Embedding Gödel propositional logic into Prior's tense logic

Stefano Aguzzoli
University of Milano,
Milano, Italy
aguzzoli@dsi.unimi.it

Brunella Gerla
University of Insubria,
Varese, Italy
brunella.gerla@uninsubria.it

Vincenzo Marra
University of Milano,
Milano, Italy
marra@dico.unimi.it

Abstract

The well-known Gödel translation embeds intuitionistic propositional logic into the modal logic S4. In this note, we use essentially the same translation to embed Gödel infinite-valued propositional logic into a schematic extension of Prior's bi-modal tense logic that allows finite chains only as flows of time. While our proofs use elementary techniques in many-valued algebraic logic, our embedding is strongly related to well-known results from the theory of modal companions to superintuitionistic logics. For the reader's convenience we include a short discussion of the latter results.

Keywords: Gödel logic, intuitionistic logic, temporal logic, modal companion.

1 Introduction

Throughout, let us fix a countably infinite set $V = \{x_1, x_2, \ldots, x_n, \ldots\}$ of propositional variables. The set $\mathcal{F}_\mathcal{G}$ of formulas of (*propositional*) *Gödel logic* $\mathcal{G}$ [6] is built as usual from V, the constant $\bot$ and the binary connectives $\wedge$ and $\to$.

As a many-valued logic, Gödel logic is the axiomatic extension of Hájek's Basic Fuzzy Logic BL [8] by means of *contraction*: $\varphi \to (\varphi \wedge \varphi)$. By [4], BL is the logic of all continuous t-norms $*$ and their residua. Contraction forces *idempotency* of $*$, that is, $x = x * x$ holds for all $x \in [0, 1]$, the real unit interval. Since there is only one continuous and idempotent t-norm, namely the *minimum* $x \wedge y = \min(x, y)$, one proves that Gödel logic is complete with respect to its *standard* fuzzy semantics that interprets formulas over the structure $\langle [0, 1], \wedge, \to_\wedge, 0 \rangle$, where the residuum $\to_\wedge$ is defined by $x \to_\wedge y = 1$ if and only if $x \leq y$, and $x \to_\wedge y = y$ otherwise. Disjunction is defined as $\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

Gödel logic can also be seen as the axiomatic extension of intuitionistic propositional logic by the *prelinearity* axiom scheme: $(\varphi \to \psi) \vee (\psi \to \varphi)$. Its algebraic semantics is therefore given by the variety of prelinear Heyting algebras, that is, *Gödel algebras*. From the point of view of Kripke semantics, Gödel logic is the logic of finite linearly ordered frames (see *e.g.* [3, Ch. 4 and 5] for background).

To explicitly relate the Kripke and algebraic (or many-valued) semantics for Gödel logic, recall that from any intuitionistic Kripke frame $(W, \leq)$ one constructs a Heyting algebra H by taking the family of upward closed subsets of the set of possible worlds W, endowed with appropriate operations. Then a formula is valid in the frame $(W, \leq)$ if and only if it is valid in the Heyting algebra H (see [3, 7.20]). At the first-order level, things are not as easy — see [2] for an in-depth investigation of the relationship between the Kripke and many-valued semantics for first-order Gödel logics.

The fact that Gödel logic is the logic of lin-

L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, pp. 992–999
Torremolinos (Málaga), June 22–27, 2008

early ordered Kripke frames, or of prelinear Heyting algebras, affords a (folklore) temporal interpretation of its semantics, as follows. Let φ and ψ be formulæ and let μ be a $[0,1]$-assignment. Then $\mu(\varphi) < \mu(\psi)$ can be read as "φ will become true (strictly) after ψ". Propositions that will become true at possibly different times can then be compared and ranked by means of Gödel implication. In §3, by way of a preliminary, we formalize this semantics in elementary terms using bit sequences. We consider assignments to Gödel formulas of sequences of Boolean values indexed by instants of time ranging in the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$. We associate with each formula φ the instant in time when φ first becomes true. Now Gödel implication $\varphi \to \psi$ is true at some instant t if after t, φ will become true not before ψ. Completeness is proved in Theorem 3.10.

Given this temporal interpretation, it is natural to examine the connections between Gödel logic and the established field of *tense logic* in the sense of Prior [9], or, more generally, (poly)modal logics [3].

It is well known that to each superintuitionistic logic[1] one can associate a *modal companion* [3, 9.6] by means of the so-called *Gödel translation* T [3, 3.3]. The latter is defined by

- $\mathsf{T}(x_i) = \Box x_i$.
- $\mathsf{T}(\bot) = \Box\bot$.
- $\mathsf{T}(\varphi \wedge \psi) = \mathsf{T}(\varphi) \wedge \mathsf{T}(\psi)$.
- $\mathsf{T}(\varphi \vee \psi) = \mathsf{T}(\varphi) \vee \mathsf{T}(\psi)$.
- $\mathsf{T}(\varphi \to \psi) = \Box(\mathsf{T}(\varphi) \to \mathsf{T}(\psi))$.

