Integral representation for divisible SMV_{Δ}-algebras

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Abstract

In this paper we propose an algebraic approach to Lebesgue integral on MV-algebras, and we show that, whenever we take an internal state σ on a divisible MV_{Δ}-algebra A, then σ can be represented by means of this more general notion of integral.

Keywords: States, MV-algebras, SMV-algebras, Lebesgue integral, integral representation.

1 Introduction

States on MV-algebras (cf [12]) represent a generalization of probability measures on boolean algebras, and in the last years many authors studied the relation among states and integrals. Here we want just to quote those ones we retain more significant for the present paper: in [9] Kroupa proves that for every state s on a semisimple MV-algebra A, there exists a unique Borel probability measure μ on the class of A-maximal ideals $\mathcal{M}(A)$, such that $s(a) = \int_{\mathcal{M}(A)} a d\mu$. In [10] Marra and Mundici characterize the Lebesgue state (that is that state defined by means of Lebesgue integral) on the n-free unital ℓ-group $\mathcal{G}_n$.

As they are, states are not internal operations on MV-algebras, because they map MV-algebras into the real unit interval $[0, 1]$. In [6] we introduce the variety $\mathbb{SMV}$ of *MV-algebras with an internal state* σ (SMV-algebras for short) and we present a method for obtaining an SMV-algebra starting from a state on an MV-algebra, and vice-versa. Clearly this shows that SMV-algebras allow a treatment of states in the context of universal algebra.

In this paper we introduce an algebraic approach to the Lebesgue integral. This generalization is obtained by considering functions and measures taking values in an abelian ℓ-group instead of the real field

Once this generalization is introduced in Section 4, the main result of this paper says us that any internal state of an MV-algebra A can be represented by means of this more general notion of integral.

2 Preliminaries

An *MV-algebra* is a system $(A, \oplus, ^*, 0)$, where $(A, \oplus, 0)$ is a commutative monoid with neutral element 0, and for each $x, y \in A$ the following equations hold: (i) $(x^*)^* = x$, (ii) $x \oplus 1 = 1$, where $1 = 0^*$, and (iii) $x \oplus (x \oplus y^*)^* = y \oplus (y \oplus x^*)^*$. The class of MV-algebras forms a variety which henceforth will be denoted by $\mathbb{MV}$.

In any MV-algebra one can define further operations as follows: $x \to y = (x^* \oplus y)$, $x \ominus y = (x \to y)^*$, $x \odot y = (x^* \oplus y^*)^*$, $x \leftrightarrow y = (x \to y) \odot (y \to x)$, $x \vee y = (x \to y) \to y$, and $x \wedge y = (x^* \vee y^*)^*$. Henceforth we shall use the following notation: for every $x \in A$ and every $n \in \mathbb{N}$, $nx = x \oplus \overset{n}{\ldots} \oplus x$, and $x^n = x \odot \overset{n}{\ldots} \odot x$.

Any MV-algebra A can be equipped with an order relation. As a matter of fact defining, for all $x, y \in A$, $x \leq y$ iff $x \to y = 1$. An MV-algebra is said *linearly ordered* (or an MV-

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chain) if the order $\leq$ is linear.

In [3] Chang showed that the variety $\mathbb{MV}$ of MV-algebras is generated by the so called *standard chain*, that is the MV-algebra $[0,1]_{MV} = ([0,1], \oplus, ^*, 0)$, based on the real unit interval $[0,1]$, with operations: for all $x, y \in [0,1]$, $x \oplus y = \min\{1, x+y\}$ and $x^* = 1-x$.

A *divisible* MV-algebra (DMV-algebra for short) is a system $(A, \{\delta_n\}_{n\in\mathbb{N}})$ where A is an MV-algebra and for each $n \in \mathbb{N}$ and each $x \in A$, $x \ominus \delta_n(x) = (n-1)\delta_n(x)$ holds. As shown in [7] the variety $\mathbb{DMV}$ of DMV-algebras is generated by the algebra $[0,1]_{DMV} = ([0,1], \oplus, ^*, \{\delta_n\}_{n\in\mathbb{N}}, 0)$, where $([0,1], \oplus, ^*, 0)$ is the standard MV-algebra, and for each $x \in [0,1]$, $\delta_n(x) = \frac{x}{n}$. In any DMV-algebra we can multiply elements by rationals in $[0,1]$: $0x = 0$, and if $0 < m \leq n$, then $\frac{m}{n}x = m\delta_n(x)$.

