On Risk-shifting Incentive Problem Based on Option Approach

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Abstract

A typical problem in financial contracting is the so-called risk-shifting problem. In this paper, we analyze the risk-shifting problem using a principal-agent framework in which the principal lends money to the agent for a finite time of period. Extending the basic intuition of early models that convexity in the borrower's payoff is responsible for risk-shifting, a contract avoiding risk-shifting is developed. In particular, we use plural assets and the sum of the all assets prices are used along with basket option approach which extends Ziegler's results.

Keywords: Incentive problem, Option approach, Game theory, Risk-shifting problem, Renegotiation-proof contract, Financing decision.

1 Introduction

In financial contracting, two typical forms of moral hazard exist, each giving rise to specific incentive issues. These problems may be very closely related to each other. One is that when the behavior of agent is not observed by principal the agent may break contract and would have incentive to invest the project which gives higher return with risk after the agent borrowed the money. This problem is the so-called risk-shifting problem and was first studied by Jensen and Meckling[5]. The other is called observability problem for which the borrower is the only person that can observe project returns at no cost. His promised payment depends positively on realized project return and he might have an incentive to underestimate project return in order to reduce his payment to the lender. This problem usually arises in the context of a lender-borrower relationship since there exist asymmetry information between them. The borrower might try to influence the return distribution of the project to increase his payoff at the expense of the lender.

These incentive problems inflict the principal a loss, who does not stand at advantage and produce social faults. In a serious situation principal would not have a contract with agent, causing financial activity and market to be withered.

Lenders that are aware of the fact that borrowers have an incentive to increase the risk of their projects can use several ways to solve [1].

In this paper, we analyze the risk-shifting problem using a principal-agent framework in which the principal lends money to the agent for a finite time of period. Extending the basic intuition of early models that convexity in the borrower's payoff is responsible for risk-shifting, a contract avoiding risk-shifting is developed. In particular, we use plural assets and the sum of the all assets prices are used along with basket option approach, which extends Ziegler's results.

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2 Risk-Shifting Problem

Consider a financial intermediary that lends money to a borrower for investment in several projects that are available only to the borrower. Assume that the lender cannot observe the borrower's project choice and hence cannot assess the risk of the projects. At initial time, each of all projects has different prices $S_i\ (i=1,...,n)$ with different risks. Assume also that the borrower can, at any time change his mind and replace some projects with others at no cost. At any time, the agent can choose to invest all funds to some of a series of projects, $i=1,...,n$, The value of the borrower's assets is assumed to follow the usual geometric Brownian motion

$$dS_i = \mu_i S_i dt + \sigma_i S_i dB_{i,t} \ (i=1,....,n). \quad (1)$$

We assume that all projects have a finite life T and random terminal values, $\overline{S}_i\ (i=1,...,n)$ are observable by both the lender and the borrower. The lender pays an amount of D_0 to the borrower in exchange for a promise by the lender to pay him $f\left(\sum_{i=1}^{n} a_i \overline{S}\right)$ at time T. $a_i\ (i=1,...,n)$ indicate the weights, in other words, which imply the number of unit for investing each asset. Then, $f\left(\sum_{i=1}^{n} a_i \overline{S}\right)$ is contingent payment for final sums of values, $\sum_{i=1}^{n} a_i \overline{S}$. Each weight, c_i to invest for each asset is described,

$$c_i = a_i S_i \Big/ \sum_{i=1}^{n} a_i S_i \ , (i=1,...,n). \quad (2)$$

Assume that there exists some correlation ρ_{ij} between $i\ th$ asset and jth asset

$$dB_{i,t} dB_{j,t} = \rho_{ij} dt . \quad (3)$$

After the financing contract is signed, the borrower chooses an investment project. At any time during the life of the contract, he can switch to another project involving higher of lower risk or to a newly weighted sums of several assets. For example, the borrower may sell ith asset, and with the money he could invest to higher or lower asset. In this context, the borrower can choose one or more assets with higher risk at any time t, which make the borrower bring some profit and at the same time make decrease the amount of payment to the lender. But, the lender can not observe the borrower's incentive behavior.