Each superintuitionistic logic L has a family of modal companions, that is, modal logics M such that

$$L \vDash \varphi \quad \text{iff} \quad M \vDash \mathsf{T}(\varphi),$$

for all formulas φ. In fact, L always has a weakest and a strongest such modal companion. It is known that the weakest modal companion of $\mathcal{G}$ is the logic S4.3, defined by adding the axiom scheme

$$\Box(\Box\varphi \to \psi) \vee \Box(\Box\psi \to \varphi)$$

to S4, while the strongest modal companion of $\mathcal{G}$ is given by extending S4.3 with the Grzegorczyk's axiom scheme:

$$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi\,.$$

The modality $\Box$ in S4.3 can be interpreted temporally as *it is true now and it always will be true* (cf. [3, p. 94]) in models where time is linear (and reflexive), while the Grzegorczyk's axiom fails in every model containing infinitely ascending chains. However, it is important to note that the use of the single modality $\Box$ does not exclude non-linear models. For instance, the frame $\langle\{a, b, c\}, \leq\rangle$, where $\leq$ is the reflexive closure of $a < c$ and $b < c$, models S4.3. Analogously, the frame $\langle\mathbb{N}, \geq\rangle$ contains an infinitely descending chain and models the Grzegorczyk's axiom. To overcome such drawbacks we turn to Prior's bimodal *minimal tense logic* K_t [9].

In §4 we consider an appropriate extension of K_t aimed at capturing precisely finite chains.[2] Specifically, we call FL the (finitely axiomatizable) extension of K_t that allows finite linear flows of time only as domains of interpretation – for details, please see §4. In Theorem 5.7 we construct a faithful embedding of Gödel logic into FL. Gödel formulas are syntactically translated into temporal formulas having the property that the set of instants in which they are true is upward closed, in close analogy with the elementary construction of §3.

2 Background on Gödel logic

The logic $\mathcal{G}$ is axiomatized by extending the system BL given in [8, 4] with the *contraction* axiom scheme: $\varphi \to (\varphi \wedge \varphi)$.

Each (fuzzy) assignment $v\colon V \to [0,1]$ to the propositional variables canonically extends to

[1] That is, a schematic extension of intuitionistic logic.

[2] Actually, disjoint unions of finite chains since if T_1 and T_2 are two models of a (poly)modal logic L then their disjoint union is a model of L, too.

$\mathcal{F}_\mathcal{G}$ as follows:

$$v(\bot) = 0 \quad (1)$$
$$v(\varphi \wedge \psi) = \min(v(\varphi), v(\psi)) \quad (2)$$
$$v(\varphi \to \psi) = \begin{cases} 1 & \text{if } v(\varphi) \leq v(\psi) \\ v(\psi) & \text{otherwise.} \end{cases} \quad (3)$$

A formula φ is *satisfied* by an assignment v if $v(\varphi) = 1$. A formula φ is a (*standard*) *tautology* of $\mathcal{G}$ if it is satisfied by all assignments. We write $\mathcal{G} \models \varphi$ to denote that φ is a (standard) tautology.

Usual derived connectives are $\neg\varphi := \varphi \to \bot$, $\top := \neg\bot$, $\varphi \vee \psi := ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$. Their standard interpretations turn out to be:

$$v(\neg\varphi) = \begin{cases} 1 & \text{if } v(\varphi) = 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$v(\varphi \vee \psi) = \max(v(\varphi), v(\psi))$ and $v(\top) = 1$.

An easy induction on the structure of formulas proves

Lemma 2.1 *For any assignment v and any formula $\varphi \in \mathcal{F}_\mathcal{G}$ with variables among $\{x_1, \ldots, x_n\}$, $v(\varphi) \in \{0, v(x_1), \ldots, v(x_n), 1\}$.*

A *partition* of a set A is a family $\{A_i\}_{i \in I}$ of nonempty subsets $A_i \subseteq A$ (called the *blocks* of the partition), such that $A = \bigcup_{i \in I} A_i$ and for each $i, j \in I$, $i \neq j$ implies $A_i \cap A_j = \emptyset$.

As a technical tool, we shall use ordered partitions (cf. the theory of chain normal forms in [1, §3.1] and the combinatorial analysis of coproducts in [5]). An *ordered partition* $\langle \{A_i\}_{i \in I}, \leq \rangle$ of a set A, is a partition $\{A_i\}_{i \in I}$ of A together with a total order relation on the set of blocks $\{A_i\}_{i \in I}$. If the index set I is finite, say $|I| = n$, the ordered partition $\langle \{A_i\}_{i \in I}, \leq \rangle$ will be displayed as $A_1 < A_2 < \cdots < A_n$.

By a *Gödel ordered partition* of $\{x_1, \ldots, x_n\}$ we mean an ordered partition of the set $\{0, x_1, \ldots, x_n, 1\}$ with the property that 0 belongs to the first block of the partition, and 1 belongs to the last one. We write Ord_n for the set of Gödel ordered partitions of $\{x_1, \ldots, x_n\}$.

