Two methods of reconstruction of generators of continuous t-norms

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Abstract

We present two methods which allow to reconstruct the multiplicative and the additive generator of a continuous Archimedean triangular norm from its partial derivatives. The methods can be used for a strict triangular norm whose multiplicative generator has a non-zero derivative at 0, resp. for a continuous Archimedean triangular norm whose additive generator has a derivative continuous at $]0,1]$ and non-zero at 1.

Keywords: Triangular norm, Archimedean triangular norm, multiplicative generator, additive generator, fuzzy logic, fuzzy conjunction, convex combination.

1 Introduction

Each continuous Archimedean triangular norm[1] (t-norm) T is characterized by its multiplicative, resp. additive, generator.

A *multiplicative generator* of a strict t-norm T is an increasing bijection $\theta\colon [0,1] \to [0,1]$ such that $T(x,y) = \theta^{-1}(\theta(x)\cdot\theta(y))$. (In the following text, we will not deal with multiplicative generators of nilpotent t-norms.)

An *additive generator* of a continuous Archimedean t-norm T is a strictly decreasing continuous function $t\colon [0,1] \to [0,\infty]$ such that $t(1) = 0$ and $T(x,y) = t^{(-1)}(t(x)+t(y))$, where the pseudoinverse $t^{(-1)}$ is defined as $t^{(-1)}(z) = t^{-1}(\min(z, t(0)))$. (For our purpose, we have simplified the definition of the pseudoinverse given in [6, 8].)

If we have a generator, the procedure of getting the t-norm is straightforward. Nevertheless, the inverse procedure, i.e., determining the generator from the t-norm, is a difficult task and so far there has not been introduced any intuitive method. It is known since [6, 8] that every continuous Archimedean t-norm has a multiplicative and an additive generator. However, the proof of this crucial theorem is difficult, despite numerous attempts to optimize it (see [1, 5, 11]).

This paper brings two methods which allow to reconstruct both multiplicative and additive generators for a subclass of continuous Archimedean t-norms. In these special cases, the generators are derived directly from partial derivatives of the t-norm.

The following lemma can be found in any book on calculus.

Lemma 1.1 *Let f be a function that is differentiable on an interval I. Let f possess an inverse function g. Each point $x \in I$ where g is differentiable and $f'(g(x)) \neq 0$ satisfies*

$$g'(x) = \frac{1}{f'(g(x))}\,.$$

The consequences will be used to determine the generators in special cases.

[1] See e.g. [3, 5] for the definition.

L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, pp. 1016–1021
Torremolinos (Málaga), June 22–27, 2008

2 Multiplicative generators of strict t-norms

We shall derive a procedure allowing to determine multiplicative generators of strict t-norms easily in a special case which, however is general enough to be of interest.

Lemma 2.1 *Let T be a strict t-norm and let θ be its multiplicative generator. Then*

$$\frac{\partial T(x,y)}{\partial y} = \frac{\theta(x)\cdot\theta'(y)}{\theta'(T(x,y))}$$

whenever both sides are defined.

Remark 2.2 *The expression $\theta'(T(x,y))$ is the value of the derivative θ' at the point $T(x,y)$, not a derivative of a compound expression.*

Proof (of Lemma 2.1): Observe that if the first partial derivative of T at (x,y) is defined, then also the first derivative of $\theta^{-1}(\theta(x)\cdot\theta(y))$ is defined.

According to Lemma 1.1, we can evaluate the expression given by the multiplicative generator:

$$\begin{aligned}\frac{\partial T(x,y)}{\partial y} &= \frac{\partial\theta^{-1}(\theta(x)\cdot\theta(y))}{\partial y}\\ &= \frac{1}{\theta'(T(x,y))}\cdot\frac{\partial(\theta(x)\cdot\theta(y))}{\partial y}\\ &= \frac{\theta(x)\cdot\theta'(y)}{\theta'(T(x,y))}\,.\end{aligned}$$

■

Theorem 2.3 *Let T be a strict t-norm which has a multiplicative generator θ satisfying $\lim_{y\to 0_+}\theta'(y) = c \in\,]0,\infty[$. Then*

$$\theta(x) = \lim_{y\to 0_+}\frac{\partial T(x,y)}{\partial y}$$

for all $x\in[0,1]$.

