Fuzzy Investment Decision Making

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Abstract

Recent literature on fuzzy numbers is rich of several approaches to approximate operations between fuzzy numbers. The desirable feature is to preserve the real shape of the fuzzy numbers resulting from the operations, without loosing in simplicity and applicability and in goodness of the approximations. In some recent papers we introduce a representation of the fuzzy numbers, based on the use of parametrized monotonic functions to model the alfa-cuts (or the membership functions) of the fuzzy numbers. We call it the LU representation, as it models directly the Lower and the Upper branches of the fuzzy numbers and it uses the parametrization to perform the arithmetic operations and more generally the fuzzy calculus. It is well known that economic and financial applications are strongly dependent on the precision of the input data and that in many cases the quality of the information becomes critical to the validity of the results. A suitable methodology to approach this problems can be based of the fuzzy calculus as it allows the description of uncertain or imprecise interest rates, volatility or prices in combination with the stochastic (risky) characters of the real world. We develop here a fuzzy investment choice process taking advantage of the good qualities of the LU parametric representation.

Keywords: Fuzzy Numbers, Parametric Representation, Capital Budgeting.

1 Introduction

We believe that contribution of fuzzy modelling to finance and economics can be relevant, especially when an efficient parametric representation is available. In [6] we introduced a representation of the fuzzy numbers, based on the use of parametrized monotonic functions to model the $\alpha - cuts$ (or the membership functions) of the fuzzy numbers (firstly introduced in [13] and strengthen by the exact analytical fuzzy mathematics and the LR representation in [3])

Afterwards in [11] and [12] we show the advantages of the use of LU-fuzzy numbers (represented through the Lower and the Upper branches) in the principal applications of fuzzy calculus: they generalize the LR-fuzzy setting in the direction of the shape preservation but also they allow easy error-controlled approximations in fuzzy calculus. The families of fuzzy numbers F^{LR} and F^{LU}, which include triangular and trapezoidal fuzzy numbers, are (in the simpler form) characterized by eight parameters and it appears that the inclusion of the slopes of lower and upper functions, even without generating piecewise monotonic approximations over subintervals (i.e. working with $N = 1$) is able to capture much more information than the linear approximation. In the simplest case, eight parameters describe the LU representation: the upper value, the lower value, the upper and the lower slope in the $\alpha - cuts$ zero and one; just eight parameters represent and model fuzzy information with high flexibility of shapes, without being constrained to strong

simplifications but allowing asymmetries or nonlinearities.

In addition, the overestimation effect that arises in interval arithmetic when a variable has more then one occurrence in the formula, is notably reduced in the LU representation, at least in the differentiable case, due to the fact that few points are in general sufficient to obtain good approximations (this is the essential gain in using the slopes to model fuzzy numbers), reducing the number of constrained problems to be solved directly.

Especially in finance, many problems are strongly dependent on the precision of the input data and that in many cases the quality of the information becomes critical when validating the results. A suitable methodology to approach this problems is based of the fuzzy calculus as it allows the description of uncertain or imprecise financial variables, when necessary also in combination with the stochastic framework. Some contributions to the fuzzy modelling in finance can be found in [1] and [5]. In [7] we approach the financial research area: we argue about the fuzzy version of the Black and Scholes option pricing model and we present a sensitivity analysis based on the study of the variations of three key parameters that are supposed to be fuzzy numbers parametrized following the LU model. We show that the Zadeh's fuzzy extension principle implies and explicit formulation of the option prices and that the Greeks play an important role in the representation of the LU-fuzzy option price.

In the second section we describe the fuzzy scenario of projects valuation: in [9] the authors support the thesis that the fuzzy approach permits variations well beyond the probability one and diverse kind of uncertainty at different levels can be involved. Decision makers, in fact, usually act in absence of precise information and the fuzzy modeling does not require rigid assumptions when taking into account uncertainty. The implementation of the LU-representation enhances the relevance of the fuzzy approach because it is possible to work with the most general shape of fuzzy numbers and the slopes can be inter-

preted as sensitivity parameters with respect of the input data. This aspect strengthen the interpretation of fuzzy models as extension of the interval analysis and as a guide to modulate the sensitivity analysis.

Some computational examples complete the section and few notes and challenging observations conclude.

2 Fuzzy Modeling in Investment Decision Making

In the framework of investments decision making there are several methods used in practice: they are extensively described and well motivated in their fuzzified form in Kuchta ([8]). Uncertainty can be involved, for example, when modeling exposure to risk, but incorrect conclusions may arise if it is not correctly incorporated. Alternatively uncertain variables (with normal or nonnormal distributions) can be incorporated in the budgeting process by specific techniques that sometimes produce applicability problems. Kuchta gives a complete survey of fuzzy equivalents for all capital budgeting methods and she obtains a set of closed intervals that correspond to the estimation of the parameters given by an expert of a certain project.

