Credibility Measure of Fuzzy Set and Applications

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Abstract

In order to measure fuzzy event, credibility measure is proposed as a non-additive set function. In this paper, this concept is extended to fuzzy sets. First, a mean measure is defined by Lebesgue integral and some properties are investigated such as the monotonicity theorem, self-duality theorem and subadditivity theorem. Furthermore, by using Sugeno integral, an equilibrium measure is proposed and studied. Finally, these concepts are applied to optimization problems, and the mean measure maximization model and equilibrium measure maximization model are proposed.

Keywords: Fuzzy set; Credibility measure; Sugeno integral.

1 Introduction

In order to measure the chance of a fuzzy event, Zadeh [10] defined a concept of possibility measure as a counterpart of probability measure in 1978. From then on, possibility theory was studied by many researchers such as Klir [3], Dubois and Prede [2]. In 1997, De Cooman [1] generalized the concept of possibility measure which takes values in the general lattice, and provided the basis for a measure- and integral-theoretic formulation of possibility theory. The necessity measure is defined as a dual part of possibility measure. However, both possibility measure and necessity measure are not self-dual. In order to get a self-duality measure, Liu and Liu [5] proposed a credibility measure as an average value of possibility measure and necessity measure in 2002, and Li and Liu [4] proved a sufficient and necessary condition for credibility measure. A good detail about credibility measure can be found in [6, 7].

As an extension of classical measure and non-additive measure, the concept of measure on fuzzy set was studied by Ralescu [9]. The purpose of this paper is to study the credibility measure on fuzzy set. For this purpose, we organize this paper as follows. Section 2 recalls some useful concepts and properties about fuzzy set and credibility measure, which are useful in the rest of this paper. In Section 3, we define the mean measure by Lebesgue integral and prove that it is increasing, self-duality and subadditivity. Section 4 proposes the equilibrium measure and the same properties are proved. As applications of these concepts, the mean measure maximization model and equilibrium measure maximization model are proposed in Section 5. At the end of this paper, a brief summary is given.

2 Preliminaries

In this section, we recalls some useful concepts about fuzzy set and credibility measure.

2.1 Fuzzy Set

A fuzzy subset $\widetilde{A}$ in a universal set U is characterized by a membership function $\mu_{\widetilde{A}}$ which associates with each element u in U a real

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number in the interval $[0,1]$. The value of the membership function at element u represents the "grade of membership" of u in $\widetilde{A}$. Thus, the nearer the value of $\mu_{\widetilde{A}}(u)$ is unity, the higher the grade of membership of u in $\widetilde{A}$. Hence, a fuzzy subset is uniquely determined by its membership function. Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy subsets with membership functions $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{B}}$, respectively. The contain relation $\widetilde{A} \subset \widetilde{B}$ is defined as $\mu_{\widetilde{A}}(u) \leq \mu_{\widetilde{B}}(u)$ for each $u \in U$. The union of $\widetilde{A}$ and $\widetilde{B}$ is determined by membership function

$$\mu_{\widetilde{A}\cup\widetilde{B}} = \mu_{\widetilde{A}} \vee \mu_{\widetilde{B}},$$

the intersection of $\widetilde{A}$ and $\widetilde{B}$ is determined by membership function

$$\mu_{\widetilde{A}\cap\widetilde{B}} = \mu_{\widetilde{A}} \wedge \mu_{\widetilde{B}},$$

and the complement of $\widetilde{A}$ is determined by the membership function

$$\mu_{\widetilde{A}^c} = 1 - \mu_{\widetilde{A}}.$$

For any $0 \leq \alpha \leq 1$, the α level set is defined as the crisp set $L_\alpha(\widetilde{A}) = \{u \in U | \mu_{\widetilde{A}}(u) \geq \alpha\}$. Let $\widetilde{A}$ be a fuzzy set with a upper semicontinuous normal membership function, Negoiţă and Ralescu [8] proved that (a) $L_0(\widetilde{A}) = U$; (b) $\alpha \leq \beta \Rightarrow L_\beta(\widetilde{A}) \subset L_\alpha(\widetilde{A})$; and (c) $L_\alpha(\widetilde{A}) = \bigcap_{\beta<\alpha} L_\beta(\widetilde{A})$. Conversely, if $\{L_\alpha | \alpha \in [0,1]\}$ is a family of crisp subsets of U with conditions (a), (b) and (c), the membership function defined by

$$\mu(u) = \sup\{0 < \alpha \leq 1 | u \in L_\alpha\} \tag{1}$$

is upper semicontinuous and normal with α level set L_α for each $0 \leq \alpha \leq 1$.