Proof : According to Lemma 2.1 and the fact that $T(x,y)\to 0$ for $y\to 0$, we obtain:

$$\begin{aligned}\lim_{y\to 0_+}\frac{\partial T(x,y)}{\partial y} &= \lim_{y\to 0_+}\frac{\theta(x)\cdot\theta'(y)}{\theta'(T(x,y))}\\ &= \frac{\theta(x)\cdot c}{c} = \theta(x)\,.\end{aligned}$$

■

Corollary 2.4 *Under the assumptions of Th. 2.3, the multiplicative generator θ of T can be also expressed as*

$$\theta(x) = \lim_{y\to 0_+}\frac{T(x,y)}{y}\,.$$

Proposition 2.5 *Let T be a strict t-norm and let $\mathcal{M}_T$ be the set of its multiplicative generators. If there exists a multiplicative generator $\theta\in\mathcal{M}_T$ satisfying the constraint $\lim_{x\to 0_+}\theta'(x) = c\in\,]0,\infty[$, then it is unique.*

Proof : Suppose we have a multiplicative generator $\theta\in\mathcal{M}_T$ satisfying $\lim_{x\to 0_+}\theta'(x) = c\in\,]0,\infty[$. It is known that any other multiplicative generator $\sigma\in\mathcal{M}_T$ is given by $\sigma=\theta^p$ for some $p\in\,]0,1[\cup]1,\infty[$. Its first derivative can be expressed as

$$\begin{aligned}\lim_{x\to 0_+}\sigma'(x) &= \lim_{x\to 0_+}(\theta^p(x))'\\ &= \lim_{x\to 0_+} p\cdot\left(\theta^{p-1}(x)\right)\cdot\theta'(x)\\ &= p\cdot c\cdot\lim_{x\to 0_+}\theta^{p-1}(x)\,.\end{aligned}$$

This limit is ∞ if $p<1$ and 0 if $p>1$. ■

Proposition 2.6 *Let T be a strict t-norm. The multiplicative generator θ of T satisfying the constraint $\lim_{x\to 0_+}\theta'(x)\in\,]0,\infty[$ exists if and only if $\lim_{x,y\to 0_+}\frac{\partial^2 T}{\partial x\,\partial y}\in\,]0,\infty[$. Note that in such a case $\lim_{x,y\to 0_+}\frac{\partial^2 T}{\partial x\,\partial y} = \lim_{x\to 0_+}\theta'(x)$.*

3 Additive generators of continuous Archimedean t-norms

Here we shall derive an analogue of Lemma 2.1 for additive generators. We shall use partial derivatives of T for one argument going to 1 instead of 0.

Lemma 3.1 *Let T be a continuous Archimedean t-norm and let t be its additive generator. Then*

$$\frac{\partial T(x,y)}{\partial y} = \frac{t'(y)}{t'(T(x,y))}$$

whenever both sides are defined and $T(x,y) \neq 0$.

Remark 3.2 *Again, $t'(T(x,y))$ is the value of the derivative t' at the point $T(x,y)$, not a derivative of a compound expression.*

Proof (of Lemma 3.1): For each $z \in [0, t(0)]$, the pseudoinverse $t^{(-1)}(z)$ coincides with the inverse $t^{-1}(z)$ and the same holds for their derivatives. Under the assumptions of the Lemma,

$$\begin{aligned} \frac{\partial T(x,y)}{\partial y} &= \frac{\partial t^{(-1)}(t(x)+t(y))}{\partial y} \\ &= \frac{\partial t^{-1}(t(x)+t(y))}{\partial y} . \end{aligned}$$

According to Lemma 1.1,

$$\begin{aligned} \frac{\partial T(x,y)}{\partial y} &= \frac{1}{t'(T(x,y))} \cdot \frac{\partial (t(x)+t(y))}{\partial y} \\ &= \frac{t'(y)}{t'(T(x,y))} . \end{aligned}$$

■

Theorem 3.3 *Let T be a continuous Archimedean t-norm with an additive generator t which has a continuous derivative at $]0,1]$ and satisfies $\lim_{y \to 1_-} t'(y) = b \in]-\infty, 0[$. Then*

$$t(x) = \int_x^1 \frac{-b}{\lim\limits_{y \to 1_-} \frac{\partial T(u,y)}{\partial y}} \, du .$$