Here we work on three main criteria and we believe that the uncertainty naturally involved in the decision process can be rigorously modelled thinking cash-flows and rate of return as fuzzy numbers.

The Net Present Value (hereafter NPV) criteria consists in choosing the investment that presents the biggest value of the discounted cash flows:

$$NPV = \sum_{k=1}^{K} \frac{F_k}{(1+R)^k} - F_0$$

The uncertainty about the future cash flows and the interest rate enters when these key variables are fuzzified and expressed through the LU parametrization:

$$F_i = \left(F_i^-, \delta F_i^-, F_i^+, \delta F_i^+ \right) \quad i = 0, 1, ..., K$$

$$R = \left(R^-, \delta R^-, R^+, \delta R^+ \right)$$

The main criticism to NPV criteria is in the fact that the choice on the interest rate R depends on the person which evaluates the investment. On the other hand this criticism can become an added value because the evaluation depends on the financial experience, on the personal feelings and on the risk propensity of the single individual: these aspects are well represented trough a fuzzy number.

The function $NPV = f(R)$ has partial derivatives defined as follows

$$\frac{\partial f}{\partial R}^{\pm} = - \sum_{k=1}^{K} F_k^{\pm} k \left(1 + R^{\mp} \right)^{-k-1}$$

where $[R]_\alpha = [R_\alpha^-, R_\alpha^+]$ and supposing the first cash flow $F_0 < 0$ and $F_i \geq 0 \ \forall i = 1, ..., K$ we have

$$\frac{\partial f}{\partial F_i} = \frac{1}{(1+R)^i} > 0, \ \frac{\partial f}{\partial F_0} = -1, \ \frac{\partial f}{\partial R} < 0$$

and the LU representation for the NPV, validated by the extension principle, becomes

$$NPV^- = \sum_{k=1}^{K} \frac{F_k^-}{(1+R^+)^k} - F_0^+$$

$$NPV^+ = \sum_{k=1}^{K} \frac{F_k^+}{(1+R^-)^k} - F_0^-$$

$$\delta NPV^- = \sum_{k=1}^{K} \frac{1}{(1+R^+)^k} \delta F_k^- - \delta F_0^+ + \frac{\partial f}{\partial R}^- \delta R^+$$

$$\delta NPV^+ = \sum_{k=1}^{K} \frac{1}{(1+R^-)^k} \delta F_k^+ - \delta F_0^- + \frac{\partial f}{\partial R}^+ \delta R^-$$

Example 1. We consider the same example as in [8] where temporal horizon is composed of three periods, the fuzzy risk-free interest rate is $(0.09, 0.1, 0.11)$ and the four fuzzy cash flows are

$$\begin{aligned} F_0 &= & (-900, 1000, 1100) \\ F_1 &= & (-90, 100, 110) \\ F_2 &= & (-180, 200, 220) \\ F_3 &= & (-1800, 2000, 2200) \end{aligned}$$
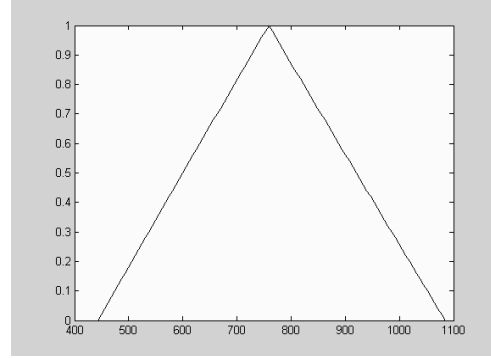

Figure 1: NPV in Example 1

The next figure shows the shape of the resulting fuzzy NPV.

In the table we report the computational result for the NPV, in particular we show that it exists an asymmetry produced by computations even if the inputs of the problem are symmetric: the asymmetry analysis can be carried on because of the properties of the LU representation.

α	NPV$^-$	NPV$^+$	δNPV$^-$	δNPV$^+$	Asym
1	758.8	320.6	758.8	-320.69	0%
0.75	678.9	318.1	839.3	-323.3	0,83%
0.5	599.7	315.5	920.5	-326.1	1,66%
0.25	521.2	312.9	1002.3	-328.7	2,5%
0	443.31	310.4	1084.9	-331.5	3,34%

The comparison between various projects is fundamental for a good decision making and it is based on the concept of ranking functions. In particular, the ordering of fuzzy numbers can be approached in many ways (see also [4] for recent literature and results); in most cases, the fuzzy numbers u are transformed into real numbers. By the use of the LU-fuzzy representation, the ranking functions can be computed, numerically or analytically, in terms of the parameters $u_i^{\pm}$ and $\delta u_i^{\pm}$ that define u.