2.2 Credibility Measure

We denote Θ a nonempty set representing the sample space, and $\mathcal{P}$ the power set of Θ. A set function Pos on $\mathcal{P}$ is called a possibility measure if $\text{Pos}\{\Theta\} = 1$; $\text{Pos}\{\emptyset\} = 0$; and $\text{Pos}\{\cup_i A_i\} = \sup_i \text{Pos}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}$. The necessity measure for $A \in \mathcal{P}$ is defined as

$$\text{Nec}\{A\} = 1 - \text{Pos}\{A^c\}. \tag{2}$$

It is easy to prove that possibility measure and necessity measure are all not self-dual. However, a self-dual measure is important in both theory and practice. In order to get a self-dual measure, Liu and Liu [5] defined a credibility measure as

$$\text{Cr}\{A\} = \frac{1}{2}\left(\text{Pos}\{A\} + \text{Nec}\{A\}\right). \tag{3}$$

In 2006, Li and Liu [4] proved that possibility measure and credibility measure are uniquely determined by each other via (3). Furthermore, for any $A \in \mathcal{P}$, we have

$$\text{Pos}\{A\} = (2\text{Cr}\{A\}) \wedge 1. \tag{4}$$

Li and Liu [4] proved that a set function Cr is a credibility measure if and only if
(a) $\text{Cr}\{\Theta\} = 1$;
(b) $\text{Cr}\{A\} \leq \text{Cr}\{B\}$, whenever $A \subset B$;
(c) Cr is self-dual, i.e., $\text{Cr}\{A\} + \text{Cr}\{A^c\} = 1$, for any $A \in \mathcal{P}$;
(d) $\text{Cr}\{\cup_i A_i\} = \sup_i \text{Cr}\{A_i\}$ for any collection $\{A_i\}$ in $\mathcal{P}$ with $\sup_i \text{Cr}\{A_i\} < 0.5$.

3 Mean Measure

Let $(\Theta, \mathcal{P}, \text{Cr})$ be a credibility space, and $\widetilde{\mathcal{P}}$ the collection of all the fuzzy subsets of Θ.

Definition 3.1 *For any fuzzy subset $\tilde{A}$ with membership function $\mu_{\tilde{A}}$, the mean measure is defined as*

$$\widetilde{\text{Cr}}\{\tilde{A}\} = \int_0^1 \text{Cr}\left\{L_\alpha(\tilde{A})\right\} \text{d}\alpha. \tag{5}$$

Remark 3.1 For any fuzzy subset $\tilde{A}$ with membership function $\mu_{\tilde{A}}$, it follows from the (5) that

$$\widetilde{\text{Cr}}\{\tilde{A}\} = \int_0^1 \text{Cr}\left\{\theta \in \Theta | \mu_{\tilde{A}}(\theta) \geq \alpha\right\} \text{d}\alpha.$$

That is, the mean measure is the expected value of fuzzy variable $\mu_{\tilde{A}}$.

Remark 3.2 For any crisp set $A \in \mathcal{P}$, we have

$$\mu_A(\theta) = \begin{cases} 1, & \text{if } \theta \in A \\ 0, & \text{if } \theta \notin A. \end{cases} \tag{6}$$

It follows from (5)that $\widetilde{\text{Cr}}\{A\} = \text{Cr}\{A\}$. That is, the mean measure is consistent with the credibility measure.

Remark 3.3 It is easy to prove that $0 \leq \widetilde{\mathrm{Cr}}\{\tilde{A}\} \leq 1$ for any fuzzy subset $\tilde{A}$.

