Credibilistic Equilibria in Extensive Game with Fuzzy Payoffs

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Abstract

This paper originally considers the finite extensive game with fuzzy payoffs. Three credibilistic approaches are introduced to define the behaviors of players in different decision situations. Accordingly, three types of credibilistic equilibria for the fuzzy extensive game are proposed. Moreover, theorems are given to confirm the existence of these new equilibria in fuzzy extensive game. At the end of this paper, two examples are given to demonstrate the importance of these new concepts.

Keywords fuzzy variable, credibility measure, extensive game, credibilistic equilibrium

1 Introduction

Game theory is a collection of mathematical models studying behaviors of people with interest conflict. Modern game theory dated from 1944 with the publication of Theory of Games and Economic Behavior by von Neumann and Morgensern [17]. The development of game theory was accelerated by Nash [12][13], Kuhn [3][4], Shapley [16] and Harsanyi [2] etc.

The theoretic game models are divided into three categories: the extensive games, the strategic games and the coalitional games. In this paper, we consider games in extensive form, which most completely describe the interactions between players in the game. The extensive form also clearly shows the various sets of information and actions for each player at every stage. The notion of extensive form was first introduced by von Neumann and Morgenstern [17] and then Kuhn [3][4] gave a more geometric definition that is popular nowadays. The theorem of Zermulo-von Neumann [17] demonstrated the existence of pure Nash equilibrium in zero-sum two-person game with perfect information. Kuhn [4] extended the result to general-sum n-person game with perfect information.

Given an extensive game, we wish to find equilibria in the game by comparing different outcomes to each strategy profile, hence, we need to know the payoff functions for every player. The equilibrium is a strategies portfolio which is assigned to players and maximizes the payoff for each player when others insist on their strategies. Traditionally, the payoffs are assumed to be deterministic and can be found by collecting and analyzing the data from analogous games played before in some circumstances. However, players in real games are often lack of statistics to clarify the precise relationship between payoffs and different combinations of strategies. Furthermore, the choices of each player or the decision stages of the game may be numerous. As a result, using similar procedures may be costly or even impossible. In such situations, fuzzy set theory provides a trustworthy and effective alternative. With it, we can make use of human experiences, personal decisions and intuitions to counterbalance the deficiency of data. In

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literature, many researchers have considered strategic games in fuzzy environment. For instance, Campos, Gonzalez and Vila [1] used linear programming to solve the fuzzy matrix games. Maeda [11] defined the Nash equilibrium in bi-matrix games with fuzzy payoffs. Nishizaki and Sakawa [14] have considered three new minimax equilibrium strategies and their properties in fuzzy matrix games. But as far as we know, all the papers about extensive games just focused on deterministic payoffs.

In this paper, we originally consider the finite extensive game with fuzzy payoffs and solve it with credibility theory, which was founded and refined by Liu [6][7] as a branch of mathematics for studying the behavior of fuzzy phenomena. For different decision situations, three credibilistic approaches are adopted to describe behaviors of variant types of players properly. We give three new definitions of Nash equilibrium in fuzzy extensive game, i.e., expected credibilistic equilibrium (ECE) for players with mean value point, $\overline{\alpha}-$credibilistic equilibrium ($\overline{\alpha}-$CE) for risk averse players and $\underline{\alpha}-$credibilistic equilibrium ($\underline{\alpha}-$CE) for risk love players. Since much of the work on extensive game is related to equilibria's properties, we also prove the existences of the credibilistic equilibria in the finite fuzzy extensive game. Lastly, we use two examples to show that these new equilibria are necessary and practical in fuzzy extensive game and each of them captures particular characteristic of equilibria in the game.

This paper is arranged as follows. In section 2, we recall the basic concepts of extensive game and credibility theory. Then in section 3, we introduce credibilistic equilibria as well as their existence theorems for finite extensive game with fuzzy payoffs (FEGF). At the end of this paper, two examples are given to illustrate the importance of our new definitions.

2 Preliminaries

2.1 Extensive Game

The usual tree structure of extensive game is given by Kuhn [3][4] and our work is based on finite extensive games with complete information and chance moves. In the following parts of this paper, we adopt the notions as Osborne and Rubinstein [15].

