An Affine Model with Decoupled Fuzzy Dynamics

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Abstract

Function approximation properties of Mamdani fuzzy model are well known. On the other hand, Takagi-Sugeno fuzzy model with affine consequent could be a local approximator of the dynamics. However, it can also be tuned to be a function approximator, but loosing its local interpretation. In this paper, an affine global model with function approximation capabilities which maintains local interpretation is introduced.

Keywords: Nonlinear systems, dynamics, fuzzy systems, modeling, fuzzy control.

1 Introduction

Mamdani and Takagi-Sugeno fuzzy models are traditionally used to model non-linear systems. Mamdani model [5] does not take into account explicitly local dynamics in the rules, but an appropriate selection of the membership functions can provide an adequate interpolation of the system dynamics.

On the other hand, Takagi-Sugeno model [10] has been widely used in fuzzy modelling for control applications, because it includes valuable information about the local dynamics. More precisely, this model interpolates among affine submodels, something which apparently should mean a good approximation of the dynamics of the non-linear system. However, a non-convex interpolation is obtained and, when the model is tuned to behave as a universal approximator, it does not fit the local dynamics [1].

In the present work we propose to obtain the best of Mamdani and Takagi-Sugeno models by using an affine model with variant coefficients which are independently governed by a zeroth order fuzzy inference system. This model may be interpreted as a generalization of Takagi-Sugeno model in which dynamic coefficients have been decoupled.

We will consider empirical models such as $x_{n+1} = f(\mathbf{x})$, being $\mathbf{x} = [x_1 \ldots x_n]^T$, and f a continuous or discrete non-linear function. Our goal is to build non-linear models able to reflect the dynamics of f.

2 Affine Local Models

In linear control theory, identification is done by exciting the system around a point $(x_1^{(0)}, \ldots, x_{n+1}^{(0)})$, and assimilating its response with the one of an affine system. After the identification process, we should obtain the same model resulted from the linearization of the empirical model:

$$\begin{aligned} x_{n+1} &\approx f(x_1^{(0)}, \ldots, x_n^{(0)}) \\ &+ \left.\frac{\partial f}{\partial x_1}\right|_{(0)} (x_1 - x_1^{(0)}) \\ &+ \ldots \\ &+ \left.\frac{\partial f}{\partial x_n}\right|_{(0)} (x_n - x_n^{(0)}) \\ &= a_0^{(0)} + a_1^{(0)} x_1 + \ldots + a_n^{(0)} x_n \quad (1) \end{aligned}$$

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As an example, let us suppose a vehicle of mass M moving along a straight line at velocity $v(t)$, due to a force $f(t)$, and under a viscous friction B (figure 1).

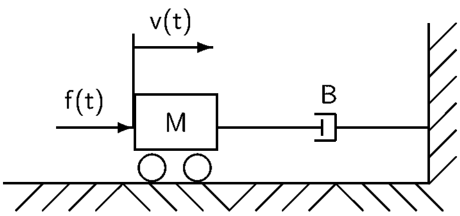

Figure 1: The vehicle

An empirical first order model of this system may be given by $M\dot{v}(t) = f(t) - Bv^2(t)$. If we choose $M = 1$ and $B = 1$, we have

$$\dot{v}(t) = f(t) - v^2(t) \tag{2}$$

A dynamical equilibrium point for this system is held at $v^{(0)} = 2$, $f^{(0)} = 4$. In order to analyze the dynamical behavior of this empirical model, we will change the force $f(t)$ to zero, so the evolution of the vehicle will follow the equation $\dot{v} = -v^2$ (see figure 2):

