Fuzzy Term Structure Equation

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Abstract

The term structure model describes the evolution of the yield curve through time, without considering the influence of risk, tax, etc. Recently, the fuzzy process was initialized and applied to option pricing. Under the assumption of fuzzy interest rate, this paper investigates the term-structure equation. The equation is first derived for valuing zero-coupon bond. Analytic solution of the fuzzy interest rate equation is given when the process for interest rate is the fuzzy counterparts of the Vasicek model.

Keywords Interest rate, equilibrium model, Brownian motion, fuzzy process, Liu process

1 Introduction

The use of Brownian motion to model the time evolution of asset prices may be dated to 1900. Until Itô introduced stochastic calculus and established the celebrated Itô formula in 1944, geometric Brownian motion became an important model for the financial market. In 1973, Black and Scholes [2] and Metron [12] used the geometric Brownian motion to construct a theory for determining the options price. And the Black-Scholes formula has been pivotal to the growth and success of financial engineering in 1980s and 1990s.

The term structure of interest rates is defined as the relationship between the yield-to-maturity on a zero-coupon bond and the bond's maturity. Under the assumption that the probability distribution of an interest rate, or some other variable at a future point is lognormal, some models are developed for valuing the zero-coupon bond. However, this assumption ignores the fact that interest rates change through time, and the corresponding models cannot be used for valuing interest at derivatives such as American-style swap options and callable bonds. In order to overcome this limitation, the short rate is described by process and equilibrium models [3][16][17], no-arbitrage models [4] and some others are proposed.

The concept of fuzzy set was introduced by Zadeh [18], and has been widely used in many areas. As a new branch of mathematics to deal with fuzzy phenomena, credibility theory was founded [8] and refined [10] by Liu. In order to model the evolution of fuzzy phenomena with time, Liu [10][11] introduced the concept of fuzzy process as well as the fuzzy calculus. Parallel to the probabilistic world, a special fuzzy process, called Liu process, plays the role of Brownian motion, and the Geometric Liu process is used to model the stock options price. The fuzzy counterpart of Black-Scholes stock model was called the Liu stock model, and the Liu formula was used for pricing the European option by Qin and Li [15].

In this paper, the short rate is described by a standard Liu process instead of stochastic process. Consequently, we present three fuzzy equilibrium models, which are the fuzzy counterpart of Rendleman-Bartter Model, Vasicek model and Cox-Ingersoll-Ross Model. In order to value the zero-coupon bond, we derive a fuzzy partial differential equation, and propose the analytic solution of the equation when the process for interest rate to be the fuzzy counterparts of the Vasicek model.

The rest of this paper is arranged as follows. In the following section, we first give a brief introduction of the fuzzy process, Liu process, geometric Liu process and Liu stock model. In Section 3, by assuming the short rate to be a standard Liu process, we present three equilibrium models. In section 4, we give the fuzzy term-structure equation for pricing the zero-coupon bond, and lastly, we show the analytic solution of the fuzzy term-structure equation when the process for interest rate is the fuzzy counterparts of the Vasicek model.

2 Preliminaries

Definition 1 (Liu [11]) *Let T be an index set and let $(\Theta, \mathcal{P}, \mathrm{Cr})$ be a credibility space. A fuzzy process is a function from $T \times (\Theta, \mathcal{P}, \mathrm{Cr})$ to the set of real numbers.*

That is, a fuzzy process $X(t;\theta)$ is a function of two variables such that the function $X(t^*;\theta)$ is a fuzzy variable for each $t^* \in T$. For each fixed θ^*, the function $X(t;\theta^*)$ is called a sample path of the fuzzy process. A fuzzy process $X(t;\theta)$ is said to be sample-continuous if the sample path is continuous for almost all θ. Instead of longer notation $X(t;\theta)$, sometimes we use the symbol X_t.

Definition 2 (Liu [11]) *A fuzzy process X_t is said to have independent increments if*

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_k} - X_{t_{k-1}} \quad (1)$$

are independent fuzzy variables for any times $t_0 < t_1 < \cdots < t_k$. A fuzzy process X_t is said to have stationary increments if, for any given $t > 0$, the $X_{s+t} - X_s$ are identically distributed fuzzy variables for all $s > 0$.

Definition 3 (Liu [11]) *A fuzzy process C_t is said to be a Liu process if*

(i) $C_0 = 0$,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normally distributed fuzzy variable with expected value et and variance $\sigma^2 t^2$.

The parameters e and σ are called the drift and diffusion coeffients, respectively. The Liu process is said to be standard if $e = 0$ and $\sigma = 1$. Any Liu process may be represented by $et + \sigma C_t$ where C_t is a standard Liu process.

Definition 4 (Liu [11]) *Let C_t be a standard Liu process. Then $et + \sigma C_t$ is a Liu process, and the fuzzy process*

$$X_t = exp(et + \sigma C_t) \quad (2)$$

is called a geometric Liu process.