Example 2.2 The following are some elements of Ord_2.

$\{0\} < \{x_2\} < \{x_1\} < \{1\}$,
$\{0\} < \{x_1, x_2\} < \{1\}$,
$\{0, x_1\} < \{x_2, 1\}$.

As mentioned in the introduction, a *Gödel algebra* is a Heyting algebra satisfying the prelinearity law $(x \to y) \vee (y \to x) = 1$ (or, equivalently, a BL-algebra satisfying the law of idempotency $x = x * x$). The standard algebra $[0, 1]$ with operations defined as in (1), (2), (3) is the main example of a Gödel algebra. Let $\mathcal{A} = \langle A, \wedge, \to, 0 \rangle$ be a Gödel algebra. Following standard usage, for any Gödel formula φ with variables among $x_1, \ldots, x_n$ and for any $a_1, \ldots, a_n \in A$, we denote by $\varphi^\mathcal{A}(a_1, \ldots, a_n)$ the element of $\mathcal{A}$ obtained by interpreting each x_i by a_i, and each connective by the corresponding operation of $\mathcal{A}$.

The standard completeness theorem for $\mathcal{G}$ can now be stated; for a proof see [8].

Theorem 2.3 *For any formula $\varphi \in \mathcal{F}_\mathcal{G}$ in the variables $x_1, \ldots, x_n$, the following are equivalent:*

(i) *φ is a theorem of $\mathcal{G}$.*

(ii) *$\varphi^\mathcal{A}(a_1, \ldots, a_n) = 1^\mathcal{A}$, for every Gödel algebra $\mathcal{A}$ and elements $a_1, \ldots, a_n \in A$.*

(iii) *φ is a standard tautology of $\mathcal{G}$.*

Given a Gödel algebra $\mathcal{A} = \langle A, \wedge, \to, 0 \rangle$ we denote by $L(\mathcal{A}) = \langle A, \wedge, \vee, 0, 1 \rangle$ its lattice reduct. The proof of the following theorem can be found in [8].

Theorem 2.4 *The order completely determines the structure of a Gödel algebra. That is:*

(i) *For every pair of Gödel algebras $\mathcal{A}$ and $\mathcal{B}$, $L(\mathcal{A})$ is isomorphic to $L(\mathcal{B})$ (as bounded lattices) if and only if $\mathcal{A}$ is isomorphic to $\mathcal{B}$ (as Gödel algebras). Each chain is the lattice reduct of a unique Gödel algebra.*

(ii) *For all integers $n \geq 1$, any two linearly ordered Gödel algebras with n elements are isomorphic.*

(iii) *Let φ be a formula of Gödel logic in the variables $\{x_1, \ldots, x_n\}$. Then $\mathcal{G} \models \varphi$ if and only if $\varphi^{\mathcal{A}}(a_1, \ldots, a_n) = 1^{\mathcal{A}}$ for all $a_1, \ldots, a_n$ in every linearly ordered Gödel algebra $\mathcal{A}$ with at most $n + 2$ elements.*

In particular, each Gödel ordered partition of $\{x_1, \ldots, x_n\}$ can be equipped uniquely with the structure of a Gödel chain having as bottom element the block containing 0 and as top element the block containing 1. Viceversa, each Gödel chain generated by $\{x_1, \ldots, x_n\}$ is isomorphic to a Gödel chain whose universe is a Gödel ordered partition of $\{x_1, \ldots, x_n\}$. For the rest of this paper, we tacitly endow Gödel ordered partitions with their unique structure of Gödel algebra whenever needed.

It is now easy to prove the following.

Theorem 2.5 *Let the variables occurring in $\varphi \in \mathcal{F}_{\mathcal{G}}$ be in $\{x_1, \ldots, x_n\}$. Then $\mathcal{G} \models \varphi$ if and only if $\varphi^P(P(x_1), \ldots, P(x_n)) = 1^P$ for every ordered partition $P \in Ord_n$, where $P(x_i)$ is the block of P containing x_i.*

3 The logic of non-decreasing bit sequences

Consider the set of classical propositional formulas built on the set of variables V, with connectives $\wedge$, $\vee$, $\to$, $\perp$. Let us denote by $\prec$ a new binary connective, and let $\mathcal{F}_{\mathcal{T}}$ be the set of formulas built from V using only the connectives $\vee, \wedge, \perp$ and $\prec$.

Definition 3.1 A *non-decreasing temporal assignment* (*NDT-assignment,* for short) is any function $v\colon V \times \mathbb{N} \to \{0, 1\}$ such that for any $x \in V$ the map $t \mapsto v(x, t)$ is *non-decreasing*, that is, if $t_1 \leq t_2$ then $v(x, t_1) \leq v(x, t_2)$.

