Some remarks on the single-stage decision problem with fuzzy utilities

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Abstract

In this paper we deal with a single-stage decision problem with imprecise utilities with some special properties. The main one is that product measurability of the utility function is not required, so that, iterated expectations are used instead of integrals over a product space. Equivalence between the two forms of the Bayesian analysis are obtained under these conditions.

Keywords: Decision support system, Bayesian analysis of single-stage decision problems, Iterated expectation, Random upper semicontinuous function, Uncertainty modeling.

1 Introduction

Uncertainty yields in the construction of many decision problems. Thus, sometimes we an action is chosen, the consequences of this election are imprecise, and the decisor is not able to express the values of the consequences of his/her choices in a real-valued scale. That is the case, for instance, of utility assessments like "safe", "risky", "satisfactory", "very expensive", "rather dangerous" and so on. Therefore, several studies have been developed to enlarge the scale to evaluate utilities. See, for instance, chronologically ordered: [18], [17], [6], [2], [3], [7] or [8].

In [7], a model for the single-stage decision problem with imprecise utilities was introduced. Under the conditions stated there, it was obtained the equivalence between the extensive and the normal forms of the Bayesian analysis. Later, in [8], this model was improved to deal with a wider class of utilities.

However, this model was based on a version of a Fubini Theorem for random upper semicontinuous functions (namely, fuzzy random variables of random fuzzy sets) which requires the product measurability.

The present work is devoted to the analysis of single-stage decision problems with imprecise utility functions when they are not necessarily product measurable. The theoretical core for the results obtained in this paper has been recently developed in [16], by studying iterated expectations of random upper semicontinuous functions without product measurability.

The paper is organized in the following sections: Section 2 gathers preliminary concepts and results; Section 3 contains the new mathematical model for single-stage decision problems and the Bayesian analysis for that problem. Finally, some possible future lines of work are outlined.

2 Preliminaries

Let $\mathcal{K}_c$ be the class of nonempty compact convex subsets of $\mathbb{R}$. This class is endowed with the Minkowski addition and the product by a scalar and becomes a semilinear space.

The *Hausdorff metric* on $\mathcal{K}_c$ is given by

$$d_H(M, N)$$

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$= \max\{\,|\inf M - \inf N|,\,|\sup M - \sup N|\,\}.$

Let $(\Omega, \mathcal{A})$ be a measurable space, a *random set* is $\mathcal{A}|\mathcal{B}_{d_H}$-measurable mapping $S : \Omega \to \mathcal{K}_c$, where $\mathcal{B}_{d_H}$ denotes the Borel σ-field generated by the topology induced by d_H on $\mathcal{K}_c$.

If $\mu : \mathcal{A} \to \mathbb{R}$ is a measure, a random set S is said to be *integrably bounded with respect to* μ, if $\|S\| \in L^1(\Omega, \mathcal{A}, \mu)$.

The *integral* of S, or *expected value* in case of μ being a probability, is given by the *Kudō-Aumann integral* (see [10] and [1]), this is,

$$\Big\{ \int_\Omega f(\omega)\, d\mu(\omega) \mid f : \Omega \to \mathbb{R},$$
$$f \in L^1(\Omega, \mathcal{A}, \mu),\ f \in S \ a.e.\,[\mu] \Big\}.$$

This set will be denoted by $E(S|\mu)$. It is easy to see that

$$E(S|\mu)$$
$$= \Big[\int_\Omega \inf S(\omega)\, d\mu(\omega), \int_\Omega \sup S(\omega)\, d\mu(\omega) \Big].$$

Let $\mathcal{F}_c$ denote the class of upper semicontinuous functions $U : \mathbb{R} \to [0,1]$ such that $U_\alpha \in \mathcal{K}_c$ if $\alpha \in [0,1]$, where $U_\alpha = \{x \in \mathbb{R} : U(x) \geq \alpha\}$ for $\alpha \in (0,1]$, and $U_0 = \mathrm{cl}\,\{x \in \mathbb{R} : U(x) > 0\}$, cl denoting the topological closure. These mappings are also referred to as *fuzzy sets* of $\mathbb{R}$, and the sets U_α as the associated *α-level sets*.