Definition 2.1 *A finite extensive game with perfect information and chance moves is a tuple $\langle N, H, P, f_c, (\succeq_i)\rangle$ consisting of the following components.*

- *A set N (the set of players).*
- *A set H of finite consequences that satisfies the following two properties.*
  - *The empty sequence $\emptyset$ is a member of H.*
  - *If $(a^k)_{k=1,\ldots,K} \in H$ and $L < K$ then $(a^k)_{k=1,\ldots,L} \in H$*

  A history $(a^k)_{k=1,\ldots,K} \in H$ is terminal if there is no a^{K+1} such that $(a^k)_{k=1,\ldots,K+1} \in H$. The set of terminal histories is denoted Z.
- *P is a function from the nonterminal histories in H to $N \cup \{c\}$. (If $P(h) = c$ then chance determines the action taken after the history h.)*
- *For each $h \in H$ with $P(h) = c$, $f_c(\cdot|h)$ is a probability measure on the action set of history h; each such probability measure is assumed to be independent of every other such measure.*
- *For each player $i \in N$, $\succeq_i$ is a preference relation on lotteries over the set of terminal histories.*

We define the outcome $O(s)$ of strategy profile $s = (s_i)_{i\in N}$ to be the terminal history which occurs when every player follows his strategy s_i, then Nash equilibrium in extensive games is defined as follows.

Definition 2.2 *A Nash equilibrium of an extensive game with perfect information and chance moves $\langle N, H, P, f_c, (\succeq_i)\rangle$ is a strategy profile s^* such that for every player $i \in N$ we have $O(s^*_{-i},\ s^*_i) \succeq_i O(s^*_{-i},\ s_i)$, for every strategy s_i of player i.*

The next lemma confirms the existence of Nash equilibrium in pure strategies.

Lemma 2.1 *A finite extensive game with complete information has a Nash equilibrium in pure strategies.*

2.2 Credibility Theory

Liu and Liu [8] presented the concept of credibility measure which is self-dual in 2002. An axiomatic foundation of credibility theory was given by Liu [6] in 2004. Here we just enumerate the basic results used in this paper.

Definition 2.3 (Liu [6]) *A fuzzy variable is defined as a function from the credibility space* $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$ *to the set of real numbers.*

Definition 2.4 (Liu [6]) *Let* ξ *be a fuzzy variable defined on the credibility space* $(\Theta, \mathcal{P}(\Theta), \mathrm{Cr})$. *Then its membership function is derived from the credibility measure by*

$$\mu(x) = (2\mathrm{Cr}\{\xi = x\}) \wedge 1, \quad x \in \Re.$$

Lemma 2.2 (Liu [6]) *Let* ξ *be a fuzzy variable with membership function* μ. *Then for any set* B *of real numbers, we have*

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B}\mu(x) + 1 - \sup_{x\in B^c}\mu(x)\right).$$

Definition 2.5 (Liu and Gao[9]) *The fuzzy variables* $\xi_1, \xi_2, \cdots, \xi_m$ *are said to be independent if and only if*

$$\mathrm{Cr}\left\{\bigcup_{i=1}^{m}\{\xi_i \in B_i\}\right\} = \max_{1\le i\le m}\mathrm{Cr}\{\xi_i \in B_i\}$$

for any sets $B_1, B_2, \cdots, B_m$ *of* $\Re$.

Definition 2.6 (Liu and Liu [8]) *Let* ξ *be a fuzzy variable. Then the expected value of* ξ *is defined by*

$$E[\xi] = \int_0^{\infty} Cr\{\xi \ge r\}dr - \int_{-\infty}^{0} Cr\{\xi \le r\}dr.$$

Lemma 2.3 (Liu and Liu [10]) *Assume that* ξ *and* η *are independent fuzzy variables with finite expected values. Then for any real numbers* a *and* b, *we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

Definition 2.7 (Liu [5]) *Let* ξ *be a fuzzy variable, and* $\alpha \in (0, 1]$. *Then*

$$\xi_{\sup}(\alpha) = \sup\{r \mid \mathrm{Cr}\{\xi \ge r\} \ge \alpha\}$$

is called the α*-optimistic value to* ξ, *and*

$$\xi_{\inf}(\alpha) = \inf\{r \mid \mathrm{Cr}\{\xi \le r\} \ge \alpha\}$$

is called the α*-pessimistic value to* ξ.

