. 1022–1029
Torremolinos (Málaga), June 22–27, 2008

etc.) are particular cases of that structure. The following definition of a residuated l-monoid is different from that given in [7]. The difference is that the operation $\rightarrow$ is not included into the signature of the residuated l-monoid (our case). However, we will show that a "natural" extension of the residuated l-monoid leads to an integral, residuated, commutative l-monoid in Höhle's sense.

Definition 1
A residuated l-monoid is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, *, 0, 1\rangle$$

such that

(i) $(L, \vee, \wedge, 0, 1)$ *is a bounded lattice,*

(ii) $(L, *, 1)$ *is a commutative monoid,*

(iii) for all $x, y, z \in L, \quad x*(y \vee z) = x*y \vee x*z$,

(iv) for each $\lambda \in L$*, the mapping* $h_\lambda : x \mapsto \lambda * x$*, is residuated, i.e. there exists an isotone mapping* $g_\lambda : L \longrightarrow L$ *such that for all* $x, y \in L$*,* $x \leq y$ *implies* $g_\lambda(x) \leq g_\lambda(y)$ *and moreover,*

$$(g_\lambda \circ h_\lambda)(x) \geq x,$$
$$(h_\lambda \circ g_\lambda)(x) \leq x.$$

2.1 Semimodules over Monoids

Modules over rings are usual algebraic structures. Semimodules over semirings were introduced in [6] and then used in [4]. In this subsection we will define a semimodule over a monoid and consider various examples of semimodules over the monoidal reduct of a residuated l-monoid. We will see that every residuated l-monoid has two different semimodules.

Definition 2
Let $\mathcal{A} = \langle A, +, \mathbf{0}\rangle$ *be a commutative monoid with the neutral element* $\mathbf{0}$ *and* $\mathcal{M} = \langle M, \cdot, 1\rangle$ *a (multiplicative) monoid. We say that* $\mathcal{A}$ *is a left semimodule over* $\mathcal{M}$ *(or* $\mathcal{M}$*-semimodule) if a left scalar multiplication by an element from* M *is defined and moreover, the following properties are fulfilled for all* $a, b \in A$ *and* $\lambda, \mu \in M$*:*

SM1. $\lambda(a + b) = \lambda a + \lambda b$,

SM2. $(\lambda \cdot \mu)a = \lambda(\mu a)$,

SM3. $1a = a$,

SM4. $\lambda\mathbf{0} = \mathbf{0}$.

The following two examples of left semimodules over the monoidal reduct of residuated l-monoid will be used in the sequel.

Example 1
Let $\mathcal{L} = \langle L, \vee, \wedge, *, 0, 1\rangle$ *be a residuated l-monoid and* $\mathcal{L}_* = \langle L, *, 1\rangle$ *a monoidal reduct of* $\mathcal{L}$*. Moreover, let* L^n $(n \geq 1)$ *be the set of* n*-dimensional vectors over* L*, and* L^X *the set of* L*-valued functions on* X *where* X *is a non-empty set.*

1. *The* $\vee$*-semilattice* $L^n_\vee = \langle L^n, \vee, \mathbf{0}\rangle$ *with the neutral element* $\mathbf{0} = (0, \ldots, 0)$ *is the left* $\mathcal{L}_*$*-semimodule where*

$$(a_1, \ldots, a_n) \vee (b_1, \ldots, b_n) = (a_1 \vee b_1, \ldots, a_n \vee b_n),$$
$$\lambda(a_1, \ldots, a_n) = (\lambda * a_1, \ldots, \lambda * a_n).$$

2. *The* $\wedge$*-semilattice* $L^n_\wedge = \langle L^n, \wedge, \mathbf{1}\rangle$ *with the neutral element* $\mathbf{1} = (1, \ldots, 1)$ *is the left* $\mathcal{L}_*$*-semimodule where*

$$(a_1, \ldots, a_n) \wedge (b_1, \ldots, b_n) = (a_1 \wedge b_1, \ldots, a_n \wedge b_n),$$
$$\lambda(a_1, \ldots, a_n) = (\lambda \rightarrow a_1, \ldots, \lambda \rightarrow a_n).$$