In [11] Mundici proved a categorical equivalence Γ between the category of MV-algebras and that of ℓ-groups with strong unit. Recall that a *lattice-ordered abelian group* (ℓ-group for short) $\mathcal{G} = (G, \wedge, \vee, +, -, 0)$ is an abelian group $(G, +, -, 0)$ equipped with a lattice structure $(G, \wedge, \vee)$ and further satisfying: $x + (y \wedge z) = (x+y) \wedge (x+z)$ for all $x, y, z \in G$. An element $u \in G$ is a *strong unit* for $\mathcal{G}$ if for all $x \in G$, there is an $n \in \mathbb{N}$ such that $nu \geq x$. An ℓ-group $\mathcal{G}$ is said *divisible* if for every $x \in G$ and for every $n \in \mathbb{N}$, there is an $y \in G$ (usually denoted by $\frac{x}{n}$) such that $ny = x$ (where ny stands for $y + \ldots + y$ n-times). Given now an ℓ-group $\mathcal{G}$ with a strong unit u, the MV-algebra $\Gamma(\mathcal{G}, u)$ has universe $\{x \in G \mid 0 \leq x \leq u\}$, and operations so defined: $x \oplus y = u \wedge (x+y)$, and $x^* = u - x$. In [7] Gerla showed that Mundici's functor Γ can be extended to prove a categorical equivalence between DMV-algebras and divisible ℓ-groups with strong unit.

MV-algebras can be naturally represented as algebras of functions, as the following theorem shows.

Theorem 2.1 (Di Nola, [5]) *Up to isomorphism, every MV-algebra A is an algebra of $[0,1]^\star$-valued functions over $Spec(A)$, where $[0,1]^\star$ is an ultrapower of the real unit interval $[0,1]$, only depending on the cardinality of A.*

A state s on MV-algebra A (cf [12]) is a map $s : A \to [0,1]$ such that $s(1) = 1$ and $s(x \oplus y) = s(x) + s(y)$, whenever $x \odot y = 0$.

By a state on an ℓ-group $\mathcal{G}$ with a strong unit u we mean a normalized positive homomorphism $h : G \to \mathbb{R}$. Precisely a state h on $\mathcal{G}$ has to satisfy: for each $x, y \in G$, $h(x+y) = h(x) + h(y)$, $h(x) \geq 0$ whenever $x \geq 0$, and $h(u) = 1$.

3 SMV-algebras

An *MV-algebra with an internal state* (SMV-algebra for short) is a pair (A, σ) where A is an MV-algebra and $\sigma : A \to A$ satisfies the following properties for each $x, y \in A$:

(σ1) $\sigma(0) = 0$,

(σ2) $\sigma(x^*) = (\sigma(x))^*$,

(σ3) $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$,

(σ4) $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$.

An SMV-algebra (A, σ) is said *faithful* if it satisfies the quasi-equation: $\sigma(x) = 0$ implies $x = 0$. Clearly the class of SMV-algebra constitutes a variety which will be henceforth denoted by $\mathbb{SMV}$.

Lemma 3.1 ([6]) *In any SMV-algebra (A, σ) the following conditions hold:*

(a) $\sigma(1) = 1$.

(b) *If $x \leq y$, then $\sigma(x) \leq \sigma(y)$.*

(c) $\sigma(x \oplus y) \leq \sigma(x) \oplus \sigma(y)$, *and if $x \odot y = 0$, then $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y)$.*

(d) $\sigma(x \ominus y) \geq \sigma(x) \ominus \sigma(y)$ *and if $y \leq x$, then $\sigma(x \ominus y) = \sigma(x) \ominus \sigma(y)$.*

(e) *Letting $d(x,y) = (x \ominus y) \oplus (y \ominus x)$, we have that $d(\sigma(x), \sigma(y)) \leq \sigma(d(x,y))$.*

(f) $\sigma(x) \odot \sigma(y) \leq \sigma(x \odot y)$. *Thus if $x \odot y = 0$, then $\sigma(x) \odot \sigma(y) = 0$.*

(g) $\sigma(\sigma(x)) = \sigma(x)$.

(h) *The image* $\sigma(A)$ *of* A *under* σ *is the domain of a MV-subalgebra of* A.