In order to discuss the influence of the risk-shifting problem we assume that all assets have the same maturity period T ,and the return on the project $\overline{S}_i$ is observed by the lender and the borrower when the contract expires at time T. Also, the lender and the borrower agree on a single, end-of-period contingent payment to the principal, $f\left(\sum_{i=1}^{n} a_i \overline{S}\right)$.

Suppose that lender and borrower have no other assets and limited liability. Then, the effective payment of the borrower to the lender at time T, whatever has been agreed upon, is given

$$\Pi_L\left(\sum_{i=1}^{n} a_i \overline{S}_i\right) = \min\left[\sum_{i=1}^{n} a_i \overline{S}_i , f\left(\sum_{i1}^{n} a_i \overline{S}_i\right)\right]. \quad (4)$$

On the other hand, the payoff to the borrower equals the difference between total project return and the amount paid out to the lender,

$$\begin{aligned}\Pi_B\left(\sum_{i=1}^{n} a_i \overline{S}_i\right) &= \sum_{i=1}^{n} a_i \overline{S}_i - \Pi_L\left(\sum_{i=1}^{n} a_i \overline{S}_i\right) \\ &= \max\left[0, \sum_{i=1}^{n} a_i \overline{S}_i - f\left(\sum_{i=1}^{n} a_i \overline{S}_i\right)\right] .\end{aligned} \quad (5)$$

Thus, we can summarize the structure of this game as below. First, the financing contract is signed. The lender pays an amount of D_0 to the borrower in exchange for a promise by the lender to pay him $f\left(\sum_{i=1}^{n} a_i \overline{S}\right)$ at the T. After receiving the money from the lender, the borrower invests in a projects and if he wishes, costlessly switch to a projects involving more or less risk at any time. Finally, when the contract expires at time T, the return on the project

$\sum_{i=1}^{n} a_i \overline{S}$ is observed by both lender and borrower pays $\min\left[\sum_{i=1}^{n} a_i \overline{S},\, f\left(\sum_{i=1}^{n} a_i \overline{S}_i\right)\right]$ to the lender.

3 Profit-Sharing Contract

We look at profit-sharing rules avoiding risk-shifting. The sharing rule showing in Fig.1 attributes everything to the lender up to a project return of X_1, plus half of any return in excess of X_2.

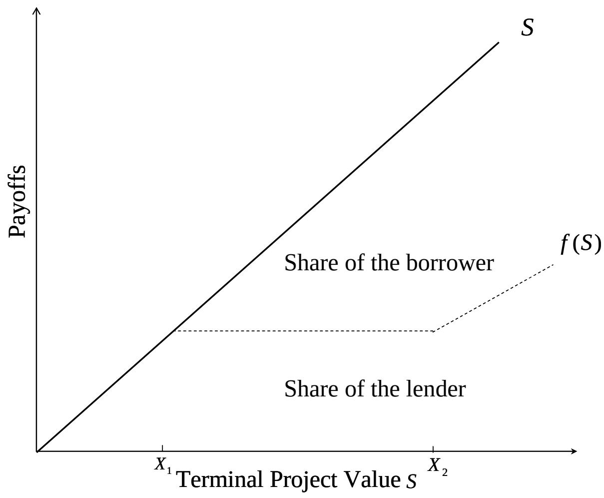

Figure1. Example of a feasible profit-sharing rule between lender and borrower

The lender and the borrower can agree on any payment, i.e. sharing rule. The sharing rule can be characterized as follows. A fixed payment from lender to borrower, D, α basket put options with strike price, X_1 and β basket call options with strike price, X_2. Thus, in our case, the underlying asset for option is described as the sum of n assets,

$$\Pi_L\left(\sum_{i=1}^{n} a_i \overline{S}_i\right) = D + \alpha \max\left[X_1 - \sum_{i=1}^{n} a_i \overline{S}_i\ ,\ 0\right] + \beta \max\left[\sum_{i=1}^{n} a_i \overline{S}_i - X_2\ ,\ 0\right] \quad . \quad (6)$$

The sharing rule attributes everything to the lender up to project returns of X_1,plus half of any returns in excess of X_2.