It was assumed that stock price follows geometric Brownian motion, and stochastic financial mathematics was then founded based on this assumption. Liu [11] presented an alternative assumption that stock price follows geometric Liu process. Based on this assumption, it is expected to reconsider option pricing, optimal stopping, portfolio selection and so on, thus producing a totally new fuzzy financial mathematics. Liu [11] presented a basic stock model for fuzzy financial market in which the bond price X_t and the stock price Y_t follow

$$\begin{aligned} X_t &= X_0 \exp(rt) \\ Y_t &= Y_0 \exp(et + \sigma C_t) \end{aligned}$$

or equivalently

$$\begin{aligned} \mathrm{d}X_t &= rX_t \mathrm{d}t \\ \mathrm{d}Y_t &= eY_t \mathrm{d}t + \sigma Y_t \mathrm{d}C_t \end{aligned} \quad (3)$$

where r is the riskless interest rate, e is the stock drift, and σ is the stock diffusion. It is just a fuzzy counterpart of Black-Scholes stock model [2].

3 Fuzzy Equilibrium Models

Equilibrium models usually start with assumptions about economic variables and derive a process for the short rate r. Then they explore what the process for r implies about the bond and option price. The short rate, r, at time t is the rate that applies to an infinitesimally short period of time at time t. It is not the process for r in the real world. Bond prices depend only on the process followed by r. We consider that in a very short time period between t and $t + \triangle t$, investors earn on average $r(t)\triangle t$. All processes for r that we present will be processes like that.

In a fuzzy equilibrium model, the process for r involves only one source of uncertainty. The short rate can be described by an standard Liu process of the form

$$dr = a(r)dt + \sigma(t)dC$$

The instantaneous drift a is assumed to be functions of r, and the instantaneous standard deviation, σ, is assumed to be a function of t.

Model 1.

Let the process for r be

$$dr = \mu r dt + \sigma r dC, \tag{4}$$

where μ and σ are constants. This means that r follows geometric Liu process. This model is the counterpart of the Rendleman-Bartter Model and the process for r was also assumed for a stock price. That is, this model assumes that the short-term interest rate behave like a stock price. Certainly, this assumption is less than ideal because this process does not incorporate **mean reversion**, which is a general economic phenomenon.

Model 2

Let the process for r be

$$dr = (a - br)dt + \sigma dC, \tag{5}$$

where a, b and σ are constants. This model is the counterpart of the Vasicek model and the process for r incorporates **mean reversion**. The short rate is pulled to a level b at rate a. Superimposed upon this "pull" is a normally distributed fuzzy term σdC.

Model 3

Let the process for r be

$$dr = (a - br)dt + \sigma\sqrt{r}dC, \tag{6}$$

where a, b and σ are constants. This model is the counterpart of the Cox-Ingersoll-Ross model. This process for r has the same mean-reverting drift as Vasicek model, but the standard deviation is proportional to $\sqrt{r}$. This means that, as the short-term rate interest increases, its standard deviation increases.

4 Fuzzy Term Structure Equation

Let $P = P(r, t, T)$ be the price at time t of a zero-coupon bond that pays off 1 at time T. In fact, for a fixed coupon T is constant parameter. Then, it follows from the fuzzy chain rule [11] that

$$\begin{aligned} & dP(r,t,T) \\ & = \frac{\partial P}{\partial r}(r,t,T)dr + \frac{\partial P}{\partial t}(r,t,T)dt \\ & = (a(r)\frac{\partial P}{\partial r}(r,t,T) + \frac{\partial P}{\partial t}(r,t,T))dt \\ & + \frac{\partial P}{\partial r}(r,t,T)\sigma(r)dC. \end{aligned} \tag{7}$$

Denote

$$\begin{aligned} & \alpha(r,t,T) \\ & = \frac{1}{P(r,t,T)}(a(r)\frac{\partial P}{\partial r}(r,t,T)\frac{\partial P}{\partial t}(r,t,T)) \end{aligned}$$

and

$$q(r,t,T) = \frac{1}{P(r,t,T)}\frac{\partial P}{\partial r}(r,t,T)\sigma(r).$$

Then equation (7) can be written as

$$\frac{dP}{P}(r,t,T) = \alpha(r,t,T)dt + q(r,t,T)dC.$$

Construct a portfolio I as N shares of bond with T_1, one share of bond with T_2 and one share of W, where where $dW(r,t) = rW(r,t)dt$.