Lemma 2.4 (Liu [6]) *Suppose that* ξ *and* η *are independent fuzzy variables, then for any* $\alpha \in (0, 1]$ *and nonnegative real numbers* a *and* b, *we have*

1. $(a\xi + b\eta)_{\sup}(\alpha) = a\xi_{\sup}(\alpha) + b\eta_{\sup}(\alpha)$;
2. $(a\xi + b\eta)_{\inf}(\alpha) = a\xi_{\inf}(\alpha) + b\eta_{\inf}(\alpha)$.

Definition 2.8 (Liu [5]) *We have three ranking criteria for fuzzy variables* ξ *and* η*:*

1. ***Expected Value Criterion:*** $\xi < \eta$ *if and only if* $E[\xi] < E[\eta]$;
2. ***Optimistic Value Criterion:*** $\xi < \eta$ *if and only if* $\xi_{\sup}(\alpha) < \eta_{\sup}(\alpha)$ *for some predetermined confidence level* $\alpha \in (0, 1]$;
3. ***Pessimistic Value Criterion:*** $\xi < \eta$ *if and only if* $\xi_{\inf}(\alpha) < \eta_{\inf}(\alpha)$ *for some predetermined confidence level* $\alpha \in (0, 1]$;

3 Credibilistic Equilibria

While considering finite extensive games before, each player's payoffs are crisp numbers, thus they can be evaluated and compared easily. But in real life, extensive games may consist of a large number of players and strategies, each player has to make decisions for several rounds. Sometimes, it is impracticable to identify the specific effect belonging to each strategy of every player $i \in N$ since the outcome is yielded by the interacting influences of a strategy profile $s = (s_i)_{i\in N}$, moreover, each strategy s_i may be constituted by numerous actions. For these reasons, making accurate or stochastic estimations about the payoffs are almost impossible for the players. We introduce fuzzy payoff functions to describe the intricate situation in finite extensive game with complete information. First, we present the definition for such games.

Definition 3.1 *A finite extensive game with fuzzy payoffs* $\langle N, H, P, f_c, (u_i)\rangle$ *(FEGF) is a finite extensive game with complete information and chance moves, the preference relation for every player* $i \in N$ *is represented by independent fuzzy payoff function* u_i.

The new definition is almost the same as the traditional one, and we will prove that equilibria in FEGF do possess similar properties as equilibria in deterministic environment.

The basic idea of any Nash equilibrium $s^* = (s_i^*)_{i\in N}$ in extensive games is that no player $i \in N$ can make himself better by choosing a strategy other than s_i^*, given that every other player j insists on s_j^*. In deterministic environment, this idea is equivalent to

$$O(s_{-i}^*,\ s_i^*) \succeq_i O(s_{-i}^*,\ s_i)$$

for every strategy s_i of player i and also equivalent to the condition that

$$u_i(s_{-i}^*,\ s_i^*) \geq u_i(s_{-i}^*,\ s_i)$$

for every strategy s_i of player i.

In FEGF, payoff functions are fuzzy variables and cannot be compared directly, hence we have to define some new versions of Nash equilibrium in FEGF.

Let the fuzzy payoff functions be $u = (u_i)_{i\in N}$, given any strategy profile $s = (s_i)_{i\in N}$, by the existence of chance moves in FEGF, the payoff $u_i(s_{-i}^*,\ s_i^*)$ is a weighted average of outcomes to some certain terminal histories $(h_1, \ldots, h_m)$, every history h_j is determined by s^*. Denote the weights as $\lambda_j \geq 0$ and $\sum_{j=1}^{m} \lambda_j = 1$, $u_i(h_j)$ is the fuzzy payoff to player $i \in N$ of terminal history h_j, we have

$$u_i(s_{-i}^*,\ s_i^*) = \lambda_1 u_i(h_1) + \ldots + \lambda_m u_i(h_m)$$

then

$$O(s_{-i}^*,\ s_i^*) \succeq_i O(s_{-i}^*,\ s_i)$$

for every strategy s_i of each player $i \in N$, if and only if fuzzy variable $u_i(s_{-i}^*,\ s_i^*)$ is greater than $u_i(s_{-i}^*,\ s_i)$ for every strategy s_i of each player $i \in N$ under certain ranking criterion.