Concerning Fig.2 there are three cases considered, depending on the size of the final value of $\sum_{i=1}^{n} a_i \overline{S}$ and the strike prices.

1) $\sum_{i=1}^{n} a_i \overline{S} < X_1$;

When the final total value is less than the strike price of put option the put option is exercised, and then the payoff to lender is described as $D + \alpha\left(X_1 - \sum_{i=1}^{n} a_i \overline{S}\right)$.

2) $X_1 < \sum_{i=1}^{n} a_i \overline{S} < X_2$;

When the final total value exists between the strike price of put option and the strike price of call option none of the option is not exercised, so that the payoff to lender is a fixed payment, D.

3) $\sum_{i=1}^{n} a_i \overline{S} > X_2$;

When the final total value is greater than the strike price of call option is exercised, and the payoff to lender is described as $D + \beta\left(\sum_{i=1}^{n} a_i \overline{S} - X_2\right)$.

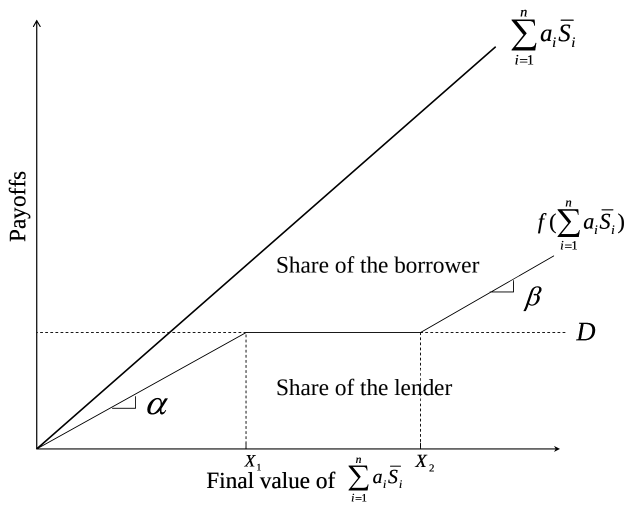

Figure2. Game theoretic structure with plural assets

4 Developing an Incentive Contract

We discuss the properties that avoid risk-shifting as in Fig. 1. The first step is to determine the value of the payment to the principal and the agent using option pricing. From the structure of the contract above the current value of the payoff to the lender can be calculated as

$$\Pi_L = e^{-r\tau} D + \alpha V_P(X_1) + \beta V_C(X_2) \,. \qquad (7)$$

where r denotes the riskless interest rate, τ the remaining life of the loan, and V_P and V_C for the Black-Scholes put and call option values with an exercise price of X_1, X_2, respectively. The evaluation formulae are given

$$V_P(X_1) = \sum_{i=1}^{n} a_i S_i \left\{ -\exp\left(\frac{1}{2}\tau\left(\sigma_x^2 - \sum_{i=1}^{n} c_i \sigma_i^2\right)\right) \times N\left(-d_1 - \sigma_x\sqrt{\tau}\right) + \tilde{K}_1 N(-d_1) \right\} \quad , \quad (8)$$

$$V_c(X_2) = \sum_{i=1}^{n} a_i S_i \left\{ \exp\left(\frac{1}{2}\tau\left(\sigma_x^2 - \sum_{i=1}^{n} c_i \sigma_i^2\right)\right) \times N\left(d_2 + \sigma_x\sqrt{\tau}\right) - \tilde{K}_2 N(d_2) \right\} \quad . \quad (9)$$

where $d_1, d_2, \tilde{K}_1, \tilde{K}_2, \sigma_x$ are defined below.