4 Lebesgue integral on MV-algebras

In this section we propose an algebraic approach to the Lebesgue integral. This approach is obtained by relaxing some conditions. More precisely:

(a) Instead of the real field, we consider a divisible and totally ordered ℓ-group $\mathcal{G} = (G, \wedge, \vee, +, -, 0)$ with a strong unit u. Therefore the structure we are cosidering needs not have a multiplication, and needs not be complete with respect to the order. Anyway, if we interpret u as 1, then we have a copy of rational numbers in G: the rational $\pm\frac{n}{m}$ is identified by $\pm(n\frac{u}{m})$. Moreover in $\mathcal{G}$ we can define the multiplication by a rational number: for each $x \in G$, and for each $\frac{n}{m} \in \mathbb{Q}$, $\pm(\frac{n}{m}x)$ can be identified by $\pm(n\frac{x}{m})$. Hence we may assume without loss of generality that the ordered group $(\mathbb{Q}, \leq, +, -, 0)$ is an ordered subgroup of G. In particular, the strong unit u will be henceforth denoted by 1. Moreover G can be regarded as a vector space over the rational field $(\mathbb{Q}, +, -, \cdot, ^{-1}, 0, 1)$.

(b) Instead of the usual measure on the reals, we have a $G \cap [0, 1]$-valued measure μ from a boolean algebra B into $G \cap [0, 1]$, that is a map $\mu : B \to G \cap [0, 1]$ such that $\mu(1) = 1$, and, if $a \wedge b = 0$, then $\mu(a \vee b) = \mu(a) + \mu(b)$. Notice that, by the Stone representation theorem B may be regarded as a family of subsets of a set U (in our picture the elements of B represent μ-measurable subsets of U). Moreover we do not require μ to be σ-additive.

(c) Finally the functions taken into consideration are not functions from $\mathbb{R}$ into $\mathbb{R}$, but functions from U into G.

An element $x \in G$ is said *bounded* if there is a rational $q > 0$ such that $|x| \leq q$ (where, for each $x \in G$, $|x|$ stands for $x \vee -x$). Every bounded element $x \in G$ has a *standard part* $st(x)$ defined by $st(x) = \sup\{q \in \mathbb{Q} \mid q \leq x\} = \inf\{q \in \mathbb{Q} \mid x \leq q\}$, where infima and suprema refer to real numbers, and not to G. Thus $st(a)$ is a real number, but not necessarily an element of G.

To each $G \cap [0, 1]$-valued measure on B we can associate a $[0, 1] \cap \mathbb{R}$ -valued measure μ_{st} letting $\mu_{st}(b) = st(\mu(b))$. We recall that by Stone's theorem B can be regarded as the set of clopen element of a compact Hausdorff space over the set $\mathcal{M}(B)$ of ultrafilters of B.

We want now to define the concept of *Lebesgue integral* of functions from U into G. For simplicity, we restrict our attention to the set of functions f which are *bounded*, that is, there is $q \in \mathbb{Q}$, $q > 0$, such that for all $x \in U$, $|f(x)| \leq q$.

Definition 4.1 *A* basic function *is a function* h *for which there are a partition* $X_1, \ldots, X_n$ *of* B *and mutually distinct rationals* $q_1, \ldots, q_n$*, such that for* $i \in \{1, \ldots, n\}$ *and for* $x \in X_i$*, one has* $h(x) = q_i$*. The* integral *of the above defined basic function* h *is defined to be*

$$I(h) = st\left(\sum_{i=1}^{n} \mu(X_i)q_i\right) = \sum_{i=1}^{n} \mu_{st}(X_i)q_i.$$

Note that $I(h)$ *is a real number.*

Now let f be a bounded function and let F^- (F^+ respectively) denote the set of all basic functions h such that $h(x) \leq f(x)$ for all $x \in U$ ($h(x) \geq f(x)$ for all $x \in U$ respectively), and let $I^-(f) = \sup\{I(h) \mid h \in F^-\}$ and $I^+(f) = \inf\{I(h) \mid h \in F^+\}$.

Definition 4.2 *We say that* f *is* Lebesgue-integrable *iff* $I^-(f) = I^+(f)$*. In this case we define the* integral

$$\int f d\mu$$

of f *to be the common value* $I^-(f) = I^+(f)$.