3 Semilinear Spaces

Let us remind that a linear space is a commutative module over a field. The notion of a semilinear space, introduced in [8], followed the same idea. Opposite to those "natural" approaches, we propose to consider a semilinear space over a monoid. However, a monoid does not have an inverse operation and thus, it is not straightforward how linear-like equations can be solved or how inverse elements could be defined. Our proposition is in enlarging the number of external operations.

3.1 Semilinear l-Spaces and Principle of Duality

In this subsection, a semilinear lattice ordered space or a *semilinear l-space*, which enjoys two mutually residuated external operations, is defined. We will show that thus defined external operations are connected by the adjunction property and moreover, a semilinear l-space could be split in two different left semimodules over one and the same monoid.

Definition 3
Let $\mathcal{M} = \langle M, \cdot, 1\rangle$ be a commutative monoid and $\langle A, \leq\rangle$ a bounded lattice where $\mathbf{0}, \mathbf{1}$ are the respective bounds. Let $\mathcal{A} = \langle A, \vee, \mathbf{0}\rangle$ be a left $\mathcal{M}$-semimodule where the scalar product of $\lambda \in M$ and $\mathbf{x} \in A$ is denoted by $\lambda\mathbf{x}$, so that for all $\lambda, \mu \in M$ and for all $\mathbf{x}, \mathbf{y} \in A$

$$\lambda(\mathbf{x} \vee \mathbf{y}) = \lambda\mathbf{x} \vee \lambda\mathbf{y},$$
$$(\lambda \cdot \mu)\mathbf{x} = \lambda(\mu\mathbf{x}),$$
$$1\mathbf{x} = \mathbf{x},$$
$$\lambda\mathbf{0} = \mathbf{0}.$$

We say that the left semimodule $\mathcal{A}$ is a semilinear l-space over $\mathcal{M}$ if for each $\lambda \in M$, the mapping $h_\lambda : \mathbf{x} \mapsto \lambda\mathbf{x}$ is residuated, i.e. there exists an isotone mapping $g_\lambda : A \longrightarrow A$ such that for all $\mathbf{x}, \mathbf{y} \in A$

$$(g_\lambda \circ h_\lambda)(\mathbf{x}) \geq \mathbf{x},$$
$$(h_\lambda \circ g_\lambda)(\mathbf{x}) \leq \mathbf{x}.$$

The elements of a semilinear l-space are called *vectors* and denoted by bold characters, and elements of L are called *scalars* and denoted by Greek characters. The scalar multiplication is used to be referred to as *external operation.*

Let $\mathcal{A}$ be a semilinear l-space over $\mathcal{M}$. It can be proved that for each $\lambda \in M$, the mapping $g_\lambda : A \longrightarrow A$ is unique. Therefore, g_λ is the residual of h_λ and thus, g_λ can be considered as the second external operation. We will call it *scalar implication* and formally denote $g_\lambda(\mathbf{x})$ by $\lambda \to \mathbf{x}$.

Example 2
Let $\mathcal{L} = \langle L, \vee, \wedge, *, 0, 1\rangle$ be a residuated l-monoid and $\mathcal{L}_* = \langle L, *, 1\rangle$ a monoidal reduct of $\mathcal{L}$. Then the left $\mathcal{L}_*$-semimodules $L^n_\vee$ from Example 1 is a semilinear l-space over $\mathcal{L}_*$ where the respective scalar implication is given by

$$\lambda \to (a_1, \ldots, a_n) = (\lambda \to a_1, \ldots, \lambda \to a_n).$$

In what follows, the above considered semilinear l-space will be denoted L^n.