$$d_g = -\frac{\log \tilde{K}_g + \frac{1}{2}\tau \sum_{i=1}^{n} c_i \sigma_i^2}{\sigma_x \sqrt{\tau}} \quad , \qquad (10)$$

$$\tilde{K}_g = \frac{X_g e^{-r\tau}}{\sum_{i=1}^{n} a_i S_i + \exp\left(\frac{1}{2}\tau\left(\sigma_x^2 - \sum_{i=1}^{n} c_i \sigma_i^2\right)\right)} - 1 \qquad g \in \{1, 2\} \quad , \qquad (11)$$

$$\sigma_x^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} c_i c_j \sigma_i \sigma_j \quad . \qquad (12)$$

and $N(\bullet)$ denotes the cumulative standard normal distribution function.

On the other hand, the current value of the payoff to the borrower is described

$$\begin{aligned} \Pi_B &= \sum_{i=1}^{n} a_i S_i - \Pi_L \\ &= \sum_{i=1}^{n} a_i S_i - e^{-r\tau} D \qquad (13) \\ &\quad - \alpha V_p(X_1) - \beta V_C(X_2) \quad . \end{aligned}$$

In order to avoid risk-shifting problem, the values of α, β, D, X_1 and X_2 must be determined so that the borrower does not have any incentive to influence the risk of the projects. We note that arbitrage-free concept comes into our discussion and hence the borrower's payoff should be independent of each of asset risks, σ_1, σ_2. Therefore, we have

$$\frac{\partial \Pi_B}{\partial \sigma_i} = -\alpha \frac{\partial V_P(X_1)}{\partial \sigma_i} - \beta \frac{\partial V_C(X_2)}{\partial \sigma_i} = 0 \,. \qquad (14)$$

where

$$\frac{\partial V_P(X_1)}{\partial \sigma_i} = \sum_{i=1}^{n} a_i S_i \left\{ A B_i \left(N\left(d_1 + \sigma_x\sqrt{\tau}\right) - N(d_1) \right) + \tilde{K}_1 e^{-d_1^2/2} C_i \right\} \quad , \qquad (15)$$

$$\frac{\partial V_C(X_2)}{\partial \sigma_i} = \sum_{i=1}^{n} a_i S_i \left\{ A B_i \left(N\left(d_2 + \sigma_x\sqrt{\tau}\right) - N(d_2) \right) + \tilde{K}_2 e^{-d_2^2/2} C_i \right\} \quad . \qquad (16)$$

and A, B_i, C_i are described as

$$A = \exp\left(\frac{1}{2}\tau\left(\sigma_x^2 - \sum_{i=1}^{n} c_i \sigma_i^2\right)\right) \quad , \qquad (17)$$

$$B_i = \tau \left(c_i \sum_{j=1}^{n} \rho_{ij} c_j \sigma_j - c_i \sigma_i \right) \quad , \tag{18}$$

$$C_i = \frac{c_i \sum_{j=1}^{n} \rho_{ij} c_j \sigma_j \sqrt{\tau}}{\sqrt{2\pi}\sigma_x} \quad . \tag{19}$$

Thus, (14) can be expressed as

$$\frac{\partial \Pi_B}{\partial \sigma_i} = -\alpha \frac{\partial V_P(X_1)}{\partial \sigma_i} - \beta \frac{\partial V_C(X_2)}{\partial \sigma_i}$$
$$= -\sum_{i=1}^{n} a_i S_i \left\{ \; AB_i \left\{ \alpha \left(N\left(d_1 + \sigma_x \sqrt{\tau}\right) - N(d_1) \right) + \beta \left(N\left(d_2 + \sigma_x \sqrt{\tau}\right) - N(d_2) \right) \right\} \right.$$
$$\left. + C_i \left(\alpha \tilde{K}_1 e^{-d_1^2/2} + \beta \tilde{K}_2 e^{-d_2^2/2} \right) \right\} = 0 \,. \tag{20}$$