Remark 4.3 *If* f *is Lebesgue integrable, then* $\int f d\mu$ *is a real number, but possibly not an element of* G*. Note also that the Lebesgue integral is* a linear and weakly monotonic *functional, in the sense that for every* $q, r \in \mathbb{Q}$

and for every pair f, g of integrable functions, we have that $qf + rg$ is integrable and $\int(qf + rg)d\mu = q\int f d\mu + r\int g d\mu$, and if $f(u) \leq g(u)$ for all $u \in U$, then $\int f d\mu \leq \int g d\mu$.

A function f is said to be *measurable* if it is bounded and for every $q \in \mathbb{Q}$, the sets $U_{f<q} = \{x \in U \mid f(x) < q\}$ and $U_{f=q} = \{x \in U \mid f(x) = q\}$ are measurable (that is they are elements of the boolean algebra B). Since measurable sets are closed under the boolean operations, it follows that if f is measurable, then for all $r, q \in \mathbb{Q}$ with $r < q$, also the set $U_{f\in[q,r)} = \{x \in U \mid q \leq f(x) < r\}$ is measurable.

Lemma 4.4 *Every measurable function is Lebesgue integrable.*

Proof. Clearly it suffices to show that for every positive rational ε there are $h \in F^-$ and $k \in F^+$ such that $I(k) - I(h) < \varepsilon$. Let $q \in \mathbb{Q}$ be such that for all $x \in U$ we have $-q < f(x) < q$. Let n be a natural number such that $\frac{2q}{n} < \varepsilon$ and let for $i = 0, \ldots, n$, $a_i = -q + \frac{2iq}{n}$. Define $h(x)$ and $k(x)$ as follows: let $x \in U$ be given and let $i(x)$ be the unique integer i with $0 \leq i < n$ such that $a_i \leq f(x) < a_{i+1}$. Then define $h(x) = a_i$ and $k(x) = a_{i+1}$. Then clearly $h(x) \leq f(x) \leq k(x)$. Moreover for every $x \in U$, we have $k(x) - h(x) = \frac{2q}{n}$. Thus

$$I(k) - I(h) = \frac{2q}{n}\sum_{i=0}^{n-1}\mu\left(U_{f\in[a_i,a_{i+1})}\right) = \frac{2q}{n} < \varepsilon.$$

This ends the proof. ∎

We want to introduce a completely algebraic treatment of Lebesgue integration of bounded functions. Thus we try to reduce all the structures we need to a unique general algebraic structure. We adopt the following conventions: first of all we restrict our attention to functions from U into $G \cap [0, 1]$. This is not a heavy restriction: modulo a linear transformation, every bounded function can be transformed into a function with values on $[0, 1]$. We stress that every object of our universe will be a function from U into $G \cap [0, 1]$. For instance the elements of $G \cap [0, 1]$ can be represented by the constant functions and the elements of B are represented by their charateristic functions, i.e., by functions from U into $\{0, 1\}$. Our language will have symbols to represent rationals in $[0, 1]$ and multiplication of a function by a rational in $[0, 1]$. When trying to internalize the definition of Lebesgue integral, we meet a difficulty: the integral $\int f d\mu$ of a function f of our universe is not a function from U into $G \cap [0, 1]$, but a real number. we shall overcome this difficulty by representing the integral of a function f by a function on U whose standard part is constantly equal to $\int f d\mu$ (such a function will be called a *non-standard approximation* of $\int f d\mu$) Of course the measure of a set $X \in B$ will be the integral (represented as shown above) of its characteristic function. The next section is devoted to the representation of this machinery in the context of a variety of universal algebras. In other words, we shall axiomatize all the properties we need in terms of identities between terms of a suitable language.

5 Integral representation for divisible SMV$_\Delta$-algebras

As we recalled in Section 3, MV-algebras can be naturally represented as algebras of functions (cf Theorem 2.1). Thus in these algebras we can represent the functions on which we want to define the Lebesgue integral.

Now we come to the second ingredient, namely the boolean algebra of measurable sets. Any MV-algebra has a largest subalgebra which is a boolean algebra, namely the subalgebra consiting of those elements x such that $x \oplus x = x$. In Di Nola's representation (see Theorem 2.1), this subalgebra consists of those functions which are $\{0, 1\}$-valued. However we do not have a term ranging over the set of boolean elements. Moreover in important MV-algebras, e.g. in free MV-algebras, the only boolean elements are 0 and 1. For this reason we shall use the operator Δ (cf [1, 8]), whose interpretation in Di Nola's representation is: $(\Delta f)(x) = 1$ if $f(x) = 1$, and $(\Delta f)(x) = 0$ otherwise.