We next give the precise definitions of three new versions of Nash equilibrium in FEGF.

Definition 3.2 *An expected credibilistic equilibrium (ECE) of FEGF=*$\langle N, H, P, f_c, (u_i)\rangle$ *is a strategy profile* s^* *such that for every player* $i \in N$ *we have*

$$E\left[u_i(s_{-i}^*,\ s_i^*)\right] \geq E\left[u_i(s_{-i}^*,\ s_i)\right]$$

for every strategy s_i *of player i.*

When we adopt the optimistic value criterion, we mean that the players are risk averse. Under a predetermined confidence level α, they want to maximize the optimistic profits, the essence of such equilibrium is similar as the maximin criterion with deterministic payoffs.

Definition 3.3 *An* $\overline{\alpha}-$*credibilistic equilibrium (*$\overline{\alpha}-$*CE) of FEGF=*$\langle N, H, P, f_c, (u_i)\rangle$ *is a strategy profile* s^* *such that for every player* $i \in N$ *we have*

$$\sup\left\{r \mid \mathrm{Cr}\{u_i(s_{-i}^*,\ s_i^*) \geq r\} \geq \alpha\right\}$$
$$\geq \sup\left\{r \mid \mathrm{Cr}\{u_i(s_{-i}^*,\ s_i) \geq r\} \geq \alpha\right\}$$

for every strategy s_i *of player i and a predetermined confidence level* $\alpha \in (0, 1]$.

In some circumstances, players may act as risk lovers, then the pessimistic criterion is a feasible means to compare the fuzzy payoffs. In detail, every player computes the pessimistic value for each fuzzy payoff, the credibility that real payoff is less than the pessimistic value is α, player then choose the strategy maximizing the pessimistic value in the game. It sounds ridiculous for the players, but in fact, although the real payoff will be less than the pessimistic value with credibility α, the difference of them maybe tiny, risk lovers will be probable and reasonable to seek high profit with great risk.

Definition 3.4 *An* $\underline{\alpha}-$*credibilistic equilibrium (*$\underline{\alpha}-$*CE) of FEGF=*$\langle N, H, P, f_c, (u_i)\rangle$ *is* *a strategy profile s^* such that for every player $i \in N$ we have*

$$\inf\{r \mid \mathrm{Cr}\{u_i(s^*_{-i},\ s^*_i) \le r\} \ge \alpha\} \\ \ge \inf\{r \mid \mathrm{Cr}\{u_i(s^*_{-i},\ s_i) \le r\} \ge \alpha\}$$

for every strategy s_i of player i and a predetermined confidence level $\alpha \in (0, 1]$.

We then generalize Lemma 2.1 of Nash equilibrium in deterministic environment to our new versions of Nash equilibrium in FEGF by the following procedure. The next theorem proves the existence of expected credibilistic equilibrium in FEGF.

Theorem 3.1 *Every FEGF=$\langle N, H, P, f_c, (u_i)\rangle$ has an ECE in pure strategies.*

Proof Let $P(\emptyset) = P_1$ and all histories with length 1 be $(h_1, h_2, \ldots, h_{r-1}, h_r)$, using the notion of subgames, we have $(\Gamma(h_1), \Gamma(h_2), \ldots, \Gamma(h_r))$. Let $u_i|_h$ be the fuzzy payoffs to player $i \in N$ in $\Gamma(h)$ and $s(j) = (s_{ij})_{i\in N}$ be pure strategies for each player in $\Gamma(h_j)$ $(1 \le j \le r)$. Then $u_i(s),\ u_i|_{h_j}(s(j))$ represent the payoffs to player i in FEGF and $\Gamma(h_j)$, respectively. See Figure 1.