Each $t \in \mathbb{N}$ is called an *instant.*

Definition 3.2 We extend NDT-assignments $v : V \times \mathbb{N} \to \{0, 1\}$ to functions $v\colon \mathcal{F}_{\mathcal{T}} \times \mathbb{N} \to \{0, 1\}$, as follows. For each instant $t \in \mathbb{N}$, let

- $v(\varphi \vee \psi, t) = \max(v(\varphi, t), v(\psi, t))$.
- $v(\varphi \wedge \psi, t) = \min(v(\varphi, t), v(\psi, t))$.
- $v(\perp, t) = 0$.
- $v(\varphi \prec \psi, t) = 1$ if $v(\varphi, s) \leq v(\psi, s)$ for all $s \geq t$.

It is easy to check that the following holds.

Lemma 3.3 *For any formula $\varphi \in \mathcal{F}_{\mathcal{T}}$, the map $t \mapsto v(\varphi, t)$ is non-decreasing.*

Remark 3.4 Notice that, while the interpretation of connectives $\wedge$ and $\vee$ is instant-wise Boolean, the interpretation of $\prec$ is not.

Fixing an instant t, in order to interpret the Boolean connective $\to$ let us extend the NDT-assignment v as $v(\varphi \to \psi, t) = 1$ iff $v(\varphi, t) \leq v(\psi, t)$. Then, in general, $v(\varphi \to \psi, t) \neq v(\varphi \prec \psi, t)$. Indeed, let $v(x_1, t) = 1$ iff $t > 10$ and $v(x_2, t) = 1$ iff $t > 5$. Then $v(x_2 \prec x_1, t) = 1$ iff $t > 10$, while the function $w(t) = v(x_2 \to x_1, t)$ is not even non-decreasing, as it evaluates to 1 iff $t \leq 5$ or $t > 10$.

Let $\mathcal{T}$ be the logical system of propositional formulas in $\mathcal{F}_{\mathcal{T}}$ specified as follows:

Definition 3.5 A formula $\varphi \in \mathcal{F}_{\mathcal{T}}$ is *valid* in $\mathcal{T}$ iff $v(\varphi, t) = 1$ for every NDT-assignment v and any instant $t \in \mathbb{N}$. The formula φ *becomes eventually satisfied by v at instant t_0* if

$$v(\varphi, t) = 1 \quad \text{iff} \quad t \geq t_0 \,.$$

The formula φ is *always satisfied by v* if it becomes eventually satisfied by v at instant 0 (hence valid formulas are exactly those always satisfied by every NDT-assignment). We write $\mathcal{T} \models \varphi$ to denote that φ is valid in $\mathcal{T}$.

Lemma 3.6 *The formula $\varphi \prec \psi \in \mathcal{F}_{\mathcal{T}}$ is always satisfied by the NDT-assignment v iff $v(\varphi, t) < v(\psi, t)$ for every $t \in T$. Further, the formula $\varphi \prec \psi$ becomes eventually satisfied by v at the instant t iff ψ becomes eventually satisfied at the instant t and φ becomes eventually satisfied at some instant $t' < t$.*

Proof. The first statement follows from Definition 3.2. Assume φ and ψ become eventually satisfied at times t' and t, respectively, with $t' < t$. Defining $v(\varphi \to \psi, s)$ as in Remark 3.4, we have $v(\varphi \to \psi, s) = 0$ for all s such that $t' \leq s < t$, while $v(\varphi \to \psi, s) = 1$ for all $s \geq t$. Hence, by Definition 3.2, $v(\varphi \prec \psi, s) = 0$ for all $s < t$ and $v(\varphi \prec \psi, t) = 1$, that is, $\varphi \prec \psi$ becomes eventually satisfied at t. On the other hand, $v(\varphi \prec \psi, r) = 1$ iff $r \geq t$. Then, for all $s < t$ there is $s \leq u < t$ such that $v(\varphi \to \psi, u) = 0$, that is, $v(\varphi, u) = 1$ and $v(\psi, u) = 0$. Since $v(\varphi \to \psi, t) = 1$, we have $v(\varphi, t) = v(\psi, t) = 1$. We conclude that ψ becomes eventually satisfied at t and φ becomes eventually satisfied at some $t' < t$. ■

Let $\varphi \in \mathcal{F}_\mathcal{T}$ be a formula over the variables $x_1, \ldots, x_n$, and let v be an NDT-assignment. We denote by t^v_φ the instant, if it exists, when φ becomes eventually satisfied by v. Otherwise, if φ never becomes eventually satisfied by v, we set $t^v_\varphi = \infty$.

For any NDT-assignment v, endow the set $W^v_n = \{t^v_{x_1}, \ldots, t^v_{x_n}\} \cup \{0, \infty\} \subset \mathbb{N} \cup \{\infty\}$ with the natural order, where $t \leq \infty$ for all t. Consider the reverse linear order $\langle W^v_n, \geq \rangle$, and let $x \sqcup y = \min\{x, y\}$ and $x \sqcap y = \max\{x, y\}$. Then, by Theorem 2.4, $\langle W^v_n, \sqcap, \sqcup, \infty, 0\rangle$ is the lattice reduct of a uniquely determined finite Gödel algebra $\mathcal{W}^v_n = \langle W^v_n, \sqcap, \Rightarrow, \infty \rangle$ with 0 as maximum and ∞ as minimum element.