Proof : According to Lemma 2.1 and the fact that $T(x,y) \to x$ for $y \to 1$, we obtain for each $x \in]0,1]$:

$$\begin{aligned} \lim_{y \to 1_-} \frac{\partial T(x,y)}{\partial y} &= \lim_{y \to 1_-} \frac{t'(y)}{t'(T(x,y))} = \frac{b}{t'(x)} , \\ t'(x) &= \frac{b}{\lim\limits_{y \to 1_-} \frac{\partial T(x,y)}{\partial y}} , \\ t(x) &= t(1) + \int_1^x \frac{b}{\lim\limits_{y \to 1_-} \frac{\partial T(u,y)}{\partial y}} \, du \\ &= 0 - \int_x^1 \frac{b}{\lim\limits_{y \to 1_-} \frac{\partial T(u,y)}{\partial y}} \, du \\ &= \int_x^1 \frac{-b}{\lim\limits_{y \to 1_-} \frac{\partial T(u,y)}{\partial y}} \, du . \end{aligned}$$

■

Proposition 3.4 *Let T be a continuous Archimedean t-norm and let $\mathcal{A}_T$ be the set of its additive generators. If there exists an additive generator $t \in \mathcal{A}_T$ satisfying the constraint $\lim_{x \to 1_-} t'(x) \in]-\infty, 0[$, then all the generators in the set $\mathcal{A}_T$ satisfy this constraint.*

Proof : Suppose we have an additive generator $t \in \mathcal{A}_T$ satisfying $\lim_{x \to 1_-} t'(x) = b \in]-\infty, 0[$. It is known that any other additive generator $s \in \mathcal{A}_T$ is given by $s = p \cdot t$ for some $p \in]0,1[\cup]1, \infty[$. We obtain:

$$\begin{aligned} \lim_{x \to 1_-} s'(x) &= \lim_{x \to 1_-} (p \cdot t(x))' \\ &= \lim_{x \to 1_-} p \cdot t'(x) \\ &= p \cdot b \in]-\infty, 0[. \end{aligned}$$

■

Corollary 3.5 *Under the assumptions of Th. 3.3 and Prop. 3.4, also the following function is an additive generator of t-norm T:*

$$\begin{aligned} t^*(x) &= \int_x^1 \frac{1}{\lim\limits_{y \to 1_-} \frac{\partial T(u,y)}{\partial y}} \, du \\ &= \int_x^1 \frac{1}{\lim\limits_{y \to 1_-} \frac{u - T(u,y)}{1-y}} \, du . \end{aligned}$$

Proposition 3.6 *Let T be an Archimedean t-norm. If the function $x \mapsto \lim_{y\to 1_-} \frac{\partial T(x,y)}{\partial y}$ is continuous and non-zero at $]0,1]$ and, moreover, $\frac{\partial T(x,y)}{\partial y}$ is differentiable at $(1,1)$, then every additive generator t of T satisfies the constraint $\lim_{x\to 1_-} t'(x) \in\,]-\infty, 0[$. Note, that in such a case*

$$\lim_{x,y\to 1_-} \frac{\partial T(x,y)}{\partial y} = 1\,.$$

4 Examples

Frank and Hamacher families[2] (except for the Hamacher product) are examples of strict t-norms where both Th. 2.3 and Th. 3.3 can be applied in order to reconstruct the multiplicative and the additive generators.

As an example, let us reconstruct a multiplicative and an additive generator of a Frank t-norm

$$T_F^\lambda(x,y) = \ln_\lambda\left(1 + \frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}\right)$$

for $\lambda \in\,]0,1[\, \cup\,]1,\infty[$. Applying Th. 2.3, we obtain

$$\begin{aligned}&\lim_{y\to 0_+} \frac{\partial T_F^\lambda(x,y)}{\partial y}\\ &= \lim_{y\to 0_+} \frac{\lambda^y\,(\lambda^x-1)}{\lambda-1+(\lambda^x-1)(\lambda^y-1)} = \frac{\lambda^x-1}{\lambda-1}\end{aligned}$$

which is a multiplicative generator of T_F^λ.

To reconstruct the additive generator, we first compute the limit at $y \to 1_-$,

$$\begin{aligned}&\lim_{y\to 1_-} \frac{\partial T_F^\lambda(x,y)}{\partial y}\\ &= \lim_{y\to 1_-} \frac{\lambda^y\,(\lambda^x-1)}{\lambda-1+(\lambda^x-1)(\lambda^y-1)}\\ &= \frac{\lambda\,(\lambda^x-1)}{\lambda^x\,(\lambda-1)} \neq 0\,,\end{aligned}$$

[2]See [5] for the definitions of the families of t-norms.

then we compute the integral from Cor. 3.5:

$$\begin{aligned}&\int_x^1 \frac{\lambda^u(\lambda-1)}{\lambda\,(\lambda^u-1)}\;du\\ &= \frac{\lambda-1}{\lambda\ln\lambda}\;[\ln|\lambda^u-1|]_{u=x}^1\\ &= \frac{\lambda-1}{\lambda\ln\lambda}\,\ln\frac{\lambda-1}{\lambda^x-1}\,.\end{aligned}$$

The obtained expression is an additive generator of T_F^λ.