We can obtain that

- the mapping g_λ preserves $\wedge$, i.e. for all $\mathbf{x}, \mathbf{y} \in A$,
$$\lambda \to (\mathbf{x} \wedge \mathbf{y}) = (\lambda \to \mathbf{x}) \wedge (\lambda \to \mathbf{y}),$$
- mappings h_λ and g_λ constitute an adjoint pair, i.e. for all $\mathbf{x}, \mathbf{y} \in A$, the adjunction property holds true:
$$\lambda\mathbf{x} \leq \mathbf{y} \quad \text{iff} \quad \mathbf{x} \leq \lambda \to \mathbf{y}. \tag{1}$$

Moreover, the equality

$$\lambda \to \mathbf{x} = \sup\{\mathbf{y} \in A \mid \lambda\mathbf{y} \leq \mathbf{x}\}$$

holds true if and only if its right-hand side exists. Therefore, the following properties of $\mathcal{A}$ can be easily proved:

$$0 \to \mathbf{x} = \mathbf{1}, \qquad 1 \to \mathbf{x} = \mathbf{x}, \qquad \lambda \to \mathbf{1} = \mathbf{1}.$$

By (1), other useful properties of semilinear l-spaces can be established in a similar way to that used in [7].

Lemma 1
Let $\mathcal{M} = \langle M, \cdot, 1\rangle$ be a commutative monoid and $\mathcal{A}$ a semilinear l-space over $\mathcal{M}$. Then for any $\lambda, \mu \in M$, $\mathbf{x} \in A$ the following

$$\lambda \to (\mu \to \mathbf{x}) = (\lambda \cdot \mu) \to \mathbf{x}.$$

holds true.

The following theorem shows that a semilinear l-space can be equivalently defined as a structure which consists of two left $\mathcal{M}$-semimodules connected by the adjunction property.

Theorem 1
Let $\mathcal{M} = \langle M, \cdot, 1\rangle$ be a commutative monoid and $\langle A, \leq\rangle$ a bounded lattice.

(i) Let the $\vee$-semilattice $\langle A, \vee, \mathbf{0}\rangle$ be a semilinear l-space over $\mathcal{M}$. Then the $\wedge$-semilattice $\langle A, \wedge, \mathbf{1}\rangle$ is a left $\mathcal{M}$-semimodule where the scalar product of $\lambda \in M$ and $\mathbf{x} \in A$ is given by the scalar implication $\lambda \to \mathbf{x}$. Moreover, the adjunction property (1) holds true.

(ii) On the other side, let the $\vee$-semilattice $\langle A, \vee, \mathbf{0}\rangle$ be a left $\mathcal{M}$-semimodule with the scalar product $\lambda\mathbf{x}$ and the $\wedge$-semilattice $\langle A, \wedge, \mathbf{1}\rangle$ be a left $\mathcal{M}$-semimodule with the scalar product $\lambda \to \mathbf{x}$. Let moreover, the scalar products are connected by (1). Then the left semimodule $\mathcal{A} = \langle A, \vee, \mathbf{0}\rangle$ is a semilinear l-space over $\mathcal{M}$.

Remark 1
By Theorem 1, each semilinear l-space over a commutative monoid is comprised of two left semimodules over the same monoid. We will further refer to them as to the left $\vee$-semimodule and the left $\wedge$-semimodule.

3.2 Homomorphisms in Semilinear Spaces

Definition 4
Let $\mathcal{M} = \langle M, \cdot, 1\rangle$ be a commutative monoid and $\mathcal{A}_1$ and $\mathcal{A}_2$ be semilinear l-spaces over $\mathcal{M}$. A mapping $H : A_1 \longrightarrow A_2$ is a homomorphism if for all $\lambda \in M$ and for all $\mathbf{a}, \mathbf{c} \in A_1$,

$$H(\mathbf{a} \vee \mathbf{c}) = H(\mathbf{a}) \vee H(\mathbf{c}), \qquad (2)$$
$$H(\lambda\mathbf{a}) = \lambda H(\mathbf{a}), \qquad (3)$$
$$H(\mathbf{0}) = \mathbf{0}. \qquad (4)$$

