Generalized coherence and connection property of imprecise conditional previsions

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Abstract

In this paper we consider imprecise conditional prevision assessments on random quantities with finite set of possible values. We use a notion of generalized coherence which is based on the coherence principle of de Finetti. We consider the checking of g-coherence, by extending some previous results obtained for imprecise conditional probability assessments. Then, we study a connection property of interval-valued g-coherent prevision assessments, by extending a result given in a previous paper for precise assessments.

Keywords: conditional random quantities, imprecise prevision assessments, generalized coherence, checking of g-coherence, connection property.

1 Introduction

In this paper we continue the study, started in [4], of imprecise conditional prevision assessments defined on families of conditional random quantities with a finite set of possible values. We use a notion of generalized coherence (g-coherence) which is based on the coherence principle of de Finetti and is equivalent to avoiding uniform loss property (AUL) introduced by Walley for lower previsions. Theoretical results and algorithms in the framework of coherence have been given by many authors (see, for instance, [5], [7], [8], [9], [15]). The checking of coherence and the extension of precise conditional prevision assessments have been studied in [6].

In the paper, after some preliminary results, we define the notion of g-coherence of interval-valued prevision assessments on conditional random quantities having a finite set of possible values. We characterize the notion of g-coherence by two different results. Then, we give an algorithm for checking g-coherence. We also examine the equivalence between g-coherence and AUL property of lower previsions. Then, we extend a result given in [4] on the connection property of precise prevision assessments to the case of g-coherent interval-valued assessments. We observe that the connection property is useful to determine imprecise prevision assessments which are intermediate between other assessments judged too extreme. We conclude the paper by some final comments.

2 Some preliminary notions

We give some preliminary notions on coherence and generalized coherence of precise and imprecise conditional prevision assessments on finite families of conditional random quantities. We denote by A^c the negation of A and by $A \vee B$ (resp., AB) the logical union (resp., intersection) of A and B. We use the same symbol to denote an event and its indicator. For each integer n, we set $J_n = \{1, 2, \dots, n\}$. We denote by $\mathcal{K}$ an arbitrary family of conditional random quantities, with finite sets of possible values.

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2.1 Precise conditional prevision assessments

Given a prevision function $\mathbb{P}$ defined on an arbitrary family of conditional random quantities $\mathcal{K}$, let $\mathcal{F}_n = \{X_i|H_i,\, i \in J_n\}$ be a finite subfamily of $\mathcal{K}$ and $\mathcal{M}_n$ the vector $(\mu_i,\, i \in J_n)$, where $\mu_i = \mathbb{P}(X_i|H_i)$ is the assessed prevision for the conditional random quantity $X_i|H_i$. With the pair $(\mathcal{F}_n, \mathcal{M}_n)$ we associate the random gain $\mathcal{G}_n = \sum_{i \in J_n} s_i H_i(X_i - \mu_i)$, where $s_1, \ldots, s_n$ are arbitrary real numbers and $H_1, \ldots, H_n$ denote the indicators of the corresponding events. We set $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$; moreover, we denote by $\mathcal{G}_n|\mathcal{H}_n$ the restriction of $\mathcal{G}_n$ to $\mathcal{H}_n$. Then, using the *betting scheme* of de Finetti, we have

Definition 1. The function $\mathbb{P}$ is coherent if and only if, $\forall\, n \geq 1$, $\forall\, \mathcal{F}_n \subseteq \mathcal{K}$, $\forall\, s_1, \ldots, s_n \in \mathbb{R}$, it is $\sup \mathcal{G}_n|\mathcal{H}_n \geq 0$.

We denote by Π_n the set of coherent conditional prevision assessments on $\mathcal{F}_n$. Given two points of Π_n,

$$\mathcal{M}' = (\mu_i',\, i \in J_n)\,, \quad \mathcal{M}'' = (\mu_i'',\, i \in J_n)\,,$$

we set

$$\begin{aligned} &\mu_i^m = \min\{\mu_i', \mu_i''\}\,, \quad \mu_i^M = \max\{\mu_i', \mu_i''\}\,, \\ &\mathcal{M}^m = \mathcal{M}' \wedge \mathcal{M}'' = (\mu_i^m,\, i \in J_n)\,, \\ &\mathcal{M}^M = \mathcal{M}' \vee \mathcal{M}'' = (\mu_i^M,\, i \in J_n)\,. \end{aligned}$$

Moreover, given any pair of points

$$\mathbf{x} = (x_i,\, i \in J_n), \quad \mathbf{y} = (y_i,\, i \in J_n)\,,$$

we set $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i$, $\forall\, i \in J_n$. Then, $\mathcal{M}^m \leq \mathcal{M}^M$, for every $\mathcal{M}', \mathcal{M}''$.
We remark that, given any point $\mathcal{M} = (\mu_i,\, i \in J_n)$, we have $\mathcal{M}^m \leq \mathcal{M} \leq \mathcal{M}^M$ if and only if there exists a vector $\Delta = (\delta_i,\, i \in J_n) \in [0,1]^n$ such that

$$\mu_i = (1 - \delta_i)\mu_i' + \delta_i \mu_i''\,, \quad i \in J_n\,.$$

In this case we say that $\mathcal{M}$ is a *generalized convex combination* of $\mathcal{M}', \mathcal{M}''$ and we write $\mathcal{M} = \mathcal{M}_\Delta$. Below, we recall (in a slightly modified version) a result given in [4], which generalizes a result obtained in [2] for conditional events.