For any formula $\varphi \in \mathcal{F}_\mathcal{G}$, let $\overline{\varphi} \in \mathcal{F}_\mathcal{T}$ be the formula obtained by replacing every occurrence of $\to$ with $\prec$.

Lemma 3.7 *For any NDT-assignment v and for any formula $\varphi \in \mathcal{F}_\mathcal{G}$ whose variables are in $\{x_1, \ldots, x_n\}$,*

$$t^v_{\overline{\varphi}} = \varphi^{\mathcal{W}^v_n}(t^v_{x_1}, \ldots, t^v_{x_n}) \,.$$

Proof. By structural induction on the formula $\varphi \in \mathcal{F}_\mathcal{G}$. If φ is a variable or $\perp$ there is nothing to prove. Suppose $\varphi = \varphi_1 \wedge \varphi_2$. Then $t^v_{\overline{\varphi}} = \max(t^v_{\overline{\varphi_1}}, t^v_{\overline{\varphi_2}})$, and $t^v_{\overline{\varphi}} = \varphi^{\mathcal{W}^v_n}$ by the induction hypothesis.

Suppose now that $\varphi = \varphi_1 \to \varphi_2$. We need to prove that

$$t^v_{\overline{\varphi}} = t^v_{\overline{\varphi_1}} \Rightarrow t^v_{\overline{\varphi_2}} = \begin{cases} 0 & \text{if } t^v_{\overline{\varphi_2}} \leq t^v_{\overline{\varphi_1}} \\ t^v_{\overline{\varphi_2}} & \text{otherwise.} \end{cases}$$

But this follows at once from Lemma 3.6. ■

Definition 3.8 Let $\sim_n$ be the binary relation on NDT-assignments such that for every assignment v and w, $v \sim_n w$ if $\mathcal{W}^v_n$ and $\mathcal{W}^w_n$ are isomorphic Gödel chains via the map

$$t^v_{x_i} \in W^v_n \mapsto t^w_{x_i} \in W^w_n \,.$$

The relation $\sim_n$ is easily shown to be an equivalence relation. Denote by J_n the set of equivalence classes of NDT-assignments defined over variables $x_1, \ldots, x_n$.

Lemma 3.9 *There is a bijection*

$$[v]_{\sim_n} \in J_n \mapsto P_v \in Ord_n$$

such that for any $[v]_{\sim_n} \in J_n$, P_v is isomorphic to $\mathcal{W}^v_n$ as Gödel algebras.

Proof. For any NDT-assignment v there is a permutation σ of $\{1, \ldots, n\}$ such that $\mathcal{W}^v_n = \{0 \preceq_0 t^v_{x_{\sigma(1)}} \preceq_1 \cdots \preceq_{n-1} t^v_{x_{\sigma(n)}} \preceq_n \infty\}$ where each $\preceq_i$ is either $<$ or $=$. The Gödel ordered partition

$$P_v = \{0 \preceq_n x_{\sigma(n)} \preceq_{n-1} \cdots \preceq_1 x_{\sigma(1)} \preceq_0 1\}$$

is isomorphic to $\mathcal{W}^v_n$ as Gödel algebras by Theorem 2.4(ii). In plain words, variables in the same block of the partition P_v become eventually valid under v at the same instant.

By Definition 3.8, the map $\phi : [v]_{\sim_n} \in J_n \mapsto P_v \in Ord_n$ is well-defined and injective. In order to show that ϕ is surjective, let $P = W_0 < \cdots < W_m$ be any Gödel ordered partition of $\{x_1, \ldots, x_n\}$, and consider instants $0 < t_1 < \cdots < t_{m-1} \in \mathbb{N}$ and the NDT-assignment v such that, if $x_i \in W_j$ with $0 < j < m$, then $v(x_i, t) = 1$ if and only if $t \geq t_{m-j}$, while if $x_i \in W_0$ then $v(x_i, t) = 0$ for every $t \in \mathbb{N}$, and if $x_i \in W_m$ then $v(x_i, t) = 1$ for every $t \in \mathbb{N}$. Hence, if $x_i \in W_j$ with $0 < j < m$ we have $t^v_{x_i} = t_{m-j}$, while if $x_i \in W_0$, $t^v_{x_i} = \infty$ and if $x_i \in W_m$, $t^v_{x_i} = 0$. Observe that:

$$\mathcal{W}^v_n = \{\infty > t_{m-1} > t_{m-2} > \cdots > t_1 > 0\} \,.$$

By definition, $P = P_v$ and hence $\phi([v]_{\sim_n}) = P$. ■

Theorem 3.10 *For any formula $\varphi \in \mathcal{F}_{\mathcal{G}}$,*

$$\mathcal{G} \models \varphi \qquad \textit{iff} \qquad \mathcal{T} \models \overline{\varphi}\,.$$