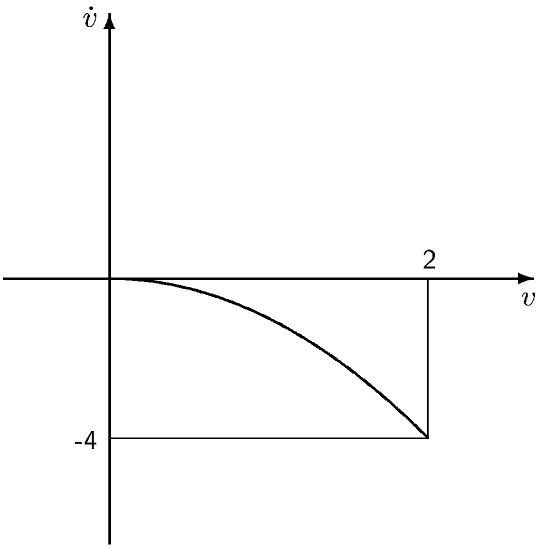

Figure 2: Empirical model of the vehicle

Figure 3 shows the dynamical evolution of the system.

In order to obtain an affine model of the vehicle, we identify the vehicle at $v^{(0)} = 2$. We should obtain the linearization of the empirical model, which is

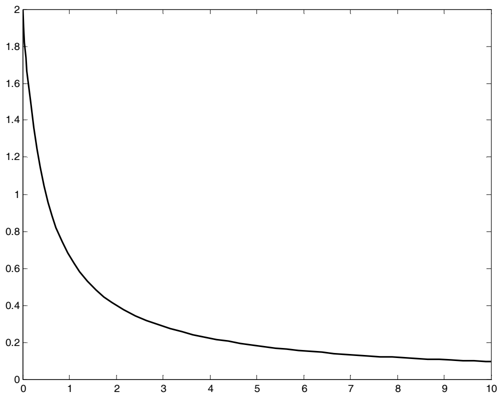

Figure 3: Dynamical evolution of the vehicle

$$\dot{v} \approx -{v^{(0)}}^2 - 2v^{(0)}\left(v - v^{(0)}\right) \tag{3}$$

$$\dot{v} \approx 4 - 4v \tag{4}$$

As it can be seen in figure 4, the error of the affine model increases as we deviate from the linearization point.

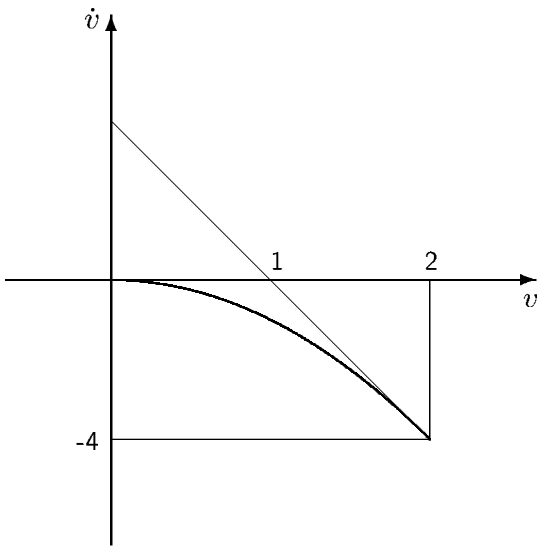

Figure 4: Affine model at $v^{(0)}$=2

Figure 5 compares the response of the affine model versus the response of the empirical model.

Now the final value is 1 instead of 0 and the

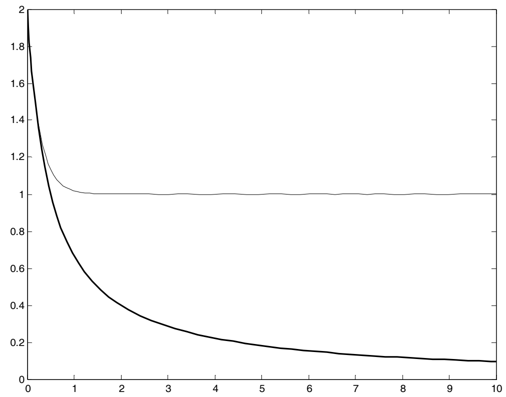

Figure 5: Vehicle model v.s. affine model response

response is much faster than the empirical system.