Theorem 1. [*Biazzo and Gilio (2007)*]. Let $\mathcal{M}' = (\mu_i',\ i \in J_n)$, $\mathcal{M}'' = (\mu_i'',\ i \in J_n)$ be two coherent prevision assessments defined on $\mathcal{F}_n = \{X_i|H_i, i \in J_n\}$. There exists a continuous curve $\mathcal{C}$ with extreme points $\mathcal{M}', \mathcal{M}''$ such that for every $\mathcal{M} \in \mathcal{C}$, we have:
(i) $\mathcal{M}$ is a coherent conditional prevision assessment on $\mathcal{F}_n$;
(ii) each $\mathcal{M} \in \mathcal{C}$ is a generalized convex combination of $\mathcal{M}', \mathcal{M}''$; i.e. $\mathcal{M}^m \leq \mathcal{M} \leq \mathcal{M}^M$.

Theorem 1 assures that, for every pair of coherent prevision assessments $\mathcal{M}', \mathcal{M}''$ on $\mathcal{F}_n$, we can construct (at least) a continuous curve $\mathcal{C} \subseteq \Pi_n$ (from $\mathcal{M}'$ to $\mathcal{M}''$) whose points are intermediate coherent prevision assessments between $\mathcal{M}'$ and $\mathcal{M}''$. Hence, the assessments $\mathcal{M}', \mathcal{M}''$ are *connected* by the intermediate prevision assessments $\mathcal{M} \in \mathcal{C}$.
We remark that in general the number of curves like $\mathcal{C}$ is infinite.

2.2 Interval-valued conditional prevision assessments

Let $\mathcal{A}_n = ([l_i, u_i],\ i \in J_n)$ be any interval-valued conditional prevision assessment on a finite family $\mathcal{F}_n = \{X_i|H_i,\ i \in J_n\} \subseteq \mathcal{K}$. We give below a notion of generalized coherence (g-coherence), already used in [1] for the case of conditional events (and simply named 'coherence' in [11]).

Definition 2. An interval-valued prevision assessment $\mathcal{A}_n = ([l_i, u_i], i \in J_n)$, defined on a family of n conditional random quantities $\mathcal{F}_n = \{X_i|H_i, i \in J_n\}$, is g-coherent if there exists a coherent precise prevision assessment $\mathcal{M}_n = (\mu_i, i \in J_n)$ on $\mathcal{F}_n$, with $\mu_i = \mathbb{P}(X_i|H_i)$, which is consistent with $\mathcal{A}_n$, that is such that $l_i \leq \mu_i \leq u_i$ for each $i \in J_n$.

We denote by $\Im_n$ the set of g-coherent interval-valued conditional prevision assessments on a family of n conditional random quantities $\mathcal{F}_n$.

3 Checking g-coherence of conditional prevision assessments

Given a family of n conditional random quantities $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$, let

$\mathcal{A}_n = ([l_j, u_j],\, j \in J_n)$ be an interval-valued prevision assessment on $\mathcal{F}_n$. We want to check g-coherence of $\mathcal{A}_n$; that is, the existence of a precise prevision assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F}_n$, where $\mu_i = \mathbb{P}(X_i|H_i)$, such that $l_i \leq \mu_i \leq u_i, \forall\, i \in J_n$. For each $i \in J_n$ we assume $X_i \in \{x_{i1}, \ldots, x_{ik_i}\}$; then, we set

$$A_{ij} = (X_i = x_{ij})\,,\; j = 1, \ldots, k_i\,,\; i \in J_n\,.$$

Of course, for each $i \in J_n$, the family $\{A_{ij}\,,\; j = 1, \ldots, k_i\}$ is a partition of the sure event Ω. Moreover, for each $i \in J_n$, the family $\{H_i^c, A_{ij}H_i\,,\; j = 1, \ldots, k_i\}$ is a partition of Ω too. Then, the constituents generated by the family $\mathcal{K}$ are (the elements of the partition of Ω) obtained by expanding the expression

$$\bigwedge_{i \in J_n} (A_{i1}H_i \vee \cdots \vee A_{ik_i}H_i \vee H_i^c)\,.$$

We set $C_0 = H_1^c \cdots H_n^c$ (it may be $C_0 = \emptyset$); moreover, we denote by $C_1, \ldots, C_m$ the constituents contained in $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Hence

$$\bigwedge_{i \in J_n} (A_{i1}H_i \vee \cdots \vee A_{ik_i}H_i \vee H_i^c) = \bigvee_{h=0}^{m} C_h\,.$$

We give below, without proof, an obvious necessary and sufficient condition for the coherence of precise conditional prevision assessments.