We prove the theorem by induction on the finite length of FEGF. Suppose the length of FEGF to be M, clearly the subgames $\Gamma(h_1), \Gamma(h_2), \ldots, \Gamma(h_r)$ have length at most $M - 1$.

1. $M = 0$, the result is trivially true;

2. $M = 1$, the only one player can simply choose the strategy which yields the maximized expected fuzzy payoffs, the pure strategy is an ECE;

3. Suppose the result holds for FEGF with length at most $M - 1$, in particular for $\Gamma(h_1), \Gamma(h_2), \ldots, \Gamma(h_r)$, let $s^*(j)$ be the ECE strategy profile in $\Gamma(h_j)$, respectively, that is

$$E\left[u_i|_{h_j}(s^*_{-i}(j),\ s^*_i(j))\right] \\ \ge E\left[u_i|_{h_j}(s^*_{-i}(j),\ s_i)\right]$$

for every strategy s_i of player i in $\Gamma(h_j)$.

we will construct an ECE strategy profile for FEGF.

Case 1. P_1 is a chance move. Let $\lambda_1, \lambda_2, \ldots, \lambda_r,\ \lambda_j \ge 0$ and $\sum_{j=1}^{r} \lambda_j = 1$, denote the probabilities for selecting subgame $\Gamma(h_1), \Gamma(h_2), \ldots, \Gamma(h_r)$. Define a strategy profile s^* of FEGF as $s^*|_{h_j} = s^*(j)$, then all the strategies of every player $i \in N$ are determined and we next prove this strategy profile is an ECE. For any strategy s_i of player $i \in N$,

$$\begin{aligned} & E\left[u_i(s^*_{-i},\ s_i)\right] \\ = \ & E\left[\sum_{j=1}^{r} \lambda_j u_i|_{h_j}(s^*_{-i}|_{h_j},\ s_i|_{h_j})\right] \\ = \ & \sum_{j=1}^{r} \lambda_j E\left[u_i|_{h_j}(s^*_{-i}(j),\ s_i|_{h_j})\right] \end{aligned} \quad (1)$$

thus

$$\begin{aligned} & E\left[u_i(s^*_{-i},\ s^*_i)\right] \\ = \ & \sum_{j=1}^{r} \lambda_j E\left[u_i|_{h_j}(s^*_{-i}(j),\ s^*_i(j))\right] \\ \ge \ & \sum_{j=1}^{r} \lambda_j E\left[u_i|_{h_j}(s^*_{-i}(j),\ s_i|_{h_j})\right] \end{aligned} \quad (2)$$

since $s^*(j)$ is an ECE strategy profile for $\Gamma(h_j)$ and (1), thus we have

$$E\left[u_i(s^*_{-i},\ s^*_i)\right] \ge E\left[u_i(s^*_{-i},\ s_i)\right]$$

i.e., s^* is an ECE strategy profile in FEGF.

Case 2. P_1 is a player in N. Without loss of generality, we can suppose P_1 to be player 1. Suppose action α taken by player 1 at the initial of the game to be the choice of $j = \alpha$ that

$$\max_{1\le j\le r} E\left[u_i|_{h_j}(s^*(j))\right]$$

is obtained. Define a strategy profile s^* of FEGF as $s^* = (s^*_{-1}, s^*_1)$ where $s^*|_{h_j} = s^*(j)$ and $s^*_1(\emptyset) = \alpha$, then all the strategies of each player $i \in N$ are designated. We then prove s^* is an ECE.