The class $\mathcal{F}_c$ is endowed with the addition and the product by a scalar, defined by means of Zadeh's extension principle (see [20]), or equivalently (see [13]) these operations can be levelwise calculated as $(U+V)_\alpha = U_\alpha + V_\alpha$ and $(\lambda U)_\alpha = \lambda U_\alpha$ for all $U, V \in \mathcal{F}_c$, $\lambda \in \mathbb{R}$ and $\alpha \in [0,1]$.

On $\mathcal{F}_c$ we consider the d_∞ metric (see [13]) given by $d_\infty(U,V) = \sup_{\alpha \in [0,1]} d_H(U_\alpha, V_\alpha)$, with $U, V \in \mathcal{F}_c$.

The *magnitude* of $U \in \mathcal{F}_c$ is defined by $\|U\| = d_\infty(U, \mathbf{1}_{\{0\}}) = d_H(U_0, \{0\})$.

Given a measurable space $(\Omega, \mathcal{A})$, a mapping $X : \Omega \to \mathcal{F}_c$ is said to be a *random upper semicontinuous function*) if $X_\alpha : \Omega \to \mathcal{K}_c$ with $X_\alpha(\omega) = (X(\omega))_\alpha$ for all $\omega \in \Omega$, is a random set for all $\alpha \in [0,1]$ (see [15]).

A random upper semicontinuous function X is said to be *integrably bounded* with respect to a measure $\mu : \mathcal{A} \to \mathbb{R}$, if the mapping $\|X\| \in L^1(\Omega, \mathcal{A}, \mu)$, where $\|X\| : \Omega \to \mathbb{R}$ is given by $\|X\|(\omega) = \|X(\omega)\|$ for all $\omega \in \Omega$.

For an integrably bounded fuzzy random variable, in [15] its *integral* is defined, denoted by $\int_\Omega X(\omega)\, d\mu(\omega)$ or $E(X|\mu)$, as the unique set in $\mathcal{F}_c$ such that

$$E(X|\mu)_\alpha = E\big(X_\alpha \big| \mu\big)$$

for every $\alpha \in [0,1]$. When $\Omega = [a,b]$, we will use also the notation $\int_a^b X(\omega)\, d\mu(\omega)$.

If μ is a probability measure, an fuzzy random variable is also referred to as a *fuzzy random variable* or *random fuzzy set* and its integral as *the fuzzy expected value of X*.

We will denote by $\mathcal{B}_\Omega$ the Borel σ-field on Ω, for any set $\Omega \subset \mathbb{R}^k$ with $k \in \mathbb{N}$. Given $(\Omega, \mathcal{B}_\Omega)$ and $m_1, m_2 : \Omega \to \mathbb{R}$ two σ-finite measures, $m_1 \ll m_2$ will indicate that m_1 is absolutely continuous with respect to m_2, and $\frac{dm_1}{dm_2}$ will denote a Radon-Nikodym derivative of m_1 with respect to m_2. If it is supposed that there exists a continuous Radon-Nikodym derivative, then $\frac{dm_1}{dm_2}$ will denote this particular function.

In [16] some results about iterated integrals are stated, and they will be the basis of the framework that we have developed here.