Theorem 3.1 *The mean measure is increasing. That is, for any fuzzy subsets $\tilde{A}_1 \subset \tilde{A}_2$, we have*

$$\widetilde{\mathrm{Cr}}\{\tilde{A}_1\} \leq \widetilde{\mathrm{Cr}}\{\tilde{A}_2\}. \tag{7}$$

Proof: Suppose that $\mu_{\tilde{A}_1}$ and $\mu_{\tilde{A}_2}$ are the membership functions of $\tilde{A}_1$ and $\tilde{A}_2$, respectively. Since $\tilde{A}_1 \subset \tilde{A}_2$, we have $\mu_{\tilde{A}_1} \leq \mu_{\tilde{A}_2}$. It follows from the definition that

$$\begin{aligned}\widetilde{\mathrm{Cr}}\{\tilde{A}_1\} &= \int_0^1 \mathrm{Cr}\{\theta \in \Theta | \mu_{\tilde{A}_1}(\theta) \geq \alpha\}\mathrm{d}\alpha \\ &\leq \int_0^1 \mathrm{Cr}\{\theta \in \Theta | \mu_{\tilde{A}_2}(\theta) \geq \alpha\}\mathrm{d}\alpha \\ &= \widetilde{\mathrm{Cr}}\{A_2\}.\end{aligned}$$

The proof is complete.

Theorem 3.2 *The mean measure is self-dual. That is, for any $\tilde{A} \in \widetilde{\mathcal{P}}$, we have*

$$\widetilde{\mathrm{Cr}}\{\tilde{A}\} + \widetilde{\mathrm{Cr}}\{\tilde{A}^c\} = 1. \tag{8}$$

Proof: suppose that $\mu_{\tilde{A}}$ is the membership function of $\tilde{A}$, then the membership function of $\tilde{A}^c$ is $1 - \mu_{\tilde{A}^c}$. It follows from the definition that

$$\begin{aligned}\widetilde{\mathrm{Cr}}\{A^c\} &= \int_0^1 \mathrm{Cr}\{\theta \in \Theta | 1 - \mu_{\tilde{A}}(\theta) \geq \alpha\}\mathrm{d}\alpha \\ &= \int_0^1 \mathrm{Cr}\{\theta \in \Theta | \mu_{\tilde{A}}(\theta) < 1 - \alpha\}\mathrm{d}\alpha \\ &= \int_0^1 \mathrm{Cr}\{\theta \in \Theta | \mu_{\tilde{A}}(\theta) < \beta\}\mathrm{d}\beta \\ &= 1 - \int_0^1 \mathrm{Cr}\{\theta \in \Theta | \mu_{\tilde{A}}(\theta) \geq \beta\}\mathrm{d}\beta \\ &= 1 - \widetilde{\mathrm{Cr}}\{A\}.\end{aligned}$$

That is, $\widetilde{\mathrm{Cr}}\{\tilde{A}\} + \widetilde{\mathrm{Cr}}\{\tilde{A}^c\} = 1$. The proof is complete.

Theorem 3.3 *The mean measure is subadditivity. That is, for any $\tilde{A}_1, \tilde{A}_2 \in \widetilde{\mathcal{P}}$, we have*

$$\widetilde{\mathrm{Cr}}\{\tilde{A}_1 \cup \tilde{A}_2\} \leq \widetilde{\mathrm{Cr}}\{\tilde{A}_1\} + \widetilde{\mathrm{Cr}}\{\tilde{A}_2\}. \tag{9}$$