Definition 5.1 *An* MV$_\Delta$-algebra *is an alge-*

bra $(A, \oplus, ^*, \Delta, 0, 1)$ *such that its* Δ*-free reduct is an MV-algebra, and* Δ *is a unary operator on* A *satisfying the following identities:*

($\Delta 1$) $\Delta(1) = 1$,

($\Delta 2$) $\Delta(x) \leq x$,

($\Delta 3$) $\Delta(\Delta(x)) = \Delta(x)$,

($\Delta 4$) $\Delta(x \to y) \leq \Delta(x) \to \Delta(y)$,

($\Delta 5$) $\Delta(x) \vee (\Delta(x))^* = 1$,

($\Delta 6$) $\Delta(x \vee y) = \Delta(x) \vee \Delta(y)$.

Any MV_Δ-algebra can be regarded as an algebra of functions from a compact Hausdorff space into the unit interval $[0,1]^\star$ of a hyperreal field. The representation is as follows: The algebra $\Delta(A)$ is a boolean algebra. Now take its dual space (U, T) (which is a compact Hausdorff space) where U is the set of ultrafilters of $\Delta(A)$ and T is the topology generated by all sets of the form $C_a = \{u \in U : a \in u\}$ for $a \in \Delta(A)$. For every $u \in U$ there is a unique ultrafilter u' of A which extends u, namely $u' = \{x \in A \mid \Delta(x) \in u\}$. Then consider the quotient A/u' of A modulo u'. We can construct an extension $[0,1]^\star$ of the standard MV-algebra $[0,1]_{MV}$ such that for every ultrafilter u' of A, A/u' embeds into $[0,1]^\star$ (one may prove this using the fact that the class of MV-chains has the amalgamation property and is closed under union of chains). Now we can associate to each $a \in A$ the function f_a on U defined for $u \in U$ by $f_a(u) = a/u'$ (the equivalence class of a modulo the congruence determined by the unique ultrafilter u' of A extending u). Operations on these functions are defined componentwise. Once again the elements of $B = \Delta(A)$ correspond to the $\{0,1\}$-valued functions.

With respect to Di Nola's representation, we have the following advantages: (1) the elements of B are precisely those of the form $\Delta(x)$, therefore we have a very simple way to express them; (2) in the case of MV_Δ-algebras, the topological space (U, T) is compact and totally disconnected.

In any MV-algebra we can somehow simulate sum, because $\oplus$ is a truncated sum, but we cannot simulate rationals and multiplication by a rational. This requirement will be satisfied taking DMV-algebras. Recall in fact in any DMV-algebra A one can multiply elements by rationals in $[0,1]$.

Clearly, measures and integrals will be expressed by means of states. A state on a DMV-algebra A is a state on the MV-reduct of A. It is easy to show that in any divisible SMV-algebra, $\sigma(\delta_n(x)) = \frac{\sigma(x)}{n}$.

As noted in the introduction, states on semisimple MV-algebras can be regarded as integrals (recall Kroupa's theorem, [9]). In the sequel we shall present a treatment of integral inside universal algebra, using the notion of DMV_Δ-algebra which we are going to define in the next lines.

Definition 5.2 *A* DMV_Δ-algebra with an internal state *($SDMV_\Delta$-algebra for short) is an algebra* $(A, \oplus, ^*, 0, \Delta, (\delta_n : n \in \omega, n > 1), \sigma)$ *such that:*

(1) $(A, \oplus, ^*, 0, \Delta)$ *is an* MV_Δ*-algebra.*

(2) $(A, \oplus, ^*, 0, \{\delta_n : n \in \mathbb{N}\})$ *is a DMV-algebra.*

(3) $(A, \oplus, ^*, \sigma, 0)$ *is an SMV-algebra further satisfying the following equation:*

($\sigma 5$) $\sigma(\Delta(\sigma(x))) = \Delta(\sigma(x))$.