Theorem 2.1. *Let $(\Omega, \mathcal{B}_\Omega, P)$ be a probability space with $\Omega \subset \mathbb{R}^k$ and let m denote the Borel measure on the interval $T = [a,b]$. For every $t \in T$, let P_t be a probability measure on $(\Omega, \mathcal{B}_\Omega)$ such that $P_t \ll P$ and there exists a continuous Radon-Nikodym derivative. For every $\omega \in \Omega$, let P_ω be a probability on $(T, \mathcal{B}_T)$ such that $P_\omega \ll m$ and there exists a continuous Radon-Nikodym derivative.*

Let $X : \Omega \times T \to \mathcal{F}_c$ be a mapping satisfying that:

i) for every $t \in T$, X_t is an integrably bounded fuzzy random variable with respect to P_t,

ii) for every $\omega \in \Omega$, X_ω is an integrably *bounded fuzzy random variable with respect to P_ω and it is continuous a.s. $[P]$,*

iii) there exists $h_1 \in L^1(\Omega, \mathcal{B}_\Omega, P)$ such that $\left\|X(\omega,t)\frac{dP_\omega}{dm}(t)\right\| \leq h_1(\omega)$ a.s. $[P]$ for every $t \in T$, and the mapping $\omega \mapsto X(\omega,t)\frac{dP_\omega}{dm}(t)$ is continuous a.e. $[m]$,

iv) there exists a mapping $g \in L^1([a,b], \mathcal{B}_{[a,b]}, m)$ such that for every $\omega \in \Omega$, $\|X(\omega,t)\frac{dP_\omega}{dm}(t)\| \leq g(t)$ a.e. $[m]$ for every $\omega \in \Omega$,

v) the mapping $t \mapsto X(\omega,t)\frac{dP_t}{dP}(\omega)$ is continuous on T a.s. $[P]$,

vi) there exists $h_2 \in L^1(\Omega, \mathcal{B}_\Omega, P)$ such that $\|X(\omega,t)\frac{dP_t}{dP}(\omega)\| \leq h_2(\omega)$ a.s. $[P]$ for every $t \in T$.

Let m' be a probability measure on $(T, \mathcal{B}_T)$ such that $m' \ll m$ and there exists a continuous Radon-Nikodym derivative. If for every $t \in T$, the equality

$$\frac{dP_\omega}{dm}(t) = \frac{dP_t}{dP}(\omega)\frac{dm'}{dm}(t) \text{ a.s. } [P]$$

holds, then

$$\int_\Omega \left(\int_a^t X(\omega,s)\, dP_\omega(s) \right) dP(\omega)$$

$$= \int_a^t \left(\int_\Omega X(\omega,s)\, dP_s(\omega) \right) dm'(s)$$

for every $t \in T$.

Theorem 2.2. *Assume the conditions in Theorem 2.1 with the interval T being not necessarily bounded, and suppose that there exists $g' \in L^1(\Omega, \mathcal{B}_\Omega, P)$ such that $\int_T \|X(\omega,s)\| dP_\omega(s) \leq g'(\omega)$ a.s. $[P]$. Then, the following equality holds,*

$$\int_\Omega \left(\int_T X(\omega,s)\, dP_\omega(s) \right) dP(\omega)$$

$$= \int_T \left(\int_\Omega X(\omega,s)\, dP_s(\omega) \right) dm'(s)\,.$$

It should be remarked that the conditions in Theorems 2.1 and 2.2 do not imply that X is an fuzzy random variable on the product measurable space as it is illustrated in [16].

3 A model for single-stage decision problems

In this section we state a model to study single-stage decision problems with imprecise utilities, and the model is valid even for not necessarily measurable utility functions. Once the model is stated, we study the conditions to guarantee the equivalence of the normal and extensive forms of the Bayesian analysis involving imprecise utilities.

In order to obtain that equivalence result, it is necessary to exchange the order of two iterated integrals, as we will see later. Results in Section 2, stated in [16], will become the main tools for this purpose. On the other hand, if Fubini type theorem could be applied to guarantee the exchange result, this case has been already studied in [7], [11] and [8]. Therefor, in this paper, we will provide alternative conditions to handle this kind of problem in a different framework.