Proposition 1. The assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F}_n$ is coherent if and only if there exists a coherent probability assessment $\mathcal{P} = (p_{ij}\,,\; j = 1, \ldots, k_i;\, i \in J_n)$ on the family of conditional events $\Phi = \{A_{ij}|H_i\,,\; j = 1, \ldots, k_i;\, i \in J_n\}$, where $p_{ij} = P(A_{ij}|H_i)$, such that $\sum_{j=1}^{k_i} p_{ij}x_{ij} = \mu_i\,, \quad i \in J_n$.

By Proposition 1, coherence of $\mathcal{M}$ could be checked by the following two steps:
(i) compute the set of solutions of the following system, in the unknowns $p_{ij}\,,\; j = 1, \ldots, k_i;\, i \in J_n$,

$$\begin{cases} \sum_{j=1}^{k_i} p_{ij}x_{ij} = \mu_i\,,\;\; i \in J_n\,; \\ \sum_{j=1}^{k_i} p_{ij} = 1\,,\; i \in J_n\,,\; p_{ij} \geq 0\,,\; \forall\, i, j\,; \end{cases}$$

(ii) find a solution $\mathcal{P} = (p_{ij}\,,\; j = 1, \ldots, k_i;\, i \in J_n)$ of the above system which, as a probability assessment on Φ, is coherent.

However, in what follows, we will avoid the explicit use of the quantities p_{ij}.
We observe that, given a conditional random quantity $X|H$, the upper prevision bound $\mathbb{P}(X|H) \leq u$ is equivalent to the lower prevision bound $\mathbb{P}(-X|H) \geq -u$; then, the g-coherence of the interval valued prevision assessment $\mathcal{A}_n = ([l_j, u_j],\, j \in J_n)$ on the family $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$ is equivalent to the g-coherence of the lower bound assessment $(l_j, -u_j,\, j \in J_n)$ on the family $\{X_1|H_1, -X_1|H_1, \ldots, X_n|H_n, -X_n|H_n\}$.
Therefore, to check g-coherence of interval-valued conditional prevision assessments, we only consider lower bounds. Given any vector of lower prevision bounds $L = (l_1, \ldots, l_n)$ on $\mathcal{F}_n$, with each constituent C_h, $h \in J_m$, we associate a vector $V_h = (v_{h1}, \ldots, v_{hn})$, where

$$v_{hi} = \begin{cases} x_{i1}\,, & C_h \subseteq A_{i1}H_i\,, \\ \cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{ik_i}\,, & C_h \subseteq A_{ik_i}H_i\,, \\ l_i\,, & C_h \subseteq H_i^c\,. \end{cases} \tag{1}$$

We observe that, in more explicit terms, for each $j \in \{1, \ldots, k_i\}$ the condition $C_h \subseteq A_{ij}H_i$ should be written

$$C_h \subseteq A_{i1}^c \cdots A_{i,j-1}^c A_{ij} A_{i,j+1}^c \cdots A_{ir}^c A_{ik_i}^c H_i\,.$$

Given any vector $\Lambda = (\lambda_h\,,\, h \in J_m)$ and any event A, we simply denote by $\sum_A \lambda_h$ the quantity $\sum_{h: C_h \subseteq A} \lambda_h$. Moreover, observing that $H_i = \bigvee_{j=1}^{k_i} A_{ij}H_i$, for each $i \in J_n$ it is

$$\begin{aligned} &\textstyle\sum_{h \in J_m} \lambda_h v_{hi} = \sum_{H_i} \lambda_h v_{hi} + \sum_{H_i^c} \lambda_h v_{hi} = \\ &= \textstyle\sum_{j=1}^{k_i} x_{ij} \sum_{A_{ij}H_i} \lambda_h + l_i \sum_{H_i^c} \lambda_h\,. \end{aligned} \tag{2}$$

Then, we examine the satisfiability of the condition

$$\sum_{h \in J_m} \lambda_h V_h \geq L\,,\; \sum_{h \in J_m} \lambda_h = 1\,,\; \lambda_h \geq 0\,,\; \forall\, h\,;$$

that is, the solvability of the following system Σ associated with the pair $(\mathcal{F}, L)$, in the nonnegative unknowns $\lambda_1, \ldots, \lambda_m$,

$$\begin{cases} \sum_{h \in J_m} \lambda_h v_{hi} \geq l_i\,,\; i \in J_n\,, \\ \sum_{h \in J_m} \lambda_h = 1\,,\; \lambda_h \geq 0\,,\; \forall\, h\,. \end{cases} \tag{3}$$