Lemma 5.3 *Let* A *be an* $SDMV_\Delta$*-algebra. Then:*

(a) If q *is a rational in* $[0,1]$ *and* $f \in A$*, then* $\sigma(qf) = q\sigma(f)$*,*

(b) The set $\sigma(A) = \{\sigma(x) : x \in A\}$ *is (the domain of) a divisible* MV_Δ*-subalgebra of* A *which is closed under* σ*.*

Proof. (a) If $q = 0$ or $q = 1$, the claim is obvious (note that $\sigma(0) = 0$ follows from ($\sigma 1$) and ($\sigma 2$)). Now suppose $q = \frac{m}{n}$ with $0 < m < n$. Then using (2) and the fact that for $i + j \leq n$ we have $(i)\delta_n(x) \odot (j)\delta_n(x) = 0$, we get that $\sigma(x) \ominus \sigma(\delta_n(x)) = \sigma(x \ominus \delta_n(x)) = \sigma((n-1)\delta_n(x)) = (n-1)\sigma(\delta_n(x))$. Thus $\sigma(\delta_n(x)) = \delta_n(\sigma(x))$, therefore $\sigma((m)\delta_n(x)) = (m)\delta_n(\sigma(x))$, as desired.

(b) We already know that $\sigma(A)$ is closed under $\oplus$, * and σ (recall Lemma 3.1). Moreover we have just proved that $\sigma(\delta_n(x)) = \delta_n(\sigma(x))$, therefore $\sigma(A)$ is closed under δ_n. It remains to prove that $\sigma(A)$ is closed under Δ. Now axiom (σ5) tells us that if $a \in \sigma(A)$, say $a = \sigma(x)$ for some x, then $\Delta(a) = \sigma(\Delta(a))$, therefore $\Delta(a) \in \sigma(A)$. This ends the proof. ■

In our representation of MV$_\Delta$-algebras as algebras of functions, an element f of $\Delta(A)$ represents the characteristic function of the set Z_f of all $u \in U$ such that $f(u) = 1$. Our idea is that if $f \in \Delta(A)$, then $\sigma(f)$ should represent the measure $\mu(Z_f)$ of Z_f and if f is an arbitrary element of A, then $\sigma(f)$ should represent $\int f d\mu$. When trying to formalize this idea, we meet a problem: in general, $\sigma(A)$ is not totally ordered, whereas the set of integrals, being a set of reals, is totally ordered. Worse than this, in Di Nola's representation of A, the elements of $\sigma(A)$ need not be constant (whereas an integral, being a number, is constant). we shall show that these problems do not occur if A is subdirectly irreducible.

We start from the following:

Definition 5.4 *A $\{\sigma, \Delta\}$-filter of an SDMV$_\Delta$-algebra is a filter of its MV-reduct which is closed under σ and Δ*

Let (A, σ) be an SDMV$_\Delta$-algebra. Then:

Lemma 5.5 *(1) The maps $\theta \mapsto F_\theta$ associating to each congruence θ the set $F_\theta = \{x \in A \mid (x, 1) \in \theta\}$ and $F \mapsto \theta_F$ mapping each $\{\sigma, \Delta\}$-filter F into $\theta_F = \{(x, y) \in A \times A \mid x \to y \in F, y \to x \in F\}$ are mutually inverse homomorphisms between the congruence lattice and the $\{\sigma, \Delta\}$-filter lattice of (A, σ).*
(2) The $\{\sigma, \Delta\}$-filter generated by an element $\sigma(a) \in \sigma(A)$ is the set $\{x : \Delta(\sigma(a)) \leq x\}$.

Proof. (1) In [6] (Theorem 4.1) it is shown that the lattice of σ-filters (that is an MV-filter closed under σ) of an SMV-algebra (A, σ) is isomorphic to the congruences lattice of (A, σ). Hence it suffices to show that an F is a $\{\sigma, \Delta\}$-filter iff θ_F is a congruence of (A, σ).

($\Rightarrow$): Suppose that F is a $\{\sigma, \Delta\}$-filter. If $(x, y) \in \theta_F$, then for every $n \in \mathbb{N}$, $\delta_n(x) \leftrightarrow \delta_n(y) \geq x \leftrightarrow y \in F$. Thus θ_F is compatible with δ_n for each $n \in \mathbb{N}$. Moreover, if $(x, y) \in \theta_F$, then $\Delta(x \leftrightarrow y) \in F$, as F is closed under Δ. Since $\Delta x \leftrightarrow \Delta y \leq \Delta(x \leftrightarrow y)$, $\Delta x \leftrightarrow y \in F$, and $(\Delta x, \Delta y) \in \theta_F$. Thus θ_F is also compatible with Δ and it is a congruence of (A, σ).