We will model *imprecise utilities* by means of fuzzy sets and imprecise utility functions by means of fuzzy random variables. Thus, in order to find the greatest utility, we will need to rank fuzzy sets. We will use the criterion introduced by [4], which is based on the following value: given $U \in \mathcal{F}_c$, its *λ, μ-average value* is defined as the real number

$$V_\mu^\lambda(U) = \int_0^1 \big(\lambda \sup U_\alpha + (1-\lambda) \inf U_\alpha\big)\, d\mu(\alpha),$$

where $\lambda \in [0,1]$ represents a degree of optimism/pessimism that is assumed by the decision maker (the greater λ the more optimistic situation in a gain context), and μ is a measure on $[0,1]$.

Then, $U \in \mathcal{F}_c$ will be said to be *greater than or equal to $W \in \mathcal{F}_c$ in the λ, μ-average sense*, denoted by $U \geq_{\lambda,\mu} W$, if, and only if, $V_\mu^\lambda(U) \geq V_\mu^\lambda(W)$.

Good properties of V_μ^λ are not only related with the way it ranks fuzzy sets, but also with its exchangeability with the fuzzy expected value, as it is stated in the following result, which can be deduced from [12].

Proposition 3.1. *Let $(\Omega, \mathcal{A}, P)$ be a prob-*

ability space and let $X : \Omega \to \mathcal{F}_c$ be an integrably bounded fuzzy random variable If $\mu : \mathcal{B}_{[0,1]} \to \mathbb{R}$ is a measure such that $\mu \ll m$, where m denotes the Borel measure on the interval $[0,1]$, *then*

$$V_\mu^\lambda \left(\int_\Omega X(\omega)\, dP(\omega) \right)$$

$$= \int_\Omega V_\mu^\lambda \left(X(\omega) \right) dP(\omega),$$

where the integral in the left-hand-side term stands for the fuzzy expected value of X with respect to P, whereas the integral in the right-hand-side term represents the usual Lebesgue integral of the random variable $V_\mu^\lambda(X)$ with respect to P.

We are formalizing now the concept of fuzzy utility function. Let us consider in the following a single-stage decision problem with state space $\Theta \subseteq \mathbb{R}$, and with action space $\mathbf{A}$. Let $\mathcal{B}_\Theta$ be the Borel σ-field on Θ, and let m denote the Borel measure on Θ.

Definition 3.1. *A mapping $U : \Theta \times \mathbf{A} \to \mathcal{F}_c$ is said to be a fuzzy utility function on $\Theta \times \mathbf{A}$ if*

i) for every $a \in \mathbf{A}$, the projection $U_a : \Theta \to \mathcal{F}_c$ is a fuzzy random variable on $(\Theta, \mathcal{B}_\Theta)$,

ii) for every pair $a_1, a_2 \in \mathbf{A}$, a_1 will be considered preferred or indifferent to a_2 with respect to a probability distribution ξ on $(\Theta, \mathcal{B}_\Theta)$, if $E(U_{a_1}|\xi) \geq_{\lambda,\mu} E(U_{a_2}|\xi)$ (for fixed $\lambda \in [0,1]$ and measure μ).

For describing the elements of the decision problem with fuzzy utilities we will use the notation $(\Theta, \mathbf{A}, U)$.

We will deal with the decision problem within a Bayesian context, therefore wi will assume the existence of a probability distribution π on $(\Theta, \mathcal{B}_\Theta)$, called *the prior distribution.* Hence, the "value" of the decision problem will be the fuzzy value $E(U_{a^\pi}|\pi)$, being a^π a *prior Bayes action* in the λ,μ-average sense, this is, $a^\pi \in \mathbf{A}$ verifies $E(U_{a^\pi}|\pi) \geq_{\lambda,\mu} E(U_a|\pi)$ for all $a \in \mathbf{A}$.