Homomorphism $H : A_1 \longrightarrow A_2$ is residuated with the residual $G : A_2 \longrightarrow A_1$ if for all $\mathbf{a} \in A_1$ and $\mathbf{b} \in A_2$,

$$(G \circ H)(\mathbf{a}) \geq \mathbf{a}, \qquad (5)$$
$$(H \circ G)(\mathbf{b}) \leq \mathbf{b}. \qquad (6)$$

Example 3
Let $\mathcal{L} = \langle L, \vee, \wedge, *, 0, 1\rangle$ be a residuated l-monoid and $\mathcal{L}_* = \langle L, *, 1\rangle$ a monoidal reduct of $\mathcal{L}$. Let L^n, $n \geq 1$, and L^m, $m \geq 1$, be semilinear vector l-spaces over $\mathcal{L}_*$ (see Example 2). Let R be an $n \times m$ matrix with elements r_{ij} from L. We define mappings $H_R : L^m \longrightarrow L^n$ and $G_R : L^n \mapsto L^m$ so that for $\mathbf{a} \in L^m$, $\mathbf{b} \in L^n$,

$$H_R(\mathbf{a})_i = \bigvee_{j=1}^{m} (r_{ij} * a_j), \quad i = 1, \ldots, n, \qquad (7)$$
$$G_R(\mathbf{b})_j = \bigwedge_{i=1}^{n} (r_{ij} \to b_i), \quad j = 1, \ldots, m. \qquad (8)$$

It is easy to verify that all properties (2)-(6) are fulfilled. Therefore, H_R is a residuated homomorphism from L^m to L^n with the residual G_R.

Further on, formulas (7) and (8) will be used in their vector forms as follows:

$$H_R(\mathbf{a}) = R \circ \mathbf{a},$$
$$G_R(\mathbf{b}) = R \to \mathbf{b}.$$

Theorem 2
Let L^n, $n \geq 1$, and L^m, $m \geq 1$, be semilinear vector l-spaces over the monoidal reduct $\mathcal{L}_* = \langle L, *, 1\rangle$ of a residuated l-monoid $\mathcal{L}$. A mapping $H : L^m \longrightarrow L^n$ is a residuated homomorphism with the residual $G : L^n \longrightarrow L^m$ if and only if there exists an $n \times m$ matrix R_H such that for all $\mathbf{a} \in L^m$, $\mathbf{b} \in L^n$, $H(\mathbf{a}) = R_H \circ \mathbf{a}$ and $G(\mathbf{b}) = R_H \to \mathbf{b}$.

4 Systems of Equations in Semilinear Spaces

In what follows, we fix a residuated l-monoid with a support L and consider semilinear vector l-spaces L^m and L^n over $\mathcal{L}_*$ (see Example 2).

Throughout this section, let $A = (a_{ij})$ be a fixed $n \times m$ matrix and $\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{d} = (d_1, \ldots, d_m)$ vectors, all have components from L. The following two systems of equations

$$a_{11} * x_1 \vee \cdots \vee a_{1m} * x_m = b_1,$$
$$\ldots\ldots\ldots\ldots$$
$$a_{n1} * x_1 \vee \cdots \vee a_{nm} * x_m = b_n,$$

and

$$(a_{11} \to y_1) \wedge \cdots \wedge (a_{n1} \to y_n) = d_1,$$
$$\cdots\cdots\cdots\cdots$$
$$(a_{1m} \to y_1) \wedge \cdots \wedge (a_{nm} \to y_n) = d_m,$$

are considered with respect to unknown vectors $\mathbf{x} = (x_1 \ldots, x_m)$ and $\mathbf{y} = (y_1 \ldots, y_n)$. Matrix A and vectors $\mathbf{b}$ and $\mathbf{d}$ will be further referred to as the *matrix of coefficients* and *vectors of right-hand sides* or *right-hand side vectors.*