The second criteria that we consider is the payback period (PBP), the time in which the investment to choose is the one that has the smallest value of the time period $T^* = k^* + \lambda^*$

such that:

$$\begin{aligned} k^* + \lambda^* &= \inf\{k + \lambda | V(k, \lambda) \geq 0, \\ \lambda &\in [0, 1], k = 0, 1, ..., K - 1\} \end{aligned}$$

where $V(k, \lambda)$ is the NPV after k periods plus a fraction λ of period $k + 1$, that is to say:

$$V(k, \lambda) = -F_0 +$$

$$+ \sum_{k=1}^{K} \left[\frac{F_k}{(1+R)^k} + \frac{\lambda F_{k+1}}{(1+R)^k (1+\lambda R)} \right]$$

The PBP criteria is often considered too simple and it depends strongly to light variations of the variables but it becomes interesting when the flows far in the future have an high degree of randomness with respect of the quite certain closer flows.

The procedure for the PBP fuzzification can be explained by the following algorithm:

For each $\alpha = \alpha_i$ $i = 0, 1, ..., N$

step 1.

Execute the crisp case to find K_i^{*-}, λ_i^{*-} using arguments $F_{0,i}^+, F_{1,i}^-, ..., F_{K,i}^-, R_i^+$

step 2.

Execute the crisp case to find K_i^{*+}, λ_i^{*+} using arguments $F_{0,i}^-, F_{1,i}^+, ..., F_{K,i}^+, R_i^-$

step 3.

Define $T_i^- = k_i^{*-} + \lambda_i^{*-}$ and $T_i^+ = k_i^{*+} + \lambda_i^{*+}$ and check $T_0^- \leq T_1^- \leq ... \leq T_N^- \leq T_N^+ \leq ... \leq T_0^+$

step 4.

To compute $\delta T_i^-, \delta T_i^+$ we use a Bernstein interpolation

$$\delta T_i^- = \sum_{j=0}^{N-1} \left(T_{j+1}^- - T_j^-\right) B_{j,N-1}(\alpha_i)$$

$$\delta T_i^+ = \sum_{j=0}^{N-1} \left(T_{j+1}^+ - T_j^+\right) B_{j,N-1}(\alpha_i)$$

where $B_{j,N-1}(\alpha) = \binom{N-1}{j} \alpha^j (1-\alpha)^{N-1-j}$.

The third criterion we take under consideration is the internal rate of return (IRR) that is based in the research of the rate R that satisfies the following equation

$$\sum_{k=1}^{K} \frac{F_k}{(1+R)^k} - F_0 = 0 \text{ , or} \quad (1)$$

$$V(R; F_0, F_1, ...F_K) = 0 \quad (2)$$

The main critical aspect of the IRR choice criteria is in the fact that the IRR may not exist or may be not unique. The interest rate can be written implicitly as a function of the cash flows

$$R = \varphi(F_0, F_1, ...F_K)$$

and, by the implicit function theorem, it is possible to write

$$\frac{\partial \varphi}{\partial F_k} = \begin{cases} \frac{-1}{\frac{\partial V}{\partial R}} & if \ k = 0 \\ -\frac{1}{(1+R)^k \frac{\partial V}{\partial R}} & if \ k = 1, 2, ..., K \end{cases}$$

where

$$\frac{\partial V}{\partial R} = -\sum_{k=1}^{K} \frac{kF_k}{(1+R)^{k+1}} < 0$$

evaluated with $R = \varphi(F_0, F_1, ...F_K)$. It follows, in the assumption of the unique first cash flow negative and the others all positive, that

$$\frac{\partial \varphi}{\partial F_0} < 0, \quad \frac{\partial \varphi}{\partial F_k} > 0 \ , \ k = 1, 2, ..., K$$

The fuzzification of the IRR criterion (an alternative approach is in [2]) presents some critical aspects that we solve by applying the fuzzy extension principle applied to the explicit form of R :

$$\begin{aligned} \varphi \ &: \ \mathbb{R}^{K+1} \to \mathbb{R}^+ \\ (F_0, F_1, ...F_K) \ &\to \ \varphi(F_0, F_1, ...F_K) \end{aligned}$$

Il follows that the $\alpha - cuts$ of the fuzzy extension of R corresponding to the $\alpha - cuts$

$$[F_k]_\alpha = \left[F_{k,\alpha}^-, F_{k,\alpha}^+\right]$$

of the fuzzy cash flows are given by:

$$[r]_\alpha = \{\varphi(F_0, F_1, ...F_K) \mid F_k \in [F_k]_\alpha, k = 0, ..., K\}$$