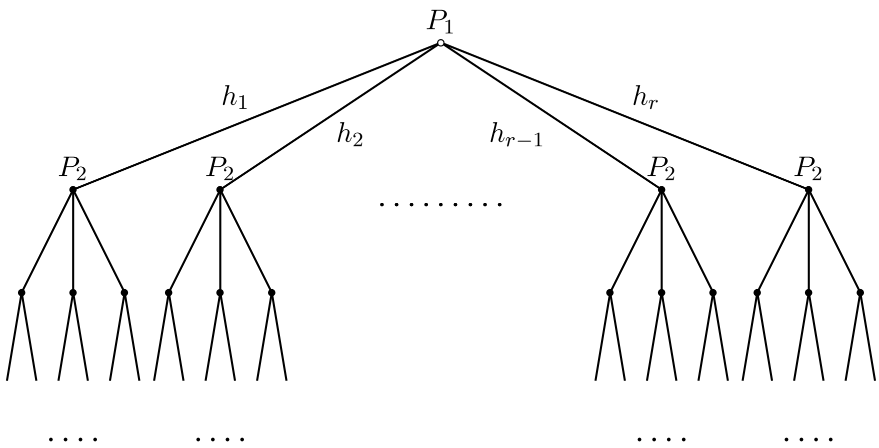

Figure 1: FEGF= $\langle N, H, P, f_c, (u_i)\rangle$

We have

$$\begin{aligned} & E\left[u_1(s^*_{-1},\ s^*_1)\right] \\ = & \ E\left[u_1|_{h_\alpha}(s^*_{-1}(\alpha),\ s^*_1(\alpha))\right] \\ \geq & \ E\left[u_1|_{h_j}(s^*_{-1}(j),\ s^*_1(j))\right],\ 1 \leq j \leq r \end{aligned}$$

And for any strategy s_1 of player 1 with $s_1(\emptyset) = j$,

$$\begin{aligned} E\left[u_1(s^*_{-1},\ s_1)\right] & = E\left[u_1|_{h_j}(s^*_{-1}|_{h_j},\ s_1|_{h_j})\right] \\ & \leq E\left[u_1|_{h_j}(s^*_{-1}(j),\ s^*_1(j)\right] \end{aligned}$$

because $s^*(j)$ is an ECE in $\Gamma(h_j)$, thus for any strategy s_1

$$E\left[u_1(s^*_{-1},\ s^*_1)\right] \geq E\left[u_1(s^*_{-1},\ s_1)\right] \quad (3)$$

For every player $i \in N,\ i \neq 1$ and any strategy s_i, we have

$$\begin{aligned} & E\left[u_i(s^*_{-i},\ s_i)\right] \\ = & \ E\left[u_i|_{h_\alpha}(s^*_{-i}|_{h_\alpha},\ s_i|_{h_\alpha})\right] \\ = & \ E\left[u_i|_{h_\alpha}(s^*_{-i}(\alpha),\ s_i|_{h_\alpha})\right] \\ \leq & \ E\left[u_i|_{h_\alpha}(s^*_{-i}(\alpha),\ s^*_i(\alpha))\right] \end{aligned}$$

since $s^*(\alpha)$ is an ECE in $\Gamma(h_\alpha)$. Hence for every player $i \neq 1$, for any strategy s_i,

$$E\left[u_i(s^*_{-1},\ s^*_1)\right] \geq E\left[u_i(s^*_{-i},\ s_i)\right] \quad (4)$$

With (3) and (4) hold, we have constructed an ECE s^* in FEGF. The proof is then finished. □

The existence of $\overline{\alpha}-$credibilistic equilibrium in FEGF can be similarly proved.

Theorem 3.2 *Every FEGF=*$\langle N, H, P, f_c, (u_i)\rangle$ *has an* $\overline{\alpha}-$*CE for any predetermined confidence level* $\alpha \in (0, 1]$ *in pure strategies.*

Proof Define the symbols as Theorem 3.1, since payoff functions are independent fuzzy variables, we can rewrite (1) and (2) as

$$\begin{aligned} & \sup\{r \mid \mathrm{Cr}\{u_i(s^*_{-i},\ s_i) \geq r\} \geq \alpha\} \\ = & \sup\left\{r \middle| \mathrm{Cr}\left\{\sum_{j=1}^r \lambda_j u_i|_{h_j}(s^*_{-i}|_{h_j},\ s_i|_{h_j}) \geq r\right\} \geq \alpha\right\} \\ = & \sum_{j=1}^r \lambda_j \sup\left\{r \middle| \mathrm{Cr}\left\{u_i|_{h_j}(s^*_{-i}(j),\ s_i|_{h_j}) \geq r\right\} \geq \alpha\right\} \end{aligned}$$