It is easy to see that both systems represent two inverse problems with respect to two homomorphisms: $H_A : L^m \mapsto L^n$ (7) and its residual $G_A : L^n \mapsto L^m$ (8), i.e.

$$A \circ \mathbf{x} = \mathbf{b}, \tag{9}$$
$$A \to \mathbf{y} = \mathbf{d}. \tag{10}$$

In the foregoing text, systems of equations will be represented shortly by (9) and (10).

We say that $\mathbf{x}^0 \in L^m$ is a solution of (9) if this (vector) equality becomes true after substitution of $\mathbf{x}^0$ for $\mathbf{x}$. Similarly, we define a solution of (10). In the literature, which is related to fuzzy sets and systems, the above considered systems are known as systems of fuzzy relation equations with sup $-*$ composition or inf $\to$ composition, see e.g. [1, 3, 5, 9, 10, 13, 12]. From these sources we took the following results which are put together in the Proposition below.

Proposition 1
Let $A = (a_{ij})$ be an $n \times m$ matrix of coefficients and $\mathbf{b} = (b_1, \ldots, b_n)$, $\mathbf{d} = (d_1, \ldots, d_m)$ vectors of of right-hand sides in (9) and (10), all have components from L. Then

(i) system (9) is solvable if and only if the vector $\hat{\mathbf{x}} = A \to \mathbf{b}$ is its solution;

(ii) if (9) is solvable then $\hat{\mathbf{x}} = A \to \mathbf{b}$ is its greatest solution;

(iii) system (10) is solvable if and only if the vector $\check{\mathbf{y}} = A \circ \mathbf{d}$ is its solution;

(iv) if (10) is solvable then $\check{\mathbf{y}} = A \circ \mathbf{d}$ is its least solution.

Further on, $\hat{\mathbf{x}}$ (respectively, $\check{\mathbf{y}}$) will always denote the vector expressed by $A \to \mathbf{b}$ (respectively, $A \circ \mathbf{d}$).

If $\mathbf{x}^1 \in L^m$ and $\mathbf{x}^2 \in L^m$ are solutions of (9) then $\mathbf{x}^1 \vee \mathbf{x}^2$ is a solution of (9) too. Therefore, the set of solutions of (9) form a $\vee$-semi-lattice with "unit" element.

If $\mathbf{y}^1 \in L^n$ and $\mathbf{y}^2 \in L^n$ are solutions of (10) then $\mathbf{y}^1 \wedge \mathbf{y}^2$ is a solution of (10) too. Therefore, the set of solutions of (10) form a $\wedge$-semi-lattice with "zero" element.

Proposition 1 turns the problem of finding solutions to (9) or (10) (with given A and $\mathbf{b}$ or given A and $\mathbf{d}$) to the investigation whether these systems are solvable (then the extreme solutions are known).

The latter problem will be considered in a new formulation which emphasizes that solvability of (9) or (10) means the expressibility of the right-hand side vectors by "linear combinations" of the column-vectors of A. The new formulation is

- Given $n \times m$ matrix A, characterize all vectors $\mathbf{b} \in L^n$ (all vectors $\mathbf{d} \in L^m$) such that (9) (respectively, (10)) is solvable.

Let us remark that the formulation above is similar to the formulation of the problem of solvability of a system of linear equations in linear algebra which puts an emphasis on a matrix of coefficients.

5 Fixed points of contraction and dilation operators

In this section, we will introduce two operators of contraction and dilation, connected with the matrix A of coefficients in systems (9) and (10). We will show that the problem of solvability of (9) and (10) is equivalent with the problem of characterizing fixed points of contraction and dilation operators. Throughout this section, $A = (a_{ij})$ will be a $n \times m$ matrix with components from L, $\mathbf{b} = (b_1, \ldots, b_n) \in L^n$, $\mathbf{d} = (d_1, \ldots, d_m) \in L^m$.