3 Mamdani Model

Let us suppose that $X_l^{(i_l)}$ are the fuzzy sets for input x_l, $\forall i_l = \{1, \ldots, r_l\}, \forall l = \{1, \ldots, n\}$. r_l is the number of fuzzy sets for x_l, and $\mu_{X_l^{(i_l)}}(x_l)$ are the corresponding membership functions (figure 6),

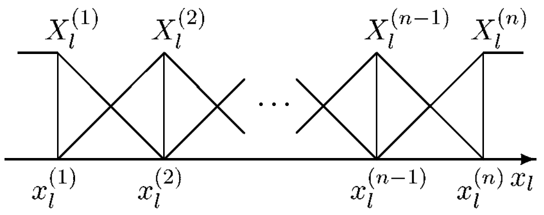

Figure 6: Membership functions for input x_l

which overlap by pairs, this means,

$$\sum_{i_l=1}^{r_l} \mu_{X_l^{(i_l)}}(x_l) = 1, \tag{5}$$

$\forall x_l^{(i_l)} \leq x_l \leq x_l^{(i_l+1)}$, $\forall l = \{1, \ldots, n\}$, where $x_l^{(i_l)}$ are guide points.

Then, each rule $R^{(i_1 \ldots i_n)}$ of Mamdani model can be defined using the centers of gravity instead of the fuzzy sets for each consequent [6]:

$$IF(x_1\ is\ X_1^{(i_1)})\ AND\ \ldots\ AND\ (x_n\ is\ X_n^{(i_n)})$$

$$THEN\ x_{n+1} = x_{n+1}^{(i_1 \ldots i_n)}$$

and the output of the system can be computed by

$$x_{n+1} = \sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \ldots i_n)}(\mathbf{x})\ x_{n+1}^{(i_1 \ldots i_n)} \tag{6}$$

where

$$w^{(i_1 \ldots i_n)}(\mathbf{x}) = \prod_{l=1}^{n} \mu_{X_l^{(i_l)}}(x_l) \tag{7}$$

represents the weight of each rule. It is known that Mamdani model can be used as function approximator [7, 8]. For example, the first order empirical model $x_2 = f(x_1)$ may be exactly approximated by a Mamdani model in the range $x_1^{(i_1)} \leq x_1 \leq x_1^{(i_1+1)}$, with two rules:

$$R^{(i_1)}\ :\ IF(x_1\ is\ X_1^{(i_1)})\ THEN\ x_2 = f(x_1^{(i_1)})$$

$$R^{(i_1+1)}\ :\ IF(x_1\ is\ X_1^{(i_1+1)})\ THEN\ x_2 = f(x_1^{(i_1+1)})$$

provided that f is strictly monotonous (increasing or decreasing) in that range. Furthermore, the fuzzy sets for perfect approximation are given by

$$\mu_{X_1^{(i_1)}}(x_1) = \frac{f(x_1^{(i_1+1)}) - f(x_1)}{f(x_1^{(i_1+1)}) - f(x_1^{(i_1)})} \tag{8}$$

$$\mu_{X_1^{(i_1+1)}}(x_1) = 1 - \mu_{X_1^{(i_1)}}(x_1) \tag{9}$$

These membership functions do not belong to $[0, 1]$ when f is not monotonous.

As an example, if we identify the vehicle at $v^{(1)} = 0$ and $v^{(2)} = 2$ we obtain that $\dot{v}^{(1)} = 0$ and $\dot{v}^{(2)} = -4$. Then, Mamdani model can be described with two rules as:

$$R^{(1)}\ :\ IF(v\ is\ V^{(1)})\ THEN\ \dot{v}^{(1)} = 0$$

$$R^{(2)}\ :\ IF(v\ is\ V^{(2)})\ THEN\ \dot{v}^{(2)} = -4$$

If we chose triangular membership functions (see figure 7):

$$\mu_{V^{(1)}}(v) = 1 - \frac{v}{2} \quad (10)$$

$$\mu_{V^{(2)}}(v) = \frac{v}{2} \quad (11)$$

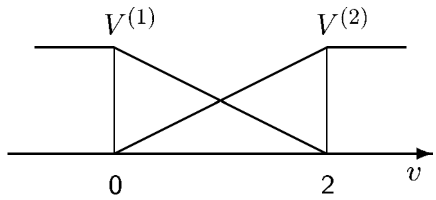

Figure 7: Membership functions for v

the output of the system, $\forall v \in [0,\ 2]$, is given by:

$$\begin{aligned} \dot{v} &= \mu_{V^{(1)}}(v)\ \dot{v}^{(1)} + \mu_{V^{(2)}}(v)\ \dot{v}^{(2)} \\ &= \left(1 - \frac{v}{2}\right) \cdot 0 + \frac{v}{2} \cdot (-4) = -2v \end{aligned} \quad (12)$$

The resultant approximation is compared with the empirical model as shown in figure 8, where we can appreciate the error between both of them. Anyway, both are convex interpolations.

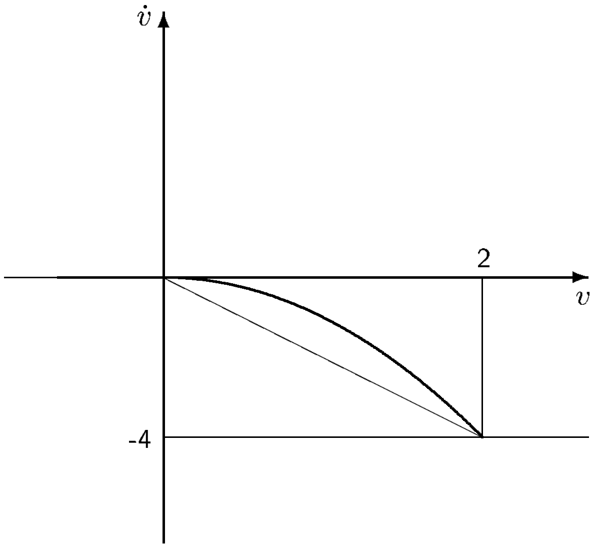

Figure 8: Mamdani model

Figure 9 compares the response of Mamdani model versus the response of the empirical model.

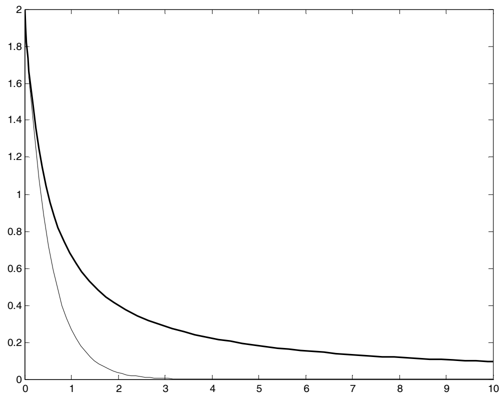

Figure 9: Empirical model v.s. Mamdani model response

The final value is the same, but the response is six times faster than the one of the empirical system. Anyway, it is always possible to select other membership functions shape, to improve the interpolation, and thus to model perfectly the dynamics of the system. Knowing that the empirical model is given by

$$\dot{v} = f(v) = -v^2 \quad (13)$$

we can choose the membership functions as follows (see figure 10):

$$\begin{aligned} \mu_{V^{(1)}}(v) &= \frac{-{v^{(2)}}^2 + v^2}{-{v^{(2)}}^2 + {v^{(1)}}^2} \\ &= 1 - \frac{v^2}{4} \end{aligned} \quad (14)$$

$$\mu_{V^{(2)}}(v) = \frac{-v^2 + {v^{(1)}}^2}{-{v^{(2)}}^2 + {v^{(1)}}^2} = \frac{v^2}{4} \quad (15)$$

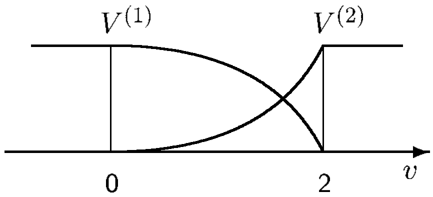

Figure 10: Parabolic membership functions for v

So the