Proof: Suppose that μ_1 and μ_2 are the membership functions of $\tilde{A}_1$ and $\tilde{A}_2$, respectively. It follows from the definition that

$$\begin{aligned}&\widetilde{\mathrm{Cr}}\{\tilde{A}_1 \cup \tilde{A}_2\} \\ &= \int_0^1 \mathrm{Cr}\{\mu_{\tilde{A}_1} \vee \mu_{\tilde{A}_2} \geq \alpha\}\mathrm{d}\alpha \\ &= \int_0^1 \mathrm{Cr}\{\{\mu_{\tilde{A}_1} \geq \alpha\} \cup \{\mu_{\tilde{A}_2} \geq \alpha\}\}\mathrm{d}\alpha \\ &\leq \int_0^1 \mathrm{Cr}\{\mu_{\tilde{A}_1} \geq \alpha\} + \mathrm{Cr}\{\mu_{\tilde{A}_2} \geq \alpha\}\mathrm{d}\alpha \\ &= \widetilde{\mathrm{Cr}}\{\tilde{A}_1\} + \widetilde{\mathrm{Cr}}\{\tilde{A}_2\}.\end{aligned}$$

The proof is complete.

Remark 3.4 In fact, we may prove that the mean measure $\widetilde{\mathrm{Cr}}$ is also countable subadditivity.

4 Equilibrium Measure

Definition 4.1 *For any fuzzy subset $\tilde{A}$ with membership function $\mu_{\tilde{A}}$, the equilibrium measure is defined as*

$$\widehat{\mathrm{Cr}}\{\tilde{A}\} = \sup_{0 \leq \alpha \leq 1} \left\{\mathrm{Cr}\{L_\alpha(\tilde{A})\} \wedge \alpha\right\}. \tag{10}$$

Remark 4.1 It is clear that the equilibrium measure is the fixed point of distribution function $\mathrm{Cr}\{\mu_{\tilde{A}} \geq \alpha\}$.

Remark 4.2 For any crisp set $A \in \Theta$, we have

$$\mu_A(\theta) = \begin{cases} 1, & \text{if } \theta \in A \\ 0, & \text{if } \theta \notin A. \end{cases} \tag{11}$$

It follows from the definition that $\widehat{\mathrm{Cr}}\{A\} = \mathrm{Cr}\{A\}$. That is, the equilibrium measure is also consistent with the credibility measure.

Remark 4.3 *It is easy to prove that $0 \leq \widehat{\mathrm{Cr}}\{\tilde{A}\} \leq 1$ for each fuzzy subset $\tilde{A}$.*

Theorem 4.1 *For any fuzzy set $\tilde{A}$ with membership function $\mu_{\tilde{A}}$, we have*

$$\widehat{\mathrm{Cr}}\{\tilde{A}\} = \inf_{0 \leq \alpha \leq 1} \left(\mathrm{Cr}\{L_\alpha(\tilde{A})\} \vee \alpha\right). \tag{12}$$

Proof: It is easy to prove that

$$\begin{aligned}&\sup_{0 \leq \alpha \leq 1} \{\mathrm{Cr}\{L_\alpha(\tilde{A})\} \wedge \alpha\} \\ &= \sup_{0 \leq \alpha \leq 1} \{\alpha | \mathrm{Cr}\{L_\alpha(\tilde{A})\} \geq \alpha\} \vee\end{aligned}$$

$$\sup_{0\le\alpha\le 1}\{\mathrm{Cr}\{L_\alpha(\tilde{A})\}|\mathrm{Cr}\{L_\alpha(\tilde{A})\}<\alpha\}$$
$$\ge \sup_{0\le\alpha\le 1}\{\alpha|\mathrm{Cr}\{L_\alpha(\tilde{A})\},$$

$$\inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{L_\alpha(\tilde{A})\}\vee\alpha\}$$
$$= \inf_{0\le\alpha\le 1}\{\alpha|\mathrm{Cr}\{L_\alpha(\tilde{A})\}<\alpha\}\vee$$
$$\inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{L_\alpha(\tilde{A})\}|\mathrm{Cr}\{L_\alpha(\tilde{A})\}\ge\alpha\}$$
$$\le \inf_{0\le\alpha\le 1}\{\alpha|\mathrm{Cr}\{L_\alpha(\tilde{A})\}<\alpha\}.$$