By the implicit function theorem, we know the partial derivatives of φ and their signs; by the continuity of φ, each $\alpha - cut$ of R is a compact interval

$$[R]_\alpha = \left[R_\alpha^-, R_\alpha^+ \right], \quad \alpha \in [0,1]$$

and the extremal values R_α^- and R_α^+ are obtained, for each α, by solving the two box-constrained optimization problems

$$R_\alpha^- = \min \left\{ \varphi\left(F_0, F_1, ...F_K \right) \mid F_{k,\alpha}^- \leq F_k \leq F_{k,\alpha}^+ \right\}$$

$$R_\alpha^+ = \max \left\{ \varphi\left(F_0, F_1, ...F_K \right) \mid F_{k,\alpha}^- \leq F_k \leq F_{k,\alpha}^+ \right\}$$

In the general case, i.e. with generic cash flows, we have to solve the last problems by appropriate procedures introduced in [12]. As φ is decreasing with respect to F_0 and increasing with respect to the remaining cash flows, we can find immediately the values R_α^- and R_α^+ given by φ evaluated at the vertices of the boxes $[F_k]_\alpha$:

$$R_\alpha^- = \varphi\left(F_{0,\alpha}^+, F_{1,\alpha}^-, ...F_{K,\alpha}^- \right)$$

$$R_\alpha^+ = \varphi\left(F_{0,\alpha}^-, F_{1,\alpha}^+, ...F_{K,\alpha}^+ \right)$$

This means that R_α^- is obtained by solving equation (1) with the cash flows $\left\{ -F_{0,\alpha}^+, F_{1,\alpha}^-, ...F_{K,\alpha}^- \right\}$ and R_α^+ is obtained by solving equation (1) with the cash flows $\left\{ -F_{0,\alpha}^-, F_{1,\alpha}^+, ...F_{K,\alpha}^+ \right\}$. The LU-representation of the fuzzy IRR is then obtained with $\alpha = \alpha_i$, $i = 0, 1, ..., N$, with R_i^- solving $V\left(R; F_{0,i}^+, F_{1,i}^-, ...F_{K,i}^- \right) = 0$, with R_i^+ solving $V\left(R; F_{0,i}^-, F_{1,i}^+, ...F_{K,i}^+ \right) = 0$ and the slopes are obtained by

$$\delta R_i^- = \sum_{k=0}^K \frac{\partial \varphi}{\partial F_k} \frac{\partial F_k}{\partial \alpha}$$

with $F_0 = F_{0,i}^+$, $F_1 = F_{1,i}^-$, ..., $F_K = F_{K,i}^-$ and

$$\delta R_i^+ = \sum_{k=0}^K \frac{\partial \varphi}{\partial F_k} \frac{\partial F_k}{\partial \alpha}$$

with $F_0 = F_{0,i}^-$, $F_1 = F_{1,i}^+$, ..., $F_K = F_{K,i}^+$.

To implement the PBP and the IRR criterion we take advantage of the LU-fuzzy calculator detailed in [10] and extended with the financial operators discussed previously.

The inputs can be chosen from a pre-defined list or introduced directly; fuzzy inputs and output can be visualized in a graphic in order to interpret their meaning in terms of asymmetry, nonlinearities, convexity or concavity of the shape.
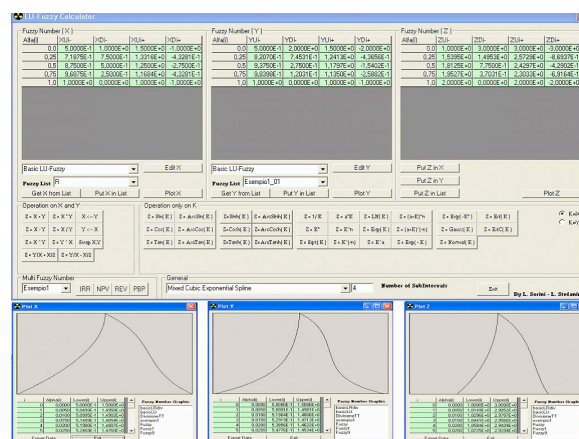

Figure 2: The layout of the hand-like fuzzy calculator implemented by a Windows-based frame for the LU model

Using the same project from Example 1 we can find the fuzzy PBP and the fuzzy IRR and interpret the shape of the results.

3 Final Remarks

We believe that fuzziness can be an extraordinary useful tool in the sensitivity analysis and in uncertainty modelling performed in the investment decision making; the presence of a fuzzy interest rate, for example, modelled in a very flexible and easy to interpret way (the LU-representation), augments the possibility to choose the best investment taking into account the real dynamics of interest rates. The next step in our investigation will be about the modelling of fuzzy real options that represent the new frontier in capital budgeting.

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