($\Leftarrow$): Suppose that θ_F is a congruence of (A, σ). Then θ_F is a congruence of the SMV-reduct of (A, σ), therefore F is a filter closed under σ. Finally, if $x \in F$, then $(x, 1) \in \theta_F$ and $(\Delta x, 1) \in \theta_F$ as θ_F is compatible with Δ. Thus $\Delta x \in F$ and F is closed under Δ. This ends the proof.

(2) Let $S = \{x : \Delta(\sigma(a)) \leq x\}$. Then the $\{\sigma, \Delta\}$-filter generated by $\sigma(a)$ must contain $\Delta(\sigma(a))$, therefore it must contain S. For the opposite direction, it suffices to show that S is a filter containing $\sigma(a)$ and closed under Δ and under σ. That $\sigma(a) \in S$ follows from the condition $\Delta(x) \leq x$. That S is upwards closed is trivial, and that S is closed under $\odot$ follows from the fact that $\Delta(x) \odot \Delta(x) = \Delta(x)$. Closure under Δ follows from the condition $\Delta(\Delta(x)) = \Delta(x)$, and closure under σ follows from condition (σ5). This ends the proof. ■

As usual, we shall interpret the elements of A as functions from the set U of ultrafilters of $\Delta(A)$ into some non-standard interval $[0, 1]^\star$. Note that MV-operations, Δ and the operations δ_n are componentwise, whilst σ is not. This is due to the fact that a congruence of the underlying MV$_\Delta$-algebra needs not be a congruence of A.

Lemma 5.6 *Let (A, σ) be a subdirectly irreducible SDMV$_\Delta$-algebra. Then:*

(1) $\sigma(A)$ is linearly ordered,

(2) Let $\mathcal{G}$ be a totally ordered abelian group with strong unit 1 such that the MV-reduct of A is isomorphic to $\Gamma(\mathcal{G}, 1)$. Then the map μ on $\Delta(A)$ defined, for $\Delta(x) \in \Delta(A)$, by $\mu(\Delta(x)) = \sigma(\Delta(x))$, is a measure on $\Delta(A)$ taking values in $G \cap [0, 1]$,

(3) *For every element f of A (represented as a function from the set of maximal Δ-filters of the MV_Δ reduct of A), $st(\sigma(f))$ is constant,*

(4) *Every $f \in A$ is a measurable function, therefore it is Lebesgue integrable (in the sense of Definition 4.2).*

Proof. (1) Let F be the minimum non-trivial $\{\sigma, \Delta\}$-filter of (A, σ). Let $c \in F$, and $c < 1$. Suppose by contradiction that $\sigma(a), \sigma(b) \in \sigma(A)$ are incomparable with respect to the order. Then by Lemma 5.5, the filter generated by $\sigma(a) \to \sigma(b)$ is $F = \{x : \Delta(\sigma(a) \to \sigma(b)) \leq x\}$. Moreover such filter is non-trivial, therefore $c \in F$, and $\Delta(\sigma(a) \to \sigma(b)) \leq c$. Similarly we can prove that $\Delta(\sigma(b) \to \sigma(a)) \leq c$, therefore $1 = \Delta(\sigma(a) \to \sigma(b)) \vee \Delta(\sigma(a) \to \sigma(b)) \leq c$, and a contradiction has been reached.

(2) It follows easily from (σ1) and (σ3).

(3) We have $\sigma(1) = 1$, $\sigma(0) = 0$ and for $0 < m < n$, $\sigma((m)\delta_n(x)) = (m)\delta_n(\sigma(x))$. It follows immediately that for every rational $q \in [0,1]$, $\sigma(q) = q$, therefore $q \in \sigma(A)$. Since $\sigma(A)$ is linearly ordered, for every $f \in A$ and for every $q \in [0,1]$ we have that either $q \leq \sigma(f)$ or $\sigma(f) \leq q$. Thus if we interpret q as the constant function $q(u)$ on U which is equal to q on each $u \in U$, we have that either for all $u \in U$, $q = q(u) \leq \sigma(f)(u)$ or for all $u \in U$, $\sigma(f)(u) \leq q(u) = q$. Thus $st(\sigma(f)(u))$ is constantly equal to $\sup\{q \in [0,1] : q \leq \sigma(f)\} = \inf\{q \in [0,1] : \sigma(f) \leq q\}$.