Denote

$$\beta=\sup\{0\le\alpha\le 1|\mathrm{Cr}\{L_\alpha(\tilde{A})\}\ge\alpha\},$$

and

$$\delta=\inf\{0\le\alpha\le 1|\mathrm{Cr}\{L_\alpha(\tilde{A})\}<\alpha\}.$$

Assume $\beta_i\uparrow\beta$ and $\delta_i\downarrow\delta$. Since $\mathrm{Cr}\{\mu_{\tilde{A}}\ge\beta_i\}\ge\beta_i$, we have $\beta_i\le\delta$, letting $i\to\infty$, we get $\beta\le\delta$. For any n, since $\mathrm{Cr}\{\mu_{\tilde{A}}\ge\delta-1/n\}\ge\delta-1/n$, we have $\delta-1/n\le\beta$, letting $n\to\infty$, we get $\delta\le\beta$. That is,

$$\sup_{0\le\alpha\le 1}\{\alpha|\mathrm{Cr}\{L_\alpha(\tilde{A})\}\ge\alpha\}$$
$$=\inf_{0\le\alpha\le 1}\{\alpha|\mathrm{Cr}\{L_\alpha(\tilde{A})\}<\alpha\}.$$

Then we have

$$\sup_{0\le\alpha\le 1}\{\mathrm{Cr}\{L_\alpha(\tilde{A})\}\wedge\alpha\}$$
$$\ge \inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{L_\alpha(\tilde{A})\}\vee\alpha\}. \quad (13)$$

On the other hand, we have $\mathrm{Cr}\{L_\alpha(\tilde{A})\}\wedge\alpha\le\mathrm{Cr}\{L_\alpha(\tilde{A})\}\vee\beta$ for any $\alpha,\beta\in[0,1]$ because if $\alpha\le\beta$, we have $\mathrm{Cr}\{L_\alpha(\tilde{A})\}\wedge\alpha\le\alpha\le\beta\le\mathrm{Cr}\{L_\beta(\tilde{A})\}\vee\beta$, if $\alpha>\beta$, we have $\mathrm{Cr}\{L_\alpha(\tilde{A})\}\le\mathrm{Cr}\{L_\beta(\tilde{A})\}$ and

$$\mathrm{Cr}\{L_\alpha(\tilde{A})\}\wedge\alpha\le\mathrm{Cr}\{L_\beta(\tilde{A})\}\vee\beta.$$

Taking supremum about α and taking infimum with respect to β, we get

$$\sup_{0\le\alpha\le 1}(\mathrm{Cr}\{L_\alpha(\tilde{A})\}\wedge\alpha)$$
$$\le \inf_{0\le\alpha\le 1}(\mathrm{Cr}\{L_\alpha(\tilde{A})\}\vee\alpha). \quad (14)$$

It follows from (13) and (14) that

$$\widehat{\mathrm{Cr}}\{\tilde{A}\}=\inf_{0\le\alpha\le 1}(\mathrm{Cr}\{L_\alpha(\tilde{A})\}\vee\alpha).$$

The proof is complete.

Theorem 4.2 *For any fuzzy subset $\tilde{A}$ with membership function $\mu_{\tilde{A}}$, we have*

$$\widehat{\mathrm{Cr}}\{\tilde{A}\}=\inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{\mu_{\tilde{A}}>\alpha\}\vee\alpha\}. \quad (15)$$

Proof: This theorem suffers to prove

$$\inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{\mu_{\tilde{A}}\ge\alpha\}\vee\alpha\}$$
$$=\inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{\mu_{\tilde{A}}>\alpha\}\vee\alpha\}. \quad (16)$$

If $\inf\{0\le\alpha\le 1|\mathrm{Cr}\{\mu_{\tilde{A}}>\alpha\}\vee\alpha\}=1$, it is clear that (16) holds. Otherwise, we have $\inf\{0\le\alpha\le 1|\mathrm{Cr}\{\mu_{\tilde{A}}>\alpha\}\vee\alpha\}<1$, for sufficient large positive integer n, let β be a point of $[0,1)$ such that

$$\mathrm{Cr}\{\mu_{\tilde{A}}>\beta\}\vee\beta$$
$$\le \inf_{0\le\alpha\le 1}(\mathrm{Cr}\{\mu_{\tilde{A}}>\alpha\}\vee\alpha)+1/n$$
$$<1-1/n.$$