Proof. We have $\mathcal{T} \models \overline{\varphi}$ if and only if for every NDT-assignment v, $t^v_{\overline{\varphi}} = 0$ if and only if (Lemma 3.7) $\varphi^{\mathcal{W}^v_n}(t^v_{x_1}, \ldots, t^v_{x_n}) = 0$ if and only if (Lemma 3.9) for every $P \in Ord_n$, $\varphi^P(P(x_1), \ldots, P(x_n)) = 1^P$. By Theorem 2.5 this is equivalent to $\mathcal{G} \models \varphi$. ∎

4 Prior's tense logic K_t and its extension FL

For background on temporal logics see [9, 7, 10]. Consider again the set of variables V. The set $\mathcal{F}_{K_t}$ of formulas of Prior's minimal tense logic K_t is built from V, from classical propositional connectives $\bot, \wedge, \to$, and from two new unary connectives G and H. The semantics of minimal tense logic is obtained by fixing a *flow of time*, that is, a set T together with a strict order (i.e. irreflexive and transitive) relation $<$, and then considering *temporal assignments*, that is, arbitrary maps $v: V \times T \to \{0, 1\}$.

Definition 4.1 Each temporal assignment v is extended to all formulas of K_t in the following way. For every $t \in T$, let

- $v(\bot, t) = 0$.
- $v(\psi_1 \wedge \psi_2, t) = \min(v(\psi_1, t), v(\psi_2, t))$.
- $v(\psi_1 \to \psi_2, t) = 1$ if and only if $v(\psi_1, t) \leq v(\psi_2, t))$.
- $v(G\varphi, t) = 1$ if and only if for every t' with $t < t'$, $v(\varphi, t') = 1$.
- $v(H\varphi, t) = 1$ if and only if for every $t' < t$, $v(\varphi, t') = 1$.

Propositional connectives $\top, \vee, \neg$ are defined as usual. Further, the defined unary connectives F and P given by

$$F\varphi := \neg G \neg \varphi\,, \qquad P\varphi := \neg H \neg \varphi$$

are such that $v(F\varphi, t) = 1$ if and only if there exists $t' > t$ such that $v(\varphi, t') = 1$, while $v(P\varphi, t) = 1$ if and only if there exists $t' < t$ such that $v(\varphi, t') = 1$.

Intuitively, the formula $G\varphi$ is true at instant t if φ is always true in the future of t, while $F\varphi$ is true at t if φ is true at some instant in the future of t. Analogously, the formula $H\varphi$ is true at instant t if φ is always true in the past of t, while $P\varphi$ is true at t if φ is true at some instant in the past of t.

Definition 4.2 A formula φ is *valid on a flow of time T* if for every temporal assignment v on T and for every $t \in T$, $v(\varphi, t) = 1$.

The axioms of K_t are those of Boolean propositional logic plus the following axiom schemata:

$$G(\varphi \to \psi) \to (G\varphi \to G\psi);$$

$$H(\varphi \to \psi) \to (H\varphi \to H\psi);$$

$$\varphi \to GP\varphi;$$

$$\varphi \to HF\varphi;$$

$$G\varphi \to GG\varphi.$$

Deduction rules are modus ponens plus the following *necessitation* rules of inference:

$$\frac{\varphi}{G\varphi}\,, \qquad \frac{\varphi}{H\varphi}\,.$$

Minimal tense logic is sound and complete with respect to *all* flows of time. We wish to strengthen K_t so as to capture *finite linear* flows of time. Let FL be the schematic extension of K_t by

(LIN1) $PF\varphi \to (P\varphi \vee \varphi \vee F\varphi)$,

(LIN2) $FP\varphi \to (F\varphi \vee \varphi \vee P\varphi)$,

(FIN1) $F\varphi \to F(\varphi \wedge G\neg\varphi)$,

(FIN2) $P\varphi \to P(\varphi \wedge H\neg\varphi)$.

Observe that (LIN1) forces flows of time to be non-branching to the future. Analogously, (LIN2) forces flows of time to be non-branching to the past. Axioms (FIN1) and (FIN2), also known as *Löb's axioms*, force flows of time to be finite.[3] Then, a variant of the completeness proof in [7, 3.4.1] yields:

Theorem 4.3 *A formula φ is valid in all finite linear flows of time, written* $\mathsf{FL} \vDash \varphi$*, if and only if φ is derivable from the axioms of* FL *using modus ponens and necessitation. Further, a flow of time T validates all formulas derivable in* FL *if and only if T is a (disjoint union of) finite chain(s).*

5 Translation of Gödel logic into FL

In this section we shall always assume that T denotes a finite linearly ordered set. Each $t \in T$ is called an *instant*. For any instant $t \in T$, we denote by $\uparrow t$ the set $\{s \in T \mid s \geq t\}$.

Definition 5.1 For every formula φ and temporal assignment v on T we let

$$(\varphi)^v = \{t \in T \mid v(\varphi, t) = 1\} \subseteq T.$$

A formula φ is (*weakly*) *increasing* with respect to v (*v-increasing*, for short) if whenever $t \in (\varphi)^v$ then $\uparrow t \subseteq (\varphi)^v$.