We remark that $X_iH_i = \sum_{j=1}^{k_i} x_{ij}A_{ij}H_i$; hence, by interpreting the vector $(\lambda_h,\, h \in J_m)$

as a probability assessment on the family $\{C_1|\mathcal{H}_n, \ldots, C_m|\mathcal{H}_n\}$, one has $\mathbb{P}(X_iH_i|\mathcal{H}_n) = \sum_{j=1}^{k_i} x_{ij} \sum_{A_{ij}H_i} \lambda_h = \mathbb{P}(X_i|H_i)P(H_i|\mathcal{H}_n)$, with $P(H_i|\mathcal{H}_n) = \sum_{H_i} \lambda_h$. Then, by decomposition formula (2), the inequality $\sum_{h \in J_m} \lambda_h v_{hi} \geq l_i$ in system (3) represents the condition $\mathbb{P}(X_iH_i|\mathcal{H}_n) \geq l_i P(H_i|\mathcal{H}_n)$.
Given a subset $J \subseteq J_n$, we set

$$\mathcal{F}_J = \{X_i|H_i\,,\, i \in J\}\,, \quad L_J = (l_i\,,\, i \in J)\,;$$

then, we denote by Σ_J, where $\Sigma_{J_n} = \Sigma$, the system like (3) associated with the pair $(\mathcal{F}_J, L_J)$. Then, we have

Theorem 2. [*General characterization of g-coherence*]. Given a family of n conditional random quantities $\mathcal{F} = \{X_1|H_1, \ldots, X_n|H_n\}$ and a vector $L = (l_1, \ldots, l_n)$, the imprecise conditional prevision assessment

$$\mathbb{P}(X_1|H_1) \geq l_1\,,\, \ldots\,,\, \mathbb{P}(X_n|H_n) \geq l_n$$

is g-coherent if and only if, for every subset $J \subseteq J_n$, defining $\mathcal{F}_J = \{X_i|H_i\,,\, i \in J\}$, $L_J = (l_i\,,\, i \in J)$, the system Σ_J associated with the pair $(\mathcal{F}_J, L_J)$ is solvable.

Proof. If the vector of lower prevision bounds L is g-coherent, then there exists a coherent assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on $\mathcal{F}$, with $\mu_i = \mathbb{P}(X_i|H_i) \geq l_i$. Then, by Proposition 1, there exists a coherent extension

$$p_{i1}, \ldots, p_{ik_i}\,,\; i \in J_n, \tag{4}$$

with $\sum_{j=1}^{k_i} p_{ij} = 1$ for each i, on the conditional events $A_{i1}|H_i, \ldots, A_{ik_i}|H_i\,,\; i \in J_n$, such that

$$p_{i1}x_{i1} + \cdots + p_{ik_i}x_{ik_i} = \mu_i \geq l_i\,,\; i \in J_n. \tag{5}$$

Considering the constituents $C_1, \ldots, C_m$ contained in $\mathcal{H}_n$, we denote by $\Lambda = (\lambda_1, \ldots, \lambda_m)$ any probability extension of (4) on the conditional events $C_1|\mathcal{H}_n, \ldots, C_m|\mathcal{H}_n$. Then, observing that

$$P(A_{ij}H_i|\mathcal{H}_n) = P(A_{ij}|H_i)P(H_i|\mathcal{H}_n)\,,$$

that is $\sum_{A_{ij}H_i} \lambda_h = p_{ij} \sum_{H_i} \lambda_h$, by (5) we obtain

$$\sum_{j=1}^{k_i} x_{ij} \sum_{A_{ij}H_i} \lambda_h \;\geq\; l_i \sum_{H_i} \lambda_h\,,\;\; i \in J_n\,,$$

and, by adding $l_i \sum_{H_i^c} \lambda_h$ to the left and the right side of the inequality, we obtain

$$\sum_{h \in J_m} \lambda_h v_{hi} \geq l_i\,,\; i \in J_n\,,$$

with $\sum_{h \in J_m} \lambda_h = 1,\; \lambda_h \geq 0,\; h \in J_m$; hence system (3) is solvable.
We observe that, for each given $J \subset J_n$, from g-coherence of L it follows that L_J is g-coherent too. Then, by the reasoning above, we obtain that the system Σ_J is solvable, $\forall\, J \subset J_n$.
Conversely, assuming that for every $J \subseteq J_n$ the system Σ_J is solvable, let S be the set of solutions $\Lambda = (\lambda_1, \ldots, \lambda_m)$ of the system (3). We set

$$\Gamma_0 = \{i:\; max_{\Lambda \in S} \sum_{H_i} \lambda_h > 0\}\,; \tag{6}$$

then, as shown in ([2], Theorem 2), there exists a vector $\Lambda^0 = (\lambda_1^0, \ldots, \lambda_m^0) \in S$ such that $\sum_{H_i} \lambda_h^0 > 0$ for every $i \in \Gamma_0$. Using Λ^0 we set

$$p_{ij} = \frac{\sum_{A_{ij}H_i} \lambda_h^0}{\sum_{H_i} \lambda_h^0}\,,\;\; i \in \Gamma_0\,;$$

then, the inequalities (5) are satisfied for every $i \in \Gamma_0$. Moreover, defining $I_0 = J_n \setminus \Gamma_0$, we set

$$\mathcal{F}_0 = \{X_i|H_i\,,\, i \in I_0\}\,, \quad L_0 = (l_i\,,\, i \in I_0)\,.$$