Then we have $\beta+1/n<1$ and

$$\mathrm{Cr}\{\mu_{\tilde{A}}>\beta\}\vee\beta$$
$$\ge\mathrm{Cr}\{\mu_{\tilde{A}}\ge\beta+1/n\}\vee\beta$$
$$\ge\mathrm{Cr}\{\mu_{\tilde{A}}\ge\beta+1/n\}\vee(\beta+1/n)-1/n$$
$$\ge\inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{\mu_{\tilde{A}}\ge\alpha\}\vee\alpha\}-1/n,$$

$$\inf_{0\le\alpha\le 1}(\mathrm{Cr}\{\mu_{\tilde{A}}>\alpha\}\vee\alpha)$$
$$\ge\mathrm{Cr}\{\mu_{\tilde{A}}>\beta\}\vee\beta-1/n$$
$$\ge\inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{\mu_{\tilde{A}}\ge\alpha\}\vee\alpha\}-2/n.$$

Letting $n\to\infty$, we get

$$\inf_{0\le\alpha\le 1}(\mathrm{Cr}\{\mu_{\tilde{A}}>\alpha\}\vee\alpha)$$
$$\ge\inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{\mu_{\tilde{A}}\ge\alpha\}\vee\alpha\}. \quad (17)$$

On the other hand, it is easy to prove that

$$\inf_{0\le\alpha\le 1}(\mathrm{Cr}\{\mu_{\tilde{A}}>\alpha\}\vee\alpha)$$
$$\le\inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{\mu_{\tilde{A}}\ge\alpha\}\vee\alpha\}. \quad (18)$$

It follows from (17) and (18) that

$$\widehat{\mathrm{Cr}}\{\tilde{A}\}=\inf_{0\le\alpha\le 1}\{\mathrm{Cr}\{\mu_{\tilde{A}}>\alpha\}\vee\alpha\}.$$

The proof is complete.

Theorem 4.3 *The equilibrium measure is increasing. That is, for any $\tilde{A} \subset \tilde{B}$, we have*

$$\widehat{\mathrm{Cr}}\{\tilde{A}\} \leq \widehat{\mathrm{Cr}}\{\tilde{B}\}. \tag{19}$$

Proof: Let $\mu_{\tilde{A}}$ and $\mu_{\tilde{B}}$ be the membership functions of $\tilde{A}$ and $\tilde{B}$. Since $\tilde{A} \subset \tilde{B}$, we have $\mu_{\tilde{A}} \leq \mu_{\tilde{B}}$ and $\mathrm{Cr}\{\mu_{\tilde{A}} \geq \alpha\} \wedge \alpha \leq \mathrm{Cr}\{\mu_{\tilde{B}} \geq \alpha\} \wedge \alpha$ for each $0 \leq \alpha \leq 1$. It follows from the definition that

$$\begin{aligned}\widehat{\mathrm{Cr}}\{\tilde{A}\} &= \sup_{0\leq\alpha\leq1} (\mathrm{Cr}\{\mu_{\tilde{A}} \geq \alpha\} \wedge \alpha)\\ &\leq \sup_{0\leq\alpha\leq1} (\mathrm{Cr}\{\mu_{\tilde{B}} \geq \alpha\} \wedge \alpha)\\ &= \widehat{\mathrm{Cr}}\{\tilde{B}\}.\end{aligned}$$

The proof is complete.