Let restrict our discussion of risk-shifting problem to the case of two assets. In order for the borrower not to have risk-shifting incentive for respective asset risk σ_1, σ_2 the following expressions must be satisfied.

$$\frac{\partial \Pi_B}{\partial \sigma_1} = -(a_1 S_1 + a_2 S_2) \left\{ \; A' B_1' \left\{ \alpha \left(N\left(d_1 + \sigma_x \sqrt{\tau}\right) - N(d_1) \right) + \beta \left(N\left(d_2 + \sigma_x \sqrt{\tau}\right) - N(d_2) \right) \right\} \right.$$
$$\left. + C_1' \left(\alpha \tilde{K}_1 e^{-d_1^2/2} + \beta \tilde{K}_2 e^{-d_2^2/2} \right) \right\} = 0 \,, \tag{21}$$

$$\frac{\partial \Pi_B}{\partial \sigma_2} = -(a_1 S_1 + a_2 S_2) \left\{ \; A' B_2' \left\{ \alpha \left(N\left(d_1 + \sigma_x \sqrt{\tau}\right) - N(d_1) \right) + \beta \left(N\left(d_2 + \sigma_x \sqrt{\tau}\right) - N(d_2) \right) \right\} \right.$$
$$\left. + C_2' \left(\alpha \tilde{K}_1 e^{-d_1^2/2} + \beta \tilde{K}_2 e^{-d_2^2/2} \right) \right\} = 0 \,. \tag{22}$$

Notice that the common terms in (21) and (22) are

$$\alpha \left(N\left(d_1 + \sigma_x \sqrt{\tau}\right) - N(d_1) \right) + \beta \left(N\left(d_2 + \sigma_x \sqrt{\tau}\right) - N(d_2) \right) \neq 0 \,, \tag{23}$$

and

$$\alpha \tilde{K}_1 e^{-d_1^2/2} + \beta \tilde{K}_2 e^{-d_2^2/2} \neq 0 \quad . \tag{24}$$

If both these expressions hold we have the following relation

$$\frac{B_1'}{B_2'} = \frac{C_1'}{C_2'} \quad . \tag{25}$$

By substituting (18) and (19) into the above

$$c_1 \left(\sigma_1 \sigma_2 - \rho_{12} \sigma_1^2 \right) + c_2 \left(\rho_{12} \sigma_2^2 - \sigma_1 \sigma_2 \right) = 0 \,. \tag{26}$$

In particular, for two assets

$$c_1 + c_2 = 1 \quad . \tag{27}$$

With (26)

$$c_1 = \frac{\rho_{12} \sigma_2^2 - \sigma_1 \sigma_2}{\rho_{12} \sigma_1^2 + \rho_{12} \sigma_2^2 - 2\sigma_1 \sigma_2} \,, \tag{28}$$

$$c_2 = \frac{\rho_{12} \sigma_1^2 - \sigma_1 \sigma_2}{\rho_{12} \sigma_1^2 + \rho_{12} \sigma_2^2 - 2\sigma_1 \sigma_2} \tag{29}$$

are obtained.