By repeating the previous reasoning, as the system Σ_{I_0} associated with the pair $(\mathcal{F}_0, L_0)$ is solvable, we determine a set $\Gamma_1 \subseteq I_0$ and a suitable vector Λ^1, by means of which we can define the probabilities p_{ij} for each $i \in \Gamma_1$. In this way, the inequalities (5) are satisfied for every $i \in \Gamma_1$, and so on. By this procedure, after a finite number of steps, we obtain a probability assessment like (4) which satisfies (5). It can be proved that the assessment obtained by the above procedure, say

$$(p_{i1}, \ldots, p_{ik_i}\,,\; i \in J_n)\,,$$

defined on the family

$$\{A_{i1}|H_i, \ldots, A_{ik_i}|H_i\,,\; i \in J_n\}\,,$$

is coherent (see, e.g., Theorem 1 in [2]). □

By the previous reasoning we easily obtain the following theorem, which generalizes an analogous result given in the case of conditional events (see, for instance, [12])

Theorem 3. [*Operative characterization of g-coherence*]. A vector of lower prevision bounds $L = (l_1, \ldots, l_n)$ on the family $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$ is g-coherent if and only if the following conditions are satisfied:
1. the system (3) is solvable ;
2. if $I_0 \neq \emptyset$, then L_0 is g-coherent.

Remark 1. Notice that, if system (3) is solvable, then it could be proved that the sub-assessment $\mathcal{A}_{\Gamma_0}$ on the subfamily $\mathcal{F}_{\Gamma_0}$ is g-coherent.

By Theorem 3, the following algorithm can be used to check the g-coherence of the imprecise assessment L on $\mathcal{F}_n$.

Algorithm 1. Let be given the triplet $(J_n, \mathcal{F}_n, L)$.
1. Construct the system (3) and check its solvability;
2. If the system (3) is not solvable then L is not g-coherent and the procedure stops, otherwise compute the set I_0;
3. If $I_0 = \emptyset$ then L is g-coherent and the procedure stops, otherwise set $(J_n, \mathcal{F}_n, L) = (I_0, \mathcal{F}_0, L_0)$ and repeat steps 1-3.

The algorithm ends after a finite number of steps, by verifying if L is g-coherent or not.

Remark 2. We observe that Theorem 3 and Algorithm 1 can be used in particular to check coherence of precise prevision assessments. More specifically, given a conditional prevision assessment $\mathcal{M} = (\mu_1, \ldots, \mu_n)$ on the family $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$, with each constituent C_h, $h \in J_m$, we associate a point $Q_h = (q_{h1}, \ldots, q_{hn})$, where

$$q_{hi} = \begin{cases} x_{i1}\,, & C_h \subseteq A_{i1}H_i\,, \\ \ldots.. & \ldots\ldots\ldots\ldots\ldots\ldots \\ x_{ik_i}\,, & C_h \subseteq A_{ik_i}H_i\,, \\ \mu_i\,, & C_h \subseteq H_i^c\,. \end{cases}$$

Then, the starting system (3) in the algorithm becomes

$$\begin{cases} \sum_{J_m} \lambda_h q_{hi} = \mu_i\,,\ i \in J_n\,; \\ \sum_{J_m} \lambda_h = 1\,,\ \ \lambda_h \geq 0\,,\ \forall\, h \in J_m\,. \end{cases} \quad (7)$$

We remark that the solvability of system (7) has the geometrical meaning that the point $\mathcal{M}$ can be represented as a linear convex combination of the points $Q_1, \ldots, Q_m$; that is

$$\mathcal{M} = \sum_{J_m} \lambda_h Q_h\,;\ \sum_{J_m} \lambda_h = 1\,;\ \lambda_h \geq 0\,,\ h \in J_m\,.$$

A geometrical approach for checking coherence of prevision assessments is also used in [13]. Concerning precise conditional prevision assessments, a characterization theorem and its application to inferential aspects have been given in [6].

3.1 Equivalence between g-coherence and AUL property of lower and upper previsions

The property of g-coherence means that there exists a dominating coherent precise prevision; hence, g-coherence is equivalent to the *avoiding uniform loss* property of lower previsions ([14]), as shown below.
We recall that a lower prevision $\underline{P}$ on a family of conditional random quantities $\mathcal{K}$ *avoids uniform loss* (AUL) if

$$\forall\, \mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\} \subseteq \mathcal{K}\,,$$

defining

$$\underline{P}(X_i|H_i) = l_i\,,\ i \in J_n\,,\ \mathcal{G}_n = \sum_{i=1}^{n} s_i H_i (X_i - l_i)\,,$$

the inequality *sup* $\mathcal{G}_n|\mathcal{H}_n \geq 0$ is satisfied for every $s_1 \geq 0, \ldots, s_n \geq 0$. By exploiting the conjugacy condition $\overline{P}(X|H) = -\underline{P}(-X|H)$, we only refer to lower previsions.
We observe that, recalling (1), the value of $\mathcal{G}_n$ associated with C_h is given by

$$g_h = \sum_{i \in J_n} s_i (v_{hi} - l_i)\,.$$

Now, given a vector of lower prevision bounds $L = (l_1, \ldots, l_n)$ on $\mathcal{F}_n$, let us consider the system (3) associated with the pair $(\mathcal{F}, L)$. We first recall a suitable alternative theorem. Let $A = (a_{hi})$ be a $m \times n-$matrix. Moreover, denote by $\mathbf{x}$ and $\mathbf{y}$, respectively, a row

$m-$vector and a column $n-$vector. The vector $\mathbf{x} = (x_1, \ldots, x_m)$ is said *semipositive* if it is nonnegative and moreover

$$x_1 + \cdots + x_m \; > \; 0 \, .$$

We have ([10], Th. 2.10)

Theorem 4. [*Gale (1960)*]. Exactly one of the following alternatives holds.
Either the inequality $\mathbf{x}A \geq 0$ has a *semipositive* solution, or the inequality $A\mathbf{y} < 0$ has a *nonnegative* solution.