Definition 5

- $AA^{\rightharpoondown} : L^n \mapsto L^n$ *is a contraction operator on* L^n *if it assigns* $(AA^{\rightharpoondown})\mathbf{b} = A\circ(A \rightarrow \mathbf{b})$ *to every element* $\mathbf{b} \in L^n$ *such that if* $\mathbf{b} = (b_1, \ldots, b_n)$ *then* $(AA^{\rightharpoondown})(\mathbf{b}) = ((AA^{\rightharpoondown})\mathbf{b}_1, \ldots, (AA^{\rightharpoondown})\mathbf{b}_n)$ *where*

$$(AA^{\rightharpoondown})\mathbf{b}_i = \bigvee_{j=1}^{m}(a_{ij} * \bigwedge_{l=1}^{n}(a_{lj} \rightarrow b_l)), \quad i = 1, \ldots, n.$$

- $A^{\rightharpoondown}A : L^m \mapsto L^m$ *is a dilation operator on* L^m *if it assigns* $(A^{\rightharpoondown}A)\mathbf{d} = A \rightarrow (A\circ\mathbf{d})$ *to every element* $\mathbf{d} \in L^m$ *such that if* $\mathbf{d} = (d_1, \ldots, d_m)$ *then* $(A^{\rightharpoondown}A)(\mathbf{d}) = ((A^{\rightharpoondown}A)\mathbf{d}_1, \ldots, (A^{\rightharpoondown}A)\mathbf{d}_m) \in L^m$ *where*

$$(A^{\rightharpoondown}A)\mathbf{d}_j = \bigwedge_{i=1}^{n}(a_{ij} \rightarrow \bigvee_{l=1}^{m}(a_{il} * d_l)), \quad j = 1, \ldots, m.$$

The following proposition easily follows from Proposition 1.

Proposition 2
Let systems of equations (9) and (10) be specified by the $n \times m$ *matrix of coefficients* $A = (a_{ij})$ *and the respective right-hand side vectors* $\mathbf{b} = (b_1, \ldots, b_n)$ *and* $\mathbf{d} = (d_1, \ldots, d_m)$*, all have components from L. Then*

(i) system $A \circ \mathbf{x} = \mathbf{b}$ *(9) is solvable if and only if*

$$(AA^{\rightharpoondown})\mathbf{b} = \mathbf{b}$$

or if and only if $\mathbf{b} \in L^n$ *is a fixed point of the operator* $AA^{\rightharpoondown}$*;*

(ii) system $A \rightarrow \mathbf{y} = \mathbf{d}$ *(10) is solvable if and only if*

$$(A^{\rightharpoondown}A)\mathbf{d} = \mathbf{d}$$

or if and only if $\mathbf{d} \in L^m$ *is a fixed point of the operator* $A^{\rightharpoondown}A$*.*

Remark 2
Easy to see that fixed points of $AA^{\rightharpoondown}$ *(respectively,* $A^{\rightharpoondown}A$*) are eigenvectors of the contraction (respectively, dilation) operator.*

Denotation. $\mathcal{F}(AA^{\rightharpoondown})$ $(\mathcal{F}(A^{\rightharpoondown}A))$ is a set of fixed points of $AA^{\rightharpoondown}$ $(A^{\rightharpoondown}A)$.

Let us remark that transformations of L^n or L^m given by contraction or dilation operators, are not completely new. They were investigated in different structures by different names: in lattices and max-plus algebras [2, 8] they are called as compositions of a mapping and its residual and vice versa. In Theorems 3 and 4 below we will combine the known and new facts and reformulate them according to our terminology.