Since $0 < c_1, c_2 < 1$ we have

$$\rho_{12} < \frac{\sigma_1}{\sigma_2} \qquad (\sigma_1 < \sigma_2), \tag{30}$$

$$\rho_{12} < \frac{\sigma_2}{\sigma_1} \qquad (\sigma_1 > \sigma_2). \tag{31}$$

Therefore, in order to avoid risk-shifting incentive of borrower it will be required at the initial time of financing contract that lender should distribute the money into the two assets with the weights given in (28) and (29), whose correlation coefficient satisfy (30) and (31). However, lender does not have any way to know the real values of σ_1, σ_2 and ρ_{12} selected by borrower. In other words, it is assumed that lender can not observe which two of the many possible assets will be selected to invest by borrower. This is a premise producing risk-shifting incentive problem. Then, it is understood that (28) and (29) don't hold. Hence, the common term of (23) and (24)must be zero. Thus, the following two expressions hold.

$$\alpha\left(N\left(d_1 + \sigma_x\sqrt{\tau}\right) - N(d_1)\right) + \beta\left(N\left(d_2 + \sigma_x\sqrt{\tau}\right) - N(d_2)\right) = 0\,, \tag{32}$$

$$\alpha\tilde{K}_1 e^{-d_1^2/2} + \beta\tilde{K}_2 e^{-d_2^2/2} = 0\,. \tag{33}$$

Also, as long as lender has nothing to do with the risk σ_i of the asset invested by borrower the conditions of (32) and (33) must be satisfied since risk incentive by borrower must be avoided. For (32) and (33) α, β, X_1 and X_2 are free to be independently chosen there exist an infinitive number of incentive compatible profit-sharing contracts at any point in time. To answer which one the lender and borrower should select we use renegotiation-proof incentive contract approach.

5 Renegotiation - Proof Incentive Contract

We extend the definition of renegotiation-proof incentive contract given by [1] to the case with n assets. A contract is renegotiation-proof if it does not give the agent risk-shifting incentives

Ⅰ) at any point in time over the life of the contract, and

Ⅱ) for any value of the underlying asset $S_i(i = 1,2,3,\cdots,n)$

Figures3 through 5 give examples of a contract that are not renegotiation-proof. The contract uses the parameters $\alpha = -0.5, \beta = 0.5$,

$X_1 = 50$, $X_2 = 100$, $a_1 = a_2 = 1$, $\sigma_1 = 0.5$, $\sigma_2 = 0.25$, $\tau = 1$, $\rho = 0.5$, $r = 0.05$. Thus, when investment is performed with two assets, if the contract is not renegotiation-proof the risk-shifting incentive changes in Figure3 through 5.

Figure3 shows that the risk-shifting incentive changes for σ_1 and σ_2 as S_1 varies when $S_2 = 25$ is fixed. Similarly, Figure4 shows that S_2 varies when $S_1 = 25$ is fixed. In the both cases, when asset values are low the borrower makes asset risk increase to obtain more profit since the borrower' risk-shifting incentive $\partial\Pi_B/\partial\sigma_i$ is positive. In other words, the borrower has incentive to invest to the asset with more risk. Contrary to that, when the value of asset increases the borrower makes the asset risk decrease to obtain more profit since $\partial\Pi_B/\partial\sigma_i$ is negative. In other words, it produces asset with less risk. Then, the value of asset goes up further $\partial\Pi_B/\partial\sigma_i$ is going to approach 0. From this it is understood that as asset value is boundlessly going up a borrower can surely obtain profit, so that he is inclined to have less incentive by considering risk. Figure5. shows changes of risk-shifting incentive as $S_1 = S_2 = 25$ is fixed and the remaining period τ moves from 1 to 0 with 0.01 interval length. At that time $\partial\Pi_B/\partial\sigma_i$ is positive and approaches 0 as the time comes closer to the maturity date.

Namely, it is concluded that this contract can not avoid risk-shifting incentive of borrower as asset value and time change so that the contract is not renegotiation-proof .