Of course, if $\mathbf{x}$ is a semipositive solution of the inequality $\mathbf{x}A \geq 0$, then the vector $\Lambda = (\lambda_1, \ldots, \lambda_m)$, where $\lambda_h = \frac{x_h}{\sum_r x_r}$, is a semipositive solution of the same inequality, with $\sum_{h \in J_m} \lambda_h = 1$. Based on Theorem 4, we have

Theorem 5. [*Equivalence between AUL property and g-coherence*]. The system (3) is solvable if and only if $sup\, \mathcal{G}_n|\mathcal{H}_n \geq 0$.

Proof. The proof is obtained by applying Theorem 4, with $A = (a_{hi})$, where

$$a_{hi} = v_{hi} - l_i \, , \; x_h = \lambda_h \geq 0 \, , \; h \in J_m \, , \; i \in J_n \, ,$$

$$\textstyle\sum_{h \in J_m} \lambda_h = 1 \, , \; y_k = s_k \geq 0 \, , \; k \in J_n \, .$$

□

Finally, the equivalence between g-coherence and AUL property follows by Theorem 2.

4 The connection property of interval-valued assessments

In this section we generalize to the case of imprecise conditional prevision assessments a result obtained in [3] concerning imprecise probability assessments on conditional events. More precisely, we prove that there exists an infinite class Υ of g-coherent interval-valued prevision assessments $\mathcal{A}_n = ([l_i, u_i], \, i \in J_n)$, defined on a family of n conditional random quantities $\mathcal{F}_n$, which are intermediate between two given g-coherent interval-valued prevision assessments

$$\mathcal{A}'_n = ([l'_i, u'_i] \, , \, i \in J_n) \, , \; \mathcal{A}''_n = ([l''_i, u''_i], \, i \in J_n) \, .$$

This means that with each $\mathcal{A}_n \in \Upsilon$ we can associate a vector $\Delta = (\delta_i, \, i \in J_n) \in [0,1]^n$ such that

$$l_i = (1-\delta_i)l'_i + \delta_i l''_i, \; u_i = (1-\delta_i)u'_i + \delta_i u''_i, \; i \in J_n.$$

We say that $\mathcal{A}_n$ is a generalized convex combination of $\mathcal{A}'_n, \mathcal{A}''_n$, also denoted by $\mathcal{A}_\Delta$.

Theorem 6. [*Connection property*]. Given n events $H_1, \ldots, H_n$ and n random quantities $X_1, \ldots, X_n$, let $\mathcal{A}'_n = ([l'_i, u'_i], \, i \in J_n)$, $\mathcal{A}''_n = ([l''_i, u''_i], \, i \in J_n)$, be two g-coherent interval-valued conditional prevision assessments on the family $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$. Then, there exists an infinite class Υ of interval-valued prevision assessments on $\mathcal{F}_n$ such that:
(i) each $\mathcal{A}_n \in \Upsilon$ is a generalized convex combination between $\mathcal{A}'_n, \mathcal{A}''_n$; i.e., $\mathcal{A}_n = \mathcal{A}_\Delta$ for some $\Delta = (\delta_i, \, i \in J_n) \in [0,1]^n$;
(ii) each $\mathcal{A}_n \in \Upsilon$ is g-coherent; i.e., $\Upsilon \subseteq \Im_n$.

Proof. Assume that $\mathcal{A}'_n, \mathcal{A}''_n$ are g-coherent; then, there exist two coherent precise conditional prevision assessments

$$\mathcal{M}' = (\mu'_1, \ldots, \mu'_n) \, , \quad \mathcal{M}'' = (\mu''_1, \ldots, \mu''_n)$$

on the family $\mathcal{F}_n = \{X_1|H_1, \ldots, X_n|H_n\}$, with $l'_j \leq \mu'_j \leq u'_j$ and $l''_j \leq \mu''_j \leq u''_j$, $j \in J_n$. Moreover, from Theorem 1, there exists a continuous curve $\mathcal{C}$ connecting $\mathcal{M}', \mathcal{M}''$, with $\mathcal{C} \subseteq \Pi_n$ and with $\mathcal{M}^m \leq \mathcal{M} \leq \mathcal{M}^M$, for every $\mathcal{M} \in \mathcal{C}$. With each $\mathcal{M} = (\mu_1, \ldots, \mu_n) \in \mathcal{C}$ we can associate a vector $\Delta = (\delta_1, \ldots, \delta_n)$ such that $\mathcal{M} = \mathcal{M}_\Delta$; hence, for each $j \in J_n$ we have $\mu_j = (1 - \delta_j)\mu'_j + \delta_j \mu''_j$. We set $\Delta_\mathcal{C} = \{\Delta : \mathcal{M}_\Delta \in \mathcal{C}\}$. Then, let