Theorem 3
Let $A = (a_{ij})$ *be an* $n \times m$ *matrix with components from L. Let* $AA^{\rightharpoondown} : L^n \mapsto L^n$ *be the corresponding contraction operator. Then the following holds true:*

a) for all $\mathbf{b} \in L^n$*,* $(AA^{\rightharpoondown})\mathbf{b} \leq \mathbf{b}$*,*

b) for all $\mathbf{x} \in L^m$*,* $A \circ \mathbf{x}$ *is a fixed point of* $AA^{\rightharpoondown}$*,*

c) for all $\mathbf{b} \in L^n$*,* $\mathbf{b}_0 = (AA^{\rightharpoondown})\mathbf{b}$ *is a fixed point of* $AA^{\rightharpoondown}$*,*

d) for each $\mathbf{b} \in L^n$ *there exists a uniquely determined fixed point* $\mathbf{b}_0 \in L^n$ *of* $AA^{\rightharpoondown}$ *such that* $A \rightarrow \mathbf{b}_0 = A \rightarrow \mathbf{b}$*.*

e) for all $\mathbf{b}_1, \mathbf{b}_2 \in L^n$*,* $\mathbf{b}_1 \leq \mathbf{b}_2$ *implies* $(AA^{\rightharpoondown})(\mathbf{b}_1) \leq (AA^{\rightharpoondown})(\mathbf{b}_2)$*,*

f) if $\mathbf{b}_1, \mathbf{b}_2 \in L^n$ *are fixed points of* $AA^{\rightharpoondown}$ *then* $\mathbf{b}_1 \vee \mathbf{b}_2$ *is a fixed point of* $AA^{\rightharpoondown}$ *too.*

Corollary 1
Let $A = (a_{ij})$ *be an* $n \times m$ *matrix with components from L. Then for all* $\mathbf{b} \in L^n$*,*

$$A \rightarrow (A \circ (A \rightarrow \mathbf{b})) = A \rightarrow \mathbf{b}.$$

Theorem 4
Let $A = (a_{ij})$ *be an* $n \times m$ *matrix with components from L. Let* $A^{\rightharpoondown}A : L^m \mapsto L^m$ *be the corresponding dilation operator. Then the following holds true:*

a) for all $\mathbf{d} \in L^m$*,* $(A^{\rightharpoondown}A)\mathbf{d} \geq \mathbf{d}$*,*

b) for all $\mathbf{y} \in L^n$*,* $A \rightarrow \mathbf{y}$ *is a fixed point of* $A^{\rightharpoondown}A$*,*

c) for all $\mathbf{d} \in L^m$, $\mathbf{d}_0 = (A^{\rightharpoondown}A)\mathbf{d}$ is a fixed point of of $A^{\rightharpoondown}A$,

d) for each $\mathbf{d} \in L^m$ there exists a uniquely determined fixed point $\mathbf{d}_0 \in L^m$ of $A^{\rightharpoondown}A$ such that $A \circ \mathbf{d}_0 = A \circ \mathbf{d}$.

e) for all $\mathbf{d}_1, \mathbf{d}_2 \in L^m$, $\mathbf{d}_1 \leq \mathbf{d}_2$ implies $(A^{\rightharpoondown}A)\mathbf{d}_1 \leq (A^{\rightharpoondown}A)\mathbf{d}_2$,

f) if $\mathbf{d}_1, \mathbf{d}_2 \in L^m$ are fixed points of $A^{\rightharpoondown}A$ then $\mathbf{d}_1 \wedge \mathbf{d}_2$ is a fixed point of $A^{\rightharpoondown}A$ too.

Corollary 2
Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L. Then for all $\mathbf{d} \in L^m$,

$$A \circ (A \rightarrow (A \circ \mathbf{d})) = A \circ \mathbf{d}.$$

The following theorem shows how fixed points of contraction and dilation operators are related.

Theorem 5
Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L.