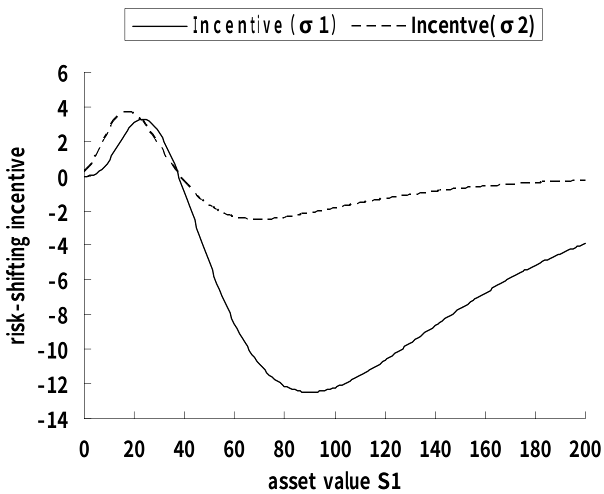

Figure3. Risk-shifting incentive for which S_2 varies from 0 to 200 with 1 width, being $S_2 = 25$ fixed

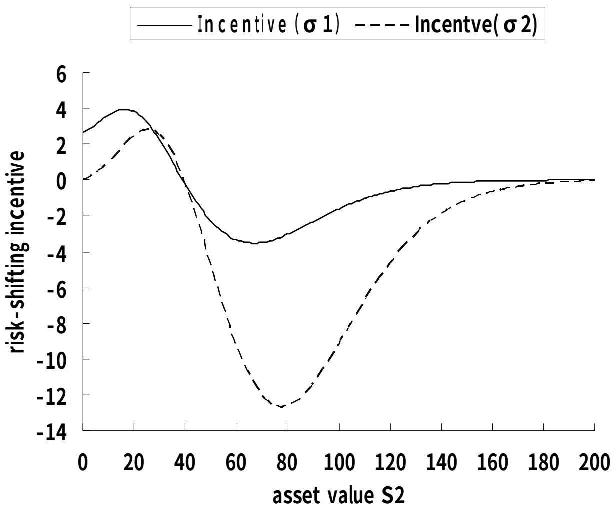

Figure4. Risk-shifting incentive for which S_2 varies from 0 to 200 with 1 width, being $S_1 = 25$ fixed

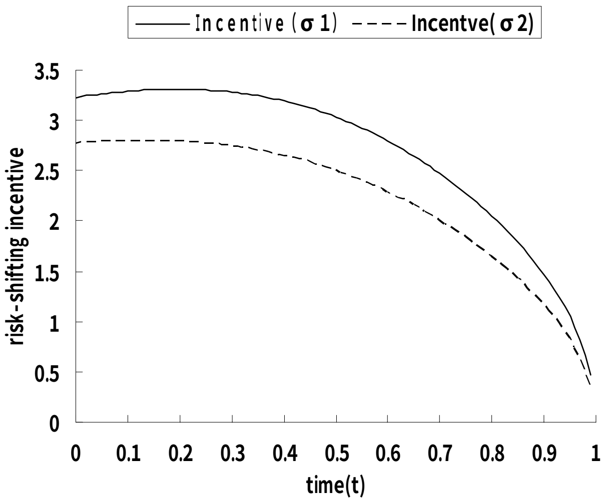

Figure5. Risk-shifting incentive for which time varies from 0 to 1 with 0.01 width, being $S_1 = S_2 = 25$ are fixed

6 Conclusion

We used game theory analysis of option to discuss risk-shifting problem on plural assets in which the sum of the all assets prices is considered the underlying asset and basket option approach was used, extending the result of a single asset of Ziegler. In order to solve risk-shifting problem for plural assets profit-sharing contract model was applied as in the case of a single asset. In other words, portfolio which consists of put option and call option as payout to lender and a fixed payment was used. In this case, the price of the underlying asset is the sum of prices of plural assets. It is found that profit-sharing contract, as solution of game, can apply not only a single asset but the case of plural assets by renegotiation-proof incentive approach.

In fact, it is not necessary for an investor to invest to a single asset when he has finance contract, so that incentive problem based on plural assets may be more realistic than the case of a single asset. In particular, we may find the conditions under which incentive contracts are negotiation-proof. As a future work, observability incentive problem under various situations may be considered to study.

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