$$\mathcal{A}_\Delta = ([l_j, u_j], \, j \in J_n) \, ,$$

be the generalized convex combination of $\mathcal{A}'_n, \mathcal{A}''_n$ associated with Δ; we set $\Upsilon = \{\mathcal{A}_\Delta : \Delta \in \Delta_\mathcal{C}\}$. Moreover, we have

$$l_j = (1-\delta_j)l'_j + \delta_j l''_j \leq (1-\delta_j)\mu'_j + \delta_j \mu''_j = \mu_j \, ,$$

$$u_j = (1-\delta_j)u'_j + \delta_j u''_j \geq (1-\delta_j)\mu'_j + \delta_j \mu''_j = \mu_j \, ;$$

that is $l_j \leq \mu_j \leq u_j \, , \quad \forall \, j \in J_n$. This means that $\mathcal{A}_\Delta$ is g-coherent, hence $\Upsilon \subseteq \Im_n$. □

By analogy with Theorem 1, we can say that $\mathcal{A}'_n, \mathcal{A}''_n$ are *connected* by the interval-valued prevision assessments contained in Υ.

4.1 A constructive procedure for determining a class Υ

We observe that Theorem 6 only shows the existence of a class Υ. We give below a constructive procedure for determining elements $\mathcal{A}_n$ of Υ, by choosing in a suitable way some continuous parameters.

(*Procedure.*) By Theorem 2, as $\mathcal{A}'_n, \mathcal{A}''_n$ are g-coherent, the following systems

$$\begin{cases} l'_i \sum_{H_i} \lambda_h \leq \sum_{j=1}^{k_i} \sum_{A_{ij}H_i} \lambda_h x_{ij} \leq u'_i \sum_{H_i} \lambda_h \,, \\ i \in J_n \,, \ \sum_{h \in J_m} \lambda_h = 1 \,, \ \lambda_h \geq 0 \,, \ \forall \, h \,, \end{cases}$$

$$\begin{cases} l''_i \sum_{H_i} \lambda_h \leq \sum_{j=1}^{k_i} \sum_{A_{ij}H_i} \lambda_h x_{ij} \leq u''_i \sum_{H_i} \lambda_h \,, \\ i \in J_n \,, \ \sum_{h \in J_m} \lambda_h = 1 \,, \ \lambda_h \geq 0 \,, \ \forall \, h \,, \end{cases}$$

respectively associated with $\mathcal{M}'$ and $\mathcal{M}''$, are solvable. We denote respectively by S' and S'' the sets of solutions of the previous systems. Then, recalling (6), we set

$$\Gamma'_0 = \{i : \ max_{\Lambda \in S'} \ \sum_{H_i} \lambda'_h > 0\} \,;$$
$$\Gamma''_0 = \{i : \ max_{\Lambda \in S''} \ \sum_{H_i} \lambda''_h > 0\} \,.$$

As it can be easily verified, there exist two vectors

$\Lambda'_0 = (\lambda'_r, r \in J_m) \in S'$, $\Lambda''_0 = (\lambda''_r, r \in J_m) \in S''$,

such that: $\sum_{H_i} \lambda'_h > 0 \,, \ \forall \, i \in \Gamma'_0$, and $\sum_{H_i} \lambda''_h > 0 \,, \ \forall \, i \in \Gamma''_0$. Given any number $\alpha_0 \in (0, 1)$, let us consider the vector

$$\Lambda_0 = (\lambda_h, h \in J_m) = (1 - \alpha_0)\Lambda'_0 + \alpha_0 \Lambda''_0 \,.$$

Of course, $\lambda_h = (1 - \alpha_0)\lambda'_h + \alpha_0 \lambda''_h$, $\forall \, h \in J_m$. Moreover, for each $i \in \Gamma^{(0)} = \Gamma'_0 \cup \Gamma''_0$ we have

$$\sum_{H_i} \lambda_h = (1 - \alpha_0) \sum_{H_i} \lambda'_h + \alpha_0 \sum_{H_i} \lambda''_h > 0 \,,$$

with $\sum_{H_i} \lambda_h = 0$, $\forall \, i \in I^{(0)} = J_n \setminus \Gamma^{(0)}$. Moreover, from g-coherence of A'_n, A''_n, for each $i \in J_n$ we have

$$l'_i \sum_{H_i} \lambda'_h \leq \sum_{j=1}^{k_i} \sum_{A_{ij}H_i} \lambda'_h x_{ij} \leq u'_i \sum_{H_i} \lambda'_h \,,$$

$$l''_i \sum_{H_i} \lambda''_h \leq \sum_{j=1}^{k_i} \sum_{A_{ij}H_i} \lambda''_h x_{ij} \leq u''_i \sum_{H_i} \lambda''_h \,.$$