and

$$\begin{aligned} & \sup\{r \mid \mathrm{Cr}\{u_i(s^*_{-i},\ s^*_i) \geq r\} \geq \alpha\} \\ = & \sum_{j=1}^r \lambda_j \sup\left\{r \middle| \mathrm{Cr}\left\{u_i|_{h_j}(s^*_{-i}(j),\ s^*_i(j)) \geq r\right\} \geq \alpha\right\} \\ \geq & \sum_{j=1}^r \lambda_j \sup\left\{r \middle| \mathrm{Cr}\left\{u_i|_{h_j}(s^*_{-i}(j),\ s_i|_{h_j}) \geq r\right\} \geq \alpha\right\} \end{aligned}$$

by Lemma 2.4. Then the result can be similarly proved as Theorem 3.1. □

Further more, the next theorem asserts that the $\underline{\alpha}-$credibilistic equilibrium exists in FEGF.

Theorem 3.3 *Every FEGF=*$\langle N, H, P, f_c, (u_i)\rangle$ *has an* $\underline{\alpha}-$*CE for any predetermined confidence level* $\alpha \in (0, 1]$ *in pure strategies.*

Proof Under the same inductive assumption, for $\Gamma(h_1), \Gamma(h_2), \ldots, \Gamma(h_r)$, let $s^*(j)$

be the $\underline{\alpha}-$CE strategy profiles, respectively. Then for any strategy s_i of player $i \in N$ in FEGF, we have

$$\inf\left\{r \mid \mathrm{Cr}\{u_i|_{h_j}(s^*_{-i}(j),\ s_i|_{h_j}) \le r\} \ge \alpha\right\}$$
$$\le \inf\left\{r \mid \mathrm{Cr}\{u_i|_{h_j}(s^*_{-i}(j),\ s^*_{-i}(j)) \le r\} \ge \alpha\right\}$$

thus

$$\begin{aligned}
& \inf\{r \mid \mathrm{Cr}\{u_i(s^*_{-i},\ s_i) \le r\} \ge \alpha\} \\
= & \inf\left\{r \middle| \mathrm{Cr}\left\{\sum_{j=1}^{r} \lambda_j u_i|_{h_j}(s^*_{-i}|_{h_j},\ s_i|_{h_j}) \le r\right\} \ge \alpha\right\} \\
= & \sum_{j=1}^{r} \lambda_j \inf\left\{r \middle| \mathrm{Cr}\left\{u_i|_{h_j}(s^*_{-i}(j),\ s_i|_{h_j}) \le r\right\} \ge \alpha\right\} \\
\le & \sum_{j=1}^{r} \lambda_j \inf\left\{r \middle| \mathrm{Cr}\left\{u_i|_{h_j}(s^*_{-i}(j),\ s^*_{-i}(j)) \le r\right\} \ge \alpha\right\} \\
= & \inf\{r \mid \mathrm{Cr}\{u_i(s^*_{-i},\ s^*_i) \le r\} \ge \alpha\}
\end{aligned}$$

by Lemma 2.4. We can take the same steps to prove this theorem as Theorem 3.1. □

We now have proved the existences of expected credibilistic equilibrium, $\overline{\alpha}-$credibilistic equilibrium and $\underline{\alpha}-$credibilistic equilibrium in FEGF.

4 Examples and Discussion

In this section, we give two numerical examples to show these new equilibria are necessary and suitable in FEGF. The first example is used to show the meanings of credibilistic equilibria in practical games. The second one is to illustrate that different credibilistic equilibria are reasonable for particular people with different preferences to decide his own strategy, none of these equilibria can be omitted.

Example 1. We first consider the simple entry game with fuzzy payoffs, the independent fuzzy payoffs are represented by triangular fuzzy variables and the challenger's payoff is the first component of each pair. See Figure 2.