Then

$$I(r,t) = NP(r,t,T_1) + P(r,t,T_2) + W(r,t) = 0,$$

We have

$$dI(r,t)$$
$$= NP(r,t,T_1)(\alpha(r,t,T_1)dt + q(r,t,T_1)dC)$$
$$+P(r,t,T_2)(\alpha(r,t,T_2)dt + q(r,t,T_2)dC)$$
$$+rW(r,t)dt.$$

Let

$$N = -\frac{P(r,t,T_2)q(r,t,T_2)}{P(r,t,T_1)q(r,t,T_1)}.$$

Then it follows from $I(r,t) = dI(r,t) = 0$ that

$$-\tfrac{P_r(r,t,T_2)}{P_r(r,t,T_1)}P(r,t,T_1)(\alpha(r,t,T_1)dt$$
$$+q(r,t,T_1)dC) + P(r,t,T_2)(\alpha(r,t,T_2)dt$$
$$+q(r,t,T_2)dC) + rWdt = 0.$$

and

$$\frac{\alpha(r,t,T_1)-r}{q(r,t,T_1)} = \frac{\alpha(r,t,T_2)-r}{q(r,t,T_2)}.$$

Let

$$\phi(r,t,T) = \frac{\alpha(r,t,T)-r}{q(r,t,T)}.$$

then

$$\tfrac{\partial P}{\partial r}(r,t,T)(a(r) - \sigma(r)\phi(r)) + \tfrac{\partial P}{\partial t}(r,t,T)$$
$$-rP(r,t,T) = 0. \tag{8}$$

Now, we will consider that the time T is constant. So we write $P(r,t)$ for short.

Then we have the **Term Structure Equation**

$$\begin{aligned}&\tfrac{\partial P}{\partial r}(r,t)(a(r) - \sigma(r)\phi(r)) + \tfrac{\partial P}{\partial t}(r,t)\\ &-rP(r,t) = 0.\end{aligned} \tag{9}$$

For different equilibrium models, we can get the solutions of the zero-coupon bond value by solving the fuzzy interest rate equation vie analytical or numerical approaches.

5 Analytic Solution

We will consider Model 2 given in Section III.

$$dr = (a - br)dt + \sigma dC.$$

This is a fuzzy counterpart of the Ornstein-Uhlenbeck process. When the interest rate r is more than a/b it will decrease averagely, and vice versa.

First, we have

$$\frac{\partial P}{\partial r}(r,t)(a(b-r)-\sigma\phi)+\frac{\partial P}{\partial t}(r,t)-rP(r,t) = 0,$$

and

$$P(r,T) = 1, a \neq 0.$$

Suppose that

$$u(r,t) = P(r, T-t),$$

$$A = a, B = -ab + \sigma\phi,$$

then we have

$$\frac{\partial u}{\partial r}(r,t)(Ar+B) + \frac{\partial u}{\partial t}(r,t) + u(r,t)r = 0.$$

and

$$u(r,0) = 1, A \neq 0.$$

After solving the equation

$$\frac{\partial r}{\partial t}(r,t) = Ar + B, r(0) = E,$$

we get

$$\int_{r(0)}^{r(t)} \frac{dr}{Ar+B} = \int_0^t dt,$$

$$\frac{1}{A}\ln(Ar(t)+B) - \frac{1}{A}\ln(AE+B) = t,$$

$$r(t) = \frac{(AE+B)e^{At} - B}{A},$$

and

$$E = (r + \frac{B}{A})e^{-At} - \frac{B}{A}.$$

Let $U(t) = u(r(t), t)$,

$$\begin{aligned}&\frac{dU(t)}{dt}\\ &= \frac{\partial u}{\partial r}\frac{dr}{dt} + \frac{\partial u}{\partial t}\\ &= \frac{\partial u}{\partial r}(Ar+B) + \frac{\partial u}{\partial t}\\ &= -ur\end{aligned}$$

$$\frac{dU(t)}{dt} = -U\frac{(AE+B)e^{At} - B}{A}.$$

and

$$U(0) = u(r(0), 0) = u(E, 0) = 1.$$

Then we have,

$$U(t) = exp(\frac{AE + B}{A^2} e^{At} - \frac{B}{A}t - \frac{AE + B}{A^2}),$$

and

$$u(r, t) = exp(\frac{Ar + B}{A^2} - \frac{B}{A}t - \frac{(Ar + B)e^{-At}}{A^2}).$$

Thus,

$$\begin{aligned} &P(r,t) \\ &= exp\left(\frac{ar - ab + \sigma\phi}{a^2} - \frac{-ab + \sigma\phi}{a}(T - t)\right. \\ &\qquad \left. - \frac{(ar - ab + \sigma\phi)e^{-a(T-t)}}{a^2}\right). \end{aligned}$$

Suppose $\phi = 0$, we have

$$\begin{aligned} &P(r,t) \\ &= exp\left(\frac{r - b}{a} + b(T - t) - \frac{(r - b)e^{-a(T-t)}}{a}\right). \end{aligned}$$

That is, P is a function of r and t.

6 Conclusion

In this paper, we started from an assumption that the short interest rate follows fuzzy process instead of stochastic process, and proposed three equilibrium models. The fuzzy interest rate equation was derived for valuing zero-coupon bond. Lastly, analytic solution of the fuzzy term structure equation is given when the process for interest rate is the fuzzy counterparts of the Vasicek model.

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