In any decision problem (despite utilities are crisp or imprecise), it becomes helpful for increasing the expected utility to add *sample information.* We will use the following notation to model this situation: $\mathbf{X}$ will be a statistical experiment characterized by a probability space $(\mathbb{X}, \mathcal{B}_\mathbb{X}, P_\theta)$, being $\theta \in \Theta$, $\mathcal{B}_\mathbb{X}$ the Borel σ-field on $\mathbb{X} \subset \mathbb{R}^k$ and the experimental distribution P_θ depends on the true unknown state θ. We will denote by P the *marginal* (also called *predictive*) distribution of the experiment.

Once the experiment is performed, if $\mathbf{X} = x$ is the available sample information, the fuzzy expected utility associated with an action $a \in \mathbf{A}$ is given by $E(U_a|\pi_x)$, being π_x the *posterior distribution* of θ given $\mathbf{X} = x$ (π_x is obtained by means of Bayes' formula). So, a *posterior Bayes action* is any $a^{\pi_x} \in \mathbf{A}$ such that $E(U_{a^{\pi_x}}|\pi_x) \geq_{\lambda,\mu} E(U_a|\pi_x)$ for every $a \in \mathbf{A}$.

The concept of *decision rule* is introduced to generalize the choice of an action for every possible sample information. A *decision rule* is a mapping from $\mathbb{X}$ to $\mathbf{A}$ satisfying conditions based on Theorems 2.1 and 2.2.

Definition 3.2. *Let $(\mathbb{X}, \mathcal{B}_\mathbb{X}, P_\theta)$ be the probability space of a statistical experiment $\mathbf{X}$ associated with the decision problem $(\Theta, \mathbf{A}, U)$. A decision rule is a mapping $d : \mathbb{X} \to \mathbf{A}$ satisfying that*

i) for every $\theta \in \Theta$, $U(\theta, d()) : \mathbb{X} \to \mathcal{F}_c$ is an integrably bounded fuzzy random variable with respect to P_θ,

ii) for every $x \in \mathbb{X}$, $U(, d(x)) : \Theta \to \mathcal{F}_c$ is an integrably bounded fuzzy random variable with respect to π_x, moreover, it is continuous a.s. $[P]$,

iii) there exists $h_1 \in L^1(\mathbb{X}, \mathcal{B}_\mathbb{X}, P)$ such that $\left\| U(\theta, d(x)) \frac{d\pi_x}{dm}(\theta) \right\| \leq h_1(x)$ a.s. $[P]$ for every $\theta \in \Theta$, and the mapping $x \mapsto U(\theta, d(x)) \frac{d\pi_x}{dm}(\theta)$ is continuous a.e. $[m]$,

iv) there exists $g \in L^1(\Omega, \mathcal{B}_\Omega, m)$ such that for every $x \in \mathbb{X}$, it holds that $\| U(\theta, d(x)) \frac{d\pi_x}{dm}(\theta) \| \leq g(\theta)$ a.e. $[m]$ for every $x \in \mathbb{X}$, *v) the mapping* $\theta \mapsto U(\theta, d(x))\frac{dP_\theta}{dP}(x)$ *is continuous on* Θ *a.s.* $[P]$,

vi) there exists $h_2 \in L^1(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, P)$ *such that* $\left\| U(\theta, d(x))\frac{dP_\theta}{dP}(x) \right\| \leq h_2(x)$ *a.s.* $[P]$ *for every* $\theta \in \Theta$,

vii) there exists $g' \in L^1(\mathbb{X}, \mathcal{B}_{\mathbb{X}}, P)$ *with* $\int_\Theta \|U(\theta, d(x))\| \, d\pi_x(\theta) \leq g'(x)$.