On the other hand, the *Hamacher product*

$$T_H(x,y) = \frac{xy}{x+y-xy}$$

is an example of a t-norm which violates the assumption of Th. 2.3. The multiplicative generators of this t-norm are of the form

$$\theta_H(x) = \left(e^{\left(1-\frac{1}{x}\right)}\right)^p\,,$$

where $p > 0$. It is easy to check that $\theta'_H(0) = 0$ for any value of the parameter p and thus the constraint $\lim_{y\to 0_+} \theta'_H(y) \in\,]0,\infty[$ cannot be satisfied. If we anyway try to apply the described method to this t-norm, we obtain

$$\begin{aligned}\lim_{y\to 0_+} \frac{\partial T_H(x,y)}{\partial y} &= \lim_{y\to 0_+} \frac{\partial}{\partial y}\left(\frac{xy}{x+y-xy}\right)\\ &= \lim_{y\to 0_+} \frac{x^2}{(x+y-xy)^2} = \frac{x^2}{x^2} = 1\end{aligned}$$

which is not a strictly increasing function and thus it cannot be a multiplicative generator. Nevertheless, the additive generator of this t-norms can still be reconstructed since the constraint of Th. 3.3 is satisfied.

Dombi family (except for the Hamacher product) and Aczél–Alsina family (except for the product t-norm) are examples of strict t-norms where both the assumptions of Th. 2.3 and Th. 3.3 are violated and therefore neither the multiplicative generator nor the additive generator can be reconstructed using the described methods.

Schweizer–Sklar and Sugeno–Weber families are examples of families of nilpotent t-norms where Th. 3.3 applies. On the contrary, Yager family of nilpotent t-norms violates the constraint. For an illustration, let us try to apply Th. 3.3 to a Yager t-norm

$$T_Y^\lambda(x,y) = 1 - \left((1-x)^\lambda + (1-y)^\lambda\right)^{\frac{1}{\lambda}} \vee 0$$

for $\lambda \in \,]0,\infty[$. Applying Th. 3.3, we obtain the expression

$$\begin{aligned}&\lim_{y\to 1_-} \frac{\partial T_Y^\lambda(x,y)}{\partial y} \\ &\quad = \lim_{y\to 1_-} \left((1-x)^\lambda + (1-y)^\lambda\right)^{\frac{1-\lambda}{\lambda}} \\ &\qquad \cdot (1-y)^{\lambda-1}\end{aligned}$$

which equals zero for $\lambda \in \,]1,\infty[$ and infinity for $\lambda \in \,]0,1[$. Let us note that this expression is non-zero and finite for $\lambda = 1$ which stands for the Łukasiewicz t-norm.

5 Conclusion

We have shown that a multiplicative generator can be reconstructed from a continuous Archimedean t-norm if the multiplicative generator satisfies the constraint $\lim\limits_{y\to 0_+} \theta'(y) \in \,]0,\infty[$. The condition of existence of such a multiplicative generator is equivalent to the constraint $\lim\limits_{x,y\to 0_+} \frac{\partial^2 T}{\partial x \partial y} \in \,]0,\infty[$; this condition may be found more useful as it refers to the shape of the surface of the t-norm.

We have furthermore shown that also an additive generator can be reconstructed from a continuous Archimedean t-norm if the additive generator has a continuous derivative at $]0,1]$ and satisfies the condition $\lim\limits_{y\to 1_-} t'(y) \in \,]-\infty,0[$. The existence of such an additive generator is implied by the constraint described in Prop. 3.6.

These results contribute also to the open question whether there is a t-norm which can be expressed as a non-trivial convex combination of continuous Archimedean t-norms (cf. [4, 7, 9, 10]). This cannot happen for the t-norms satisfying the assumptions of Ths. 2.3 and 3.3. This extends the partial solution given in [12]. Further generalizations of these results will be subject to future study.

Acknowledgement. The authors acknowledge the support of the grant 201/07/1136 of the Czech Science Foundation.

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