Let the confidence level be 0.8 for $\overline{\alpha}-$credibilistic equilibria, 0.6 for $\underline{\alpha}-$credibilistic equilibria. By computation, we have $E[(0,1,2)] = 1$, $E[(1,2,3)] = 2$, $(0,1,2)_{\sup}(0.8) = 0.4$, $(1,2,3)_{\sup}(0.8) = 1.4$, $(0,1,2)_{\inf}(0.6) = 1.2$, $(1,2,3)_{\inf}(0.6) = 2.2$. In this entry game, the ECE, the $\overline{\alpha}-$CE and

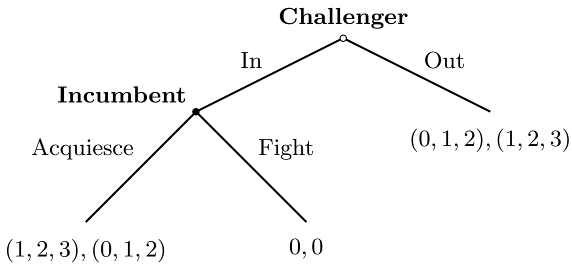

Figure 2: Entry game with fuzzy payoffs.

the $\underline{\alpha}-$CE are the same, (In, Acquiesce) and (Out, Fight).

Example 2. In this example, we illustrate that under different ranking criteria, the equilibria will probably be different, let the payoffs be independent triangular fuzzy variables and player 1's fuzzy payoff be the first component of each pair. See Figure 3.

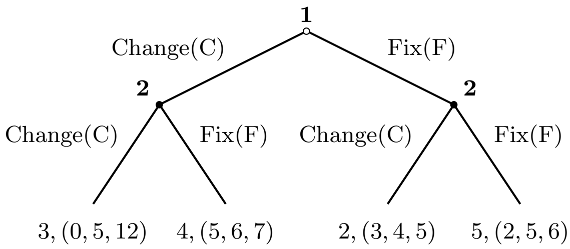

Figure 3: An FEGF with two players.

First we list the pure strategies for the two players in this FEGF as Table 1.

Table 1: Pure strategies for the two players

Player 1	Player 2
C	I. If 1 selects C, C; If 1 selects F, C
F	II. If 1 selects C, C; If 1 selects F, F
	III. If 1 selects C, F; If 1 selects F, C
	IV. If 1 selects C, F; If 1 selects F, F

Given confidence level 0.8 for $\overline{\alpha}-$credibilistic equilibria, 0.6 for $\underline{\alpha}-$credibilistic equilibria, we use the strategic forms of this FEGF to calculate the equilibria as in deterministic environment, see Table 2, 3 and 4.

Then the ECE are (C, III), (F, II) and (F, IV); the $\overline{\alpha}-$CE is (C, III); the $\underline{\alpha}-$CE are (C, I), (F, II) and (F, IV).

Table 2: Strategic Form for Expected Payoffs

		Player 2			
		I	II	III	IV
Player 1	C	$(3, 5.5)$	$(3, 5.5)$	$(4, 6)$	$(4, 6)$
	F	$(2, 4)$	$(5, 4.5)$	$(2, 4)$	$(5, 4.5)$

Table 3: Strategic Form for Optimistic Payoffs

		Player 2			
		I	II	III	IV
Player 1	C	$(3, 2)$	$(3, 2)$	$(4, 5.4)$	$(4, 5.4)$
	F	$(2, 3.4)$	$(5, 3.2)$	$(2, 3.4)$	$(5, 3.2)$

Table 4: Strategic Form for Pessimistic Payoffs

		Player 2			
		I	II	III	IV
Player 1	C	$(3, 6.4)$	$(3, 6.4)$	$(4, 6.2)$	$(4, 6.2)$
	F	$(2, 4.2)$	$(5, 5.2)$	$(2, 4.2)$	$(5, 5.2)$

5 Conclusion

In this paper, we firstly used fuzzy variables to characterize payoffs in extensive games due to the incompleteness of information. Then we proposed new concepts of credibilistic equilibria in FEGF as Nash equilibrium in deterministic environment. Furthermore, we proved the existence theorems that affirm these new equilibria do exist in finite extensive games with fuzzy payoffs. Starting from them, we can do further researches on the properties of credibilistic equilibria in FEGF. At the end of this paper, we gave two numerical examples to illustrate the rationality and necessity of these new equilibria in FEGF.

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