Lemma 5.2 *For any temporal assignment v on T and for any formula φ, $\varphi \wedge G\varphi$ is v-increasing. Further, if φ is v-increasing then $(\varphi)^v = (\varphi \wedge G\varphi)^v$.*

Proof. By definition, $(\varphi \wedge G\varphi)^v = \{t \in T \mid v(\varphi \wedge G\varphi, t) = 1\} = \{t \in T \mid v(\varphi, t) = 1 \text{ and for all } s > t, v(\varphi, s) = 1\} = \{t \in T \mid v(\varphi, s) = 1 \text{ for every } s \geq t\}$. If $t \in (\varphi \wedge G\varphi)^v$ then $\uparrow t \subseteq (\varphi \wedge G\varphi)^v$ and so $\varphi \wedge G\varphi$ is v-increasing.

Note that in general $(\varphi \wedge G\varphi)^v \subseteq (\varphi)^v$. If φ is v-increasing and $t \in (\varphi)^v$ then $\uparrow t \in (\varphi)^v$, hence $t \in (\varphi \wedge G\varphi)^v$ and $(\varphi)^v \subseteq (\varphi \wedge G\varphi)^v$. ■

Example 5.3 Let v be the temporal assignment on $T = \{0, 1, 2, 3, 4, 5, 6, 7\}$ such that $v(x_1, t) = 1$ if and only if $t \geq 3$, while $v(x_2, t) = 1$ if and only if $t \leq 1$ or $t \geq 5$ and $v(x_3, t) = 1$ if and only if t is an even number. Then $v(x_1 \wedge Gx_1, t) = v(x_1, t)$ for every $t \in T$, while $(x_2 \wedge Gx_2)^v = \{t \in T \mid t \geq 5\}$ and $v(x_3 \wedge Gx_3, t) = 0$ for every $t \in T$.

We translate a Gödel formula φ into a formula $\widehat{\varphi}$ of $\mathcal{F}_{K_t}$ in the following way:

- If $\varphi = \bot$ then $\widehat{\varphi} = \bot$
- If $\varphi = x_i$ then $\widehat{\varphi} = x_i \wedge Gx_i$
- If $\varphi = \varphi_1 \wedge \varphi_2$ then $\widehat{\varphi} = \widehat{\varphi_1} \wedge \widehat{\varphi_2}$
- If $\varphi = \varphi_1 \rightarrow \varphi_2$ then $\widehat{\varphi} = (\widehat{\varphi_1} \rightarrow \widehat{\varphi_2}) \wedge G(\widehat{\varphi_1} \rightarrow \widehat{\varphi_2})$.

This is essentially the translation used in [3, 3.89], itself a variant of the Gödel translation.

Let v be a temporal assignment on T, and let V_n^v be defined as:

$$V_n^v = \{(\widehat{x}_1)^v, \ldots, (\widehat{x}_n)^v\} \cup \{\emptyset, T\} \subseteq 2^T.$$

Endow V_n^v with the order given by inclusion. Since all $\widehat{x}_i$ are v-increasing by Lemma 5.2, the order on V_n^v is total. By Theorem 2.4(i), $\langle V_n^v, \cap, \cup, \emptyset, T\rangle$ is the lattice reduct of a uniquely determined finite Gödel chain $\mathcal{V}_n^v$ with $\emptyset$ as minimum and T as maximum element, respectively.

Lemma 5.4 *Let v be a temporal assignment on T. Then for any $\varphi \in \mathcal{F}_{\mathcal{G}}$,*

$$(\widehat{\varphi})^v = \varphi^{\mathcal{V}_n^v}((\widehat{x}_1)^v, \ldots, (\widehat{x}_n)^v) \in V_n^v.$$

Proof. The proof follows by an easy induction, using Lemma 5.2. ■

Example 5.5 Let v be as in Example 5.3. Then, using the notation of ordered partitions,

$$V_3^v = \{\{\emptyset, (\widehat{x}_3)^v\} < \{(\widehat{x}_2)^v\} < \{(\widehat{x}_1)^v\} < T\}.$$

If $\varphi = (x_1 \rightarrow x_2) \vee x_3$, then $\widehat{\varphi} = ((\widehat{x}_1 \rightarrow \widehat{x}_2) \wedge G(\widehat{x}_1 \rightarrow \widehat{x}_2)) \vee \widehat{x}_3$ and $(\widehat{\varphi})^v = (\widehat{x}_2)^v \in V_3^v$.

Lemma 5.6 *For any flow of time T with at least $n+2$ elements and for any temporal assignment v on T, there is an NDT-assignment $v' : \mathcal{F}_{\mathcal{T}} \times \mathbb{N} \to \{0, 1\}$ such that the Gödel chains $\mathcal{V}_n^v$ and $\mathcal{W}_n^{v'}$ are isomorphic via the map $\Theta \colon V_n^v \to W_n^{v'}$ defined by $\Theta(\emptyset) = \infty$, $\Theta(T) = 0$ and $\Theta((\widehat{x}_i)^v) = t_{x_i}^{v'}$.*

[3] Note that (FIN1) fails in infinite ascending chains, while (FIN2) fails in infinite descending chains. It follows that Löb's axioms force irreflexivity, too.