(4) Let $q \in [0,1]$. Then $U_{f<q} = \Delta(f \to q) \wedge (\Delta(q \to f))^*$, and $U_{f=q} = \Delta(f \to q) \wedge \Delta(q \to f)$. Since $\Delta(A)$ is closed under all MV-operations, we have that $U_{f<q}$ and $U_{f=q}$ belong to $\Delta(A)$, the algebra of measurable sets. ∎

Theorem 5.7 *Under the assumptions of Lemma 5.6, we have $\int f d\mu = st(\sigma(f))$.*

Proof. By Lemma 5.6, (4), $\int f d\mu$ exists, therefore we only have to prove that $\int f d\mu = st(\sigma(f))$. Clearly, it suffices to prove that for every (arbitrarily small) positive real number ε, there are $h \in F^-$ and $k \in F^+$ such that $I(h) \leq \sigma(f) \leq I(k)$ and $I(k) - I(h) < \varepsilon$. Now let $\varepsilon > 0$ be given, and let $n \in \omega$ be such that $\frac{1}{n} < \varepsilon$. Let

$$h = \Delta(f) \oplus \left\{ \bigoplus_{i=0}^{n-1} \frac{i}{n} \left[\Delta\left(\frac{i}{n} \to f\right) \wedge \right.\right.$$

$$\left.\left. \wedge \left(\Delta\left(\frac{i+1}{n} \to f\right)\right)^* \right] \right\}$$

and

$$k = \Delta(f) \oplus \left\{ \bigoplus_{i=0}^{n-1} \frac{i+1}{n} \left[\Delta\left(\frac{i}{n} \to f\right) \wedge \right.\right.$$

$$\left.\left. \wedge \left(\Delta\left(\frac{i+1}{n} \to f\right)\right)^* \right] \right\}.$$

Note that:

(1) If $f(u) = 1$, then $h(u) = k(u) = 1$.

(2) If $\frac{i}{n} \leq f \leq \frac{i+1}{n}$ $(i = 0, \ldots, n-1)$, then $h(u) = \frac{i}{n}$ and $k(u) = \frac{i+1}{n}$. Thus $h \leq f \leq k$, therefore $\sigma(h) \leq \sigma(f) \leq \sigma(k)$. Moreover

$$I(k) - I(h) = \sum_{i=0}^{n-1} \frac{1}{n} \mu_{st}\left(U_{f \in [\frac{i}{n}, \frac{i+1}{n})}\right) = \frac{1}{n} < \varepsilon.$$

In order to get the claim, it suffices to prove that $st(\sigma(h)) = I(h)$ and $st(\sigma(k)) = I(k)$. Now let for $i = 0, \ldots, n-1$, $t_i = (\Delta(\frac{i}{n} \to f) \wedge (\Delta(\frac{i+1}{n} \to f))^*$ and let $t_n = \Delta(f)$. Then for $i, j = 0, \ldots, n$, if $i \neq j$, then $t_i \odot t_j = 0$, (because if $i < n$, then t_i is the characteristic function of $U_{f \in [\frac{i}{n}, \frac{i+1}{n})}$ and t_n is the characteristic function of $U_{f=1}$). Thus by Lemma 3.1, (b) and (c),

$$\sigma(h) = \sum_{i=0}^{n-1} \frac{i}{n} \sigma(t_i) \text{ and } \sigma(k) = \sum_{i=0}^{n-1} \frac{i+1}{n} \sigma(t_i),$$

therefore

$$\begin{aligned} st(\sigma(h)) &= \sum_{i=0}^{n-1} \frac{i}{n}(st(\sigma(t_i))) \\ &= \sum_{i=0}^{n-1} \frac{i}{n}(\mu_{st}\left(U_{f \in [\frac{i}{n}, \frac{i+1}{n})}\right) \\ &= I(h), \end{aligned}$$

and

$$\begin{aligned} st(\sigma(k)) &= \sum_{i=0}^{n-1} \frac{i+1}{n}(st(\sigma(t_i))) \\ &= \sum_{i=0}^{n-1} \frac{i+1}{n}(\mu_{st}\left(U_{f\in[\frac{i}{n},\frac{i+1}{n})}\right) \\ &= I(k). \end{aligned}$$

This ends the proof. ■

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