Theorem 4.4 *The equilibrium measure is self-dual. That is, for any $\tilde{A} \in \widetilde{\mathcal{P}}$, we have*

$$\widehat{\mathrm{Cr}}\{\tilde{A}\} + \widehat{\mathrm{Cr}}\{\tilde{A}^c\} = 1. \tag{20}$$

Proof: Assume that $\mu_{\tilde{A}}$ is the membership function of $\tilde{A}$. Then we have

$$\begin{aligned}\widehat{\mathrm{Cr}}\{\tilde{A}^c\} &= \sup_{0\leq\alpha\leq1} (\mathrm{Cr}\{1-\mu_{\tilde{A}} \geq \alpha\} \wedge \alpha)\\ &= \sup_{0\leq\alpha\leq1} (\mathrm{Cr}\{\mu_{\tilde{A}} \leq 1-\alpha\} \wedge \alpha)\\ &= \sup_{0\leq\beta\leq1} (\mathrm{Cr}\{\mu_{\tilde{A}} \leq \beta\} \wedge (1-\beta))\\ &= \sup_{0\leq\beta\leq1} ((1-\mathrm{Cr}\{\mu_{\tilde{A}} > \beta\}) \wedge (1-\beta))\\ &= 1 - \inf_{0\leq\beta\leq1} (\mathrm{Cr}\{\mu_{\tilde{A}} > \beta\} \vee \beta)\\ &= 1 - \widehat{\mathrm{Cr}}\{\tilde{A}\}.\end{aligned}$$

That is, $\widehat{\mathrm{Cr}}\{\tilde{A}\} + \widehat{\mathrm{Cr}}\{\tilde{A}^c\} = 1$. The proof is complete.

Theorem 4.5 *The equilibrium measure is subadditivity. That is, for any $\tilde{A}_1, \tilde{A}_2 \in \widetilde{\mathcal{P}}$, we have*

$$\widehat{\mathrm{Cr}}\{\tilde{A}_1 \cup \tilde{A}_2\} \leq \widehat{\mathrm{Cr}}\{\tilde{A}_1\} + \widehat{\mathrm{Cr}}\{\tilde{A}_2\}. \tag{21}$$

Proof: Suppose that $\mu_{\tilde{A}_1}$ and $\mu_{\tilde{A}_1}$ are the membership functions of $\tilde{A}_1$ and $\tilde{A}_2$, respectively. It follows from the definition that

$$\begin{aligned}&\widehat{\mathrm{Cr}}\{\tilde{A}_1 \cup \tilde{A}_2\}\\ &= \sup_{0\leq\alpha\leq1} (\mathrm{Cr}\{\mu_{\tilde{A}_1} \vee \mu_{\tilde{A}_2} \geq \alpha\} \wedge \alpha)\\ &= \sup_{0\leq\alpha\leq1} (\mathrm{Cr}\{\{\mu_{\tilde{A}_1} \geq \alpha\} \cup \{\mu_{\tilde{A}_2} \geq \alpha\}\} \wedge \alpha)\\ &\leq \sup_{0\leq\alpha\leq1} ((\mathrm{Cr}\{\mu_{\tilde{A}_1} \geq \alpha\} +\\ &\quad \mathrm{Cr}\{\mu_{\tilde{A}_2} \geq \alpha\}) \wedge \alpha)\\ &\leq \sup_{0\leq\alpha\leq1} (\mathrm{Cr}\{\mu_{\tilde{A}_1} \geq \alpha\} \wedge \alpha) +\\ &\quad \sup_{0\leq\alpha\leq1} (\mathrm{Cr}\{\mu_{\tilde{A}_2} \geq \alpha\} \wedge \alpha)\\ &= \widehat{\mathrm{Cr}}\{\tilde{A}_1\} + \widehat{\mathrm{Cr}}\{\tilde{A}_2\}.\end{aligned}$$

The proof is complete.

Remark 4.4 In fact, we may prove that the equilibrium measure $\widehat{\mathrm{Cr}}$ is also countable subadditivity.