Proof. Let $v\colon \mathcal{F}_{K_t} \times T \to \{0,1\}$ be a temporal assignment and list the elements of V_n^v as $\{\emptyset \preceq_0 (\widehat{x}_{\sigma(1)})^v \preceq_1 \cdots \preceq_{n-1} (\widehat{x}_{\sigma(n)})^v \preceq_n T\}$ where σ is a permutation of $\{1,\ldots,n\}$ and each $\preceq_i$ is either $<$ or $=$.

Let $0 \preceq_n t_n \preceq_{n-1} \cdots \preceq_1 t_1 \preceq_0 \infty \in \mathbb{N} \cup \{\infty\}$. We set, for every $m \in \mathbb{N}$, $v'(x_{\sigma(i)}, m) = 1$ if and only if $m \geq t_i$.

Then it is immediate to check that v' is an NDT-assignment and $t_i = t^{v'}_{x_{\sigma(i)}}$. An easy induction shows that for any $\varphi \in \mathcal{F}_{\mathcal{G}}$, $\Theta((\widehat{\varphi})^v) = t^{v'}_{\overline{\varphi}}$. The proof follows by Lemmas 3.7 and 5.4. ■

Theorem 5.7 *For any formula $\varphi \in \mathcal{F}_{\mathcal{G}}$,*

$$\mathcal{G} \models \varphi \qquad \text{if and only if} \qquad \mathsf{FL} \models \widehat{\varphi}\,.$$

Proof. As a consequence of Lemmas 5.6 and 5.4, for any $\varphi \in \mathcal{F}_{\mathcal{G}}$ and temporal assignment v, we have $\Theta((\widehat{\varphi})^v) = t^{v'}_{\overline{\varphi}}$. We prove that $\mathsf{FL} \models \widehat{\varphi}$ if and only if $\mathcal{T} \models \overline{\varphi}$. The claim then follows by Theorem 3.10. Indeed, suppose $\mathcal{T} \models \overline{\varphi}$. Then for every temporal assignment v, letting v' be the NDT-assignment of Lemma 5.6, we have $v'(\overline{\varphi}, m) = 1$ for every $m \in \mathbb{N}$ hence $t^{v'}_{\overline{\varphi}} = 0$ and applying Θ, $(\widehat{\varphi})^v = T$. Then $v(\widehat{\varphi}, t) = 1$ for every $t \in T$ and $\mathsf{FL} \models \widehat{\varphi}$.

On the other hand, if $\mathcal{T} \not\models \overline{\varphi}$ then there is an NDT-assignment v' and $t_0 \in \mathbb{N}$ such that $v'(\overline{\varphi}, t_0) = 0$. Consider the flow of time $T = \{0, t^{v'}_{x_1}, \ldots, t^{v'}_{x_n}, \infty\} \subseteq \mathbb{N} \cup \{\infty\}$. Then the temporal assignment $v\colon \mathcal{F}_{K_t} \times T \to \{0,1\}$ defined by $v(x_i, \infty) = 0$ and $v(x_i, t) = v'(x_i, t)$ for any $t \in \mathbb{N}$, is such that, by Lemma 5.2, $v(\widehat{\varphi}, t_0) = v'(\overline{\varphi}, t_0) = 0$. Hence $\mathsf{FL} \not\models \widehat{\varphi}$. ■

References

[1] Baaz, M. and Veith, H. (1999). Interpolation in fuzzy logic. *Arch. Math. Logic 38*, 461–489.

[2] Beckmann, A. and Preining, N. (2007). Linear Kripke frames and Gödel logics. *J. Symbolic Logic 72*, 26–44.

[3] Chagrov, A. and Zakharyaschev, M. (1997). *Modal Logic.* Oxford Logic Guides, *35.* Clarendon Press, Oxford.

[4] Cignoli, R., Esteva, F., Godo, L., and Torrens, A. (2000). Basic Fuzzy Logic is the logic of continuous t-norms and their residua. *Soft Computing 4*, 106–112.

[5] D'Antona, O. and Marra, V. (2006). Computing coproducts of finitely presented Gödel algebras. *Ann. Pure and Appl. Logic 142*, 202–211.

[6] Dummett, M. (1959). A propositional calculus with denumerable matrix. *J. of Symbolic Logic 24*, 97–106.

[7] Gabbay, D., Hodkinson, I. and Reynolds, M. (1994). *Temporal Logic. Mathematical Foundations and Computational Aspects, Volume 1.* Oxford Logic Guides, *28.* Clarendon Press, Oxford.

[8] Hájek, P. (1998). *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht.

[9] Prior, A. (1957). *Time and Modality.* Clarendon Press, Oxford.

[10] Venema, Y. (2001). *Temporal Logic.* The Blackwell guide to philosophical logic. Blackwell, Oxford, 203–223.