Therefore, on one hand we can consider the *normal Bayesian analysis*. It consists of finding a *Bayes decision rule*, this is, a rule d_B such that

$$\int_\Theta \left(\int_{\mathbb{X}} U(\theta, d_B(x)) \, dP_\theta(x) \right) d\pi(\theta)$$

$$\geq_{\lambda,\mu} \int_\Theta \left(\int_{\mathbb{X}} U(\theta, d(x)) \, dP_\theta(x) \right) d\pi(\theta)$$

for every decision rule d. In this case, the "value" of the problem is

$$\int_\Theta \left(\int_{\mathbb{X}} U(\theta, d_B(x)) \, dP_\theta(x) \right) d\pi(\theta).$$

On the other hand the *extensive Bayesian analysis* is considered. It consists of, for each sample outcome x, obtaining a posterior Bayes action a^{π_x}, and considering the decision rule $x \mapsto a^{\pi_x}$. Now, the "value" of the experiment $\mathbf{X}$ is quantified by the fuzzy expected terminal utility, which is defined as follows:

Definition 3.3. *Given* $(\Theta, \mathbf{A}, U)$ *a decision problem and* $\mathbf{X} = (\mathbb{X}, \mathcal{B}_{\mathbb{X}}, P_\theta)$, *an associated experiment, the fuzzy expected terminal utility of* $\mathbf{X}$ *is given by*

$$U_t(\mathbf{X}) = \int_{\mathbb{X}} \left(\int_\Theta U(\theta, a^{\pi_x}) \, d\pi_x(\theta) \right) dP(x).$$

The following result states conditions to guarantee the exchange of the iterated integrals appearing in the problem.

Theorem 3.2. *Let* $(\Theta, \mathbf{A}, U)$ *be a decision problem, let* $\Theta \subset \mathbb{R}$ *and let* π *be a prior probability on* $(\Theta, \mathcal{B}_\Theta)$ *such that* $\pi \ll m$ *with a continuous Radon-Nikodym derivative. Let* $\mathbf{X} = (\mathbb{X}, \mathcal{B}_{\mathbb{X}}, P_\theta)$ *be an associated experiment, and let* P *be the marginal distribution. For every* $\theta \in \Theta$, *suppose that* $P_\theta \ll P$ *and there exists a continuous Radon-Nikodym derivative.*

For every $x \in \mathbb{X}$, *let* π_x *be the posterior distribution on* $(\Theta, \mathcal{B}_\Theta)$ *such that* $\pi_x \ll m$ *with a continuous Radon-Nikodym derivative.*

If for every $\theta \in \Theta$, *it holds that* $\frac{d\pi_x}{dm}(\theta) = \frac{dP_\theta}{dP}(x)\frac{d\pi}{dm}(\theta)$ *a.s.* $[P]$, *then*

$$\int_{\mathbb{X}} \left(\int_\Theta U(\theta, d(x)) \, d\pi_x(\theta) \right) dP(x)$$

$$= \int_\Theta \left(\int_{\mathbb{X}} U(\theta, d(x)) \, dP_\theta(x) \right) d\pi(\theta)$$

whatever the decision rule $d : \mathbb{X} \to \mathbf{A}$ *may be.*

The next result is the natural consequence of the previous development, and it is the main result in this paper, stating the equivalence between the two forms, normal and extensive, of the Bayesian analysis.

Theorem 3.3. *Under conditions in Theorem 3.2, consider the mapping associating to each sample information* $x \in \mathbb{X}$ *a posterior Bayes action* a^{π_x}. *If this mapping satisfies the definition of decision rule, then it is a Bayes decision rule. Moreover,* $U_t(\mathbf{X})$ *is equal, in the* λ, μ*-average sense, to the fuzzy expected utility associated with any Bayes decision rule, or, equivalently:*

$$U_t(\mathbf{X})$$

$$=_{\lambda,\mu} \int_{\mathbb{X}} \left(\int_\Theta U(\theta, d_B(x)) \, d\pi_x(\theta) \right) dP(x).$$

4 Future lines

Some open problems could be suggested from this paper. Some of then are based on using different criteria for ranking fuzzy sets, maybe not crisp criteria but based on fuzzy preferences. Another interesting problem is the study of the case in which not only the integrated random elements (in particular, utilities) but also the probability assessments are imprecise (see for instance [19] or [5]).

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