5 Applications to Fuzzy Optimization Problems

Suppose that ξ is a fuzzy variable from credibility space $(\Theta, \mathcal{P}, \mathrm{Cr})$. For any upper semicontinuous normal fuzzy subset $\widetilde{B}$ of $\Re$, define fuzzy set $\xi^{-1}(\tilde{B})$ by the α-cut sets

$$L_\alpha\left(\xi^{-1}(\widetilde{B})\right) = \xi^{-1}\left(L_\alpha(\widetilde{B})\right).$$

Then we get

$$\mu_{\xi^{-1}(\widetilde{B})}(\theta) = \mu_{\widetilde{B}}(\xi(\theta)).$$

Suppose that ξ is a fuzzy variable with membership function μ and $f(x, y)$ is a two dimensional function. For any fuzzy subset $\widetilde{B}$ of $\Re$ and fixed point x, the inverse image set $(f(x, \xi))^{-1}(\widetilde{B})$ is a fuzzy subset of $\Re$ with membership function

$$\mu_{(f(x,\xi))^{-1}(\widetilde{B})} = \mu_{\widetilde{B}}(f(x, y)), \quad y \in \Re.$$

Then the mean measure of $(f(x, \xi))^{-1}(\widetilde{B})$ is

$$\int_0^1 \mathrm{Cr}\{y \in \Re | \mu_{\widetilde{B}}(f(x, y)) \geq \alpha\} \mathrm{d}\alpha,$$

and equilibrium measure of $(f(x, \xi))^{-1}(\widetilde{B})$ is

$$\sup_{0\leq\alpha\leq1} \left(\mathrm{Cr}\{y \in \Re | \mu_{\widetilde{B}}(f(x, y)) \geq \alpha\} \wedge \alpha\right).$$

Let $f(x, \xi)$ and $g_j(x, \xi)$ be functions from $\Re^2$ to $\Re$, where ξ is a fuzzy variable. If f is the profit function of an investment project and $g_j(x,\xi)$ are the constraint functions for $j = 1, 2, \cdots$ The credibility measure maximization model is

$$\begin{cases} \max\limits_x \mathrm{Cr}\{f(x,\xi) \in A\} \\ \text{subject to:} \\ \quad \mathrm{Cr}\{g_j(x,\xi) \in A_j, j = 1, \cdots, p\} \geq \beta \end{cases} \tag{22}$$

where A and A_j for $j = 1, 2, \cdots$ are crisp subsets of $\Re$ and β is a given predetermined confidence level. For example, if we take $B = [10^6, \infty)$, the objective is to maximize the credibility that the profit is more than or equal to 10^6. However, if A and A_j for $j = 1, 2, \cdots$ are fuzzy subsets instead of crisp sets, in this case, we can define the following mean measure maximization model

$$\begin{cases} \max\limits_x \widetilde{\mathrm{Cr}}\{f(x,\xi) \in \widetilde{B}\} \\ \text{subject to:} \\ \quad \widetilde{\mathrm{Cr}}\{g_j(x,\xi) \in \widetilde{A}_j, j = 1, \cdots, p\} \geq \beta \end{cases} \tag{23}$$

where $\widetilde{A}$ and $\widetilde{A}_j$ for $j = 1, 2, \cdots$ are fuzzy subsets of $\Re$ and β is a given predetermined confidence level. For example, if we take $\widetilde{B} =$ "about 10^6", the objective is to maximize the mean measure that the profit is about 10^6. If we use equilibrium measure instead of mean measure, we get the following equilibrium measure maximization model

$$\begin{cases} \max\limits_x \widehat{\mathrm{Cr}}\{f(x,\xi) \in \widetilde{B}\} \\ \text{subject to:} \\ \quad \widehat{\mathrm{Cr}}\{g_j(x,\xi) \in \widetilde{A}_j, j = 1, \cdots, p\} \geq \beta \end{cases} \tag{24}$$

where $\widetilde{A}$ and $\widetilde{A}_j$ for $j = 1, 2, \cdots$ are fuzzy subsets of $\Re$ and β is a given predetermined confidence level. For example, if we take $\widetilde{B} =$ "about 10^6", the objective is to maximize the equilibrium measure that the profit is about 10^6.

6 Conclusions

In this paper, the mean measure and equilibrium measure for fuzzy subset were defined as extensions of credibility measure. Furthermore, some useful properties about were discussed such as monotonicity theorem, self-duality theorem and subadditivity theorem. Finally, these concepts were applied to fuzzy optimization problems, and the mean measure maximization model and equilibrium measure maximization model were proposed.

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