Now, let us consider the interval-valued assessment $A_{\Gamma^{(0)}} = ([l_i, u_i], \, i \in \Gamma^{(0)})$, where

$$\begin{aligned} l_i &= (1 - \delta^0_i) l'_i + \delta^0_i l''_i \,, \\ u_i &= (1 - \delta^0_i) u'_i + \delta^0_i u''_i \,, \\ \delta^0_i &= \frac{\alpha_0 \sum_{H_i} \lambda''_h}{(1-\alpha_0)\sum_{H_i} \lambda'_h + \alpha_0 \sum_{H_i} \lambda''_h} = \\ &= \frac{\alpha_0 \sum_{H_i} \lambda''_h}{\sum_{H_i} \lambda_h} \ \in [0, 1] \,. \end{aligned} \qquad (8)$$

From (8), for each $i \in \Gamma^{(0)}$ we have

$$\sum_{j=1}^{k_i} \sum_{A_{ij}H_i} \lambda_h x_{ij} =$$
$$= \sum_{j=1}^{k_i} \sum_{A_{ij}H_i} [(1 - \alpha_0)\lambda'_h + \alpha_0 \lambda''_h] x_{ij} =$$

$$= (1 - \alpha_0) \sum_{j=1}^{k_i} \sum_{A_{ij}H_i} \lambda'_h x_{ij} +$$
$$+ \alpha_0 \sum_{j=1}^{k_i} \sum_{A_{ij}H_i} \lambda''_h x_{ij} \ \geq$$

$$\geq (1 - \alpha_0) l'_i \sum_{H_i} \lambda'_h + \alpha_0 l''_i \sum_{H_i} \lambda''_h =$$

$$= \left[\frac{(1-\alpha_0)\sum_{H_i} \lambda'_h}{\sum_{H_i} \lambda_h} \, l'_i + \frac{\alpha_0 \sum_{H_i} \lambda''_h}{\sum_{H_i} \lambda_h} \, l''_i \right] \sum_{H_i} \lambda_h =$$

$$= [(1 - \delta^0_i) l'_i + \delta^0_i l''_i] \sum_{H_i} \lambda_h = l_i \sum_{H_i} \lambda_h \,.$$

By a similar reasoning, from (8), we have

$$\sum_{j=1}^{k_i} \sum_{A_{ij}H_i} \lambda_h x_{ij} \ \leq \ u_i \sum_{H_i} \lambda_h \,;$$

hence, for each $i \in \Gamma^{(0)}$, it is

$$l_i \sum_{H_i} \lambda_h \ \leq \ \sum_{j=1}^{k_i} \sum_{A_{ij}H_i} \lambda_h x_{ij} \ \leq \ u_i \sum_{H_i} \lambda_h \,.$$

Now, given any quantities

$$\delta^0_i \in [0, 1], \ \ i \in I^{(0)} = J_n \setminus \Gamma^{(0)} \,, \qquad (9)$$

let us consider the assessment $\mathcal{A}_n = ([l_i, u_i], \, i \in J_n)$, where, for each $i \in J_n$, it is

$$l_i = (1 - \delta^0_i) l'_i + \delta^0_i l''_i, \ u_i = (1 - \delta^0_i) u'_i + \delta^0_i u''_i \,,$$

and where δ^0_i is defined by (8) for $i \in \Gamma^{(0)}$ and by (9) for $i \in I^{(0)}$. For each $i \in J_n$ we have

$$l_i \sum_{H_i} \lambda_h \ \leq \ \sum_{j=1}^{k_i} \sum_{A_{ij}H_i} \lambda_h x_{ij} \ \leq \ u_i \sum_{H_i} \lambda_r \,;$$

hence, Λ_0 is a solution of the system like (3) and, considering the sets Γ_0 (as defined by (6)) and $I_0 = J_n \setminus \Gamma_0$, we have $I_0 \subseteq I^{(0)}$, $\Gamma^{(0)} \subseteq \Gamma_0$. Then, by Remark 1, the assessment $\mathcal{A}_{\Gamma_0}$ on $\mathcal{F}_{\Gamma_0}$ is g-coherent.

By iterating the previous reasoning, after a finite number $k+1$ of steps, with $k \leq n-1$, we construct a g-coherent interval-valued assessment $\mathcal{A}_\Delta = (\mathcal{A}_{\Gamma_0}, \mathcal{A}_{\Gamma_1}, \ldots, \mathcal{A}_{\Gamma_k})$ on $\mathcal{F}_n$, which is intermediate between $\mathcal{A}'_n$, $\mathcal{A}''_n$.

5 Conclusions

We have considered imprecise prevision assessments on conditional random quantities with finite sets of possible values. We have examined the checking of g-coherence and the equivalence between g-coherence and Walley's AUL property of lower previsions. Then, we have studied the connection property of interval-valued g-coherent prevision assessments, by extending a result given in a previous paper for precise assessments. A further development of the research should deepen the study of imprecise prevision assessments on more general conditional random quantities.

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