(i) *If $\mathbf{b}_0 \in L^n$ is a fixed point of $AA^{\rightharpoondown}$ on L^n then there exists a uniquely determined fixed point $\mathbf{d}_0 \in L^m$ of $A^{\rightharpoondown}A$ on L^m such that $A \circ \mathbf{d}_0 = \mathbf{b}_0$ and $\mathbf{d}_0 = A \rightarrow \mathbf{b}_0$.*

(ii) *If $\mathbf{d}_0 \in L^m$ is a fixed point of $A^{\rightharpoondown}A$ on L^m then there exists a uniquely determined fixed point $\mathbf{b}_0 \in L^n$ of $AA^{\rightharpoondown}$ on L^n such that $A \rightarrow \mathbf{b}_0 = \mathbf{d}_0$ and $\mathbf{b}_0 = A \circ \mathbf{d}_0$.*

Corollary 3
Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L. Let $H_A \mid_{\mathcal{F}(A^{\rightharpoondown}A)}$, $G_A \mid_{\mathcal{F}(AA^{\rightharpoondown})}$ be restrictions of the respective homomorphisms on the respective sets of fixed points. Then $H_A \mid_{\mathcal{F}(A^{\rightharpoondown}A)}$ isomorphically maps $\mathcal{F}(A^{\rightharpoondown}A)$ onto $\mathcal{F}(AA^{\rightharpoondown})$. Moreover, $G_A \mid_{\mathcal{F}(AA^{\rightharpoondown})}$ is inverse to $H_A \mid_{\mathcal{F}(A^{\rightharpoondown}A)}$ so that for all $\mathbf{d}_0 \in \mathcal{F}(A^{\rightharpoondown}A)$, $\mathbf{b}_0 \in \mathcal{F}(AA^{\rightharpoondown})$

$$A \circ \mathbf{d}_0 = \mathbf{b}_0 \quad \textit{iff} \quad A \rightarrow \mathbf{b}_0 = \mathbf{d}_0.$$

Remark 3
It is worth to be noticed that homomorphic images of fixed points are fixed points too. This is not always true for preimages. This means that if a fixed point $\mathbf{b}_0 \in L^n$ of $AA^{\rightharpoondown}$ is represented by $A \circ \mathbf{x} = \mathbf{b}_0$ (or $\mathbf{b}_0$ is an image of $\mathbf{x}$) then $\mathbf{x} \in L^m$ is not necessarily a fixed point of $A^{\rightharpoondown}A$. Similarly for a fixed point $\mathbf{d}_0 \in L^m$ of $A^{\rightharpoondown}A$ and its representation $A \rightarrow \mathbf{y} = \mathbf{d}_0$.

5.1 Fixed Points of $AA^{\rightharpoondown}$ and $A^{\rightharpoondown}A$ as Subsemimodules

Let $A = (a_{ij})$ be an $n \times m$ matrix with components from L. In the following theorems we will characterize the set of fixed points of $AA^{\rightharpoondown}$ as a $\vee$-subsemimodule over $\mathcal{L}_\vee$ and the set of fixed points of $A^{\rightharpoondown}A$ as a dual $\wedge$-subsemimodule over $\mathcal{L}_\wedge$.

Theorem 6
Let $A = (a_{ij})$ be a $n \times m-$ matrix with components from L, $AA^{\rightharpoondown} : L^n \mapsto L^n$ the corresponding contraction operator. Then $\mathcal{F}(AA^{\rightharpoondown})$ is a $\vee$-subsemimodule over $\mathcal{L}_\vee$.

Theorem 7
Let $A = (a_{ij})$ be a $n \times m-$ matrix with components from L, $A^{\rightharpoondown}A : L^m \mapsto L^m$ the corresponding dilation operator. Then $\mathcal{F}(A^{\rightharpoondown}A)$ is a dual $\wedge$-subsemimodule over $\mathcal{L}_\wedge$.

Acknowledgements

The author has been partially supported by the project MSM 6198898701 of the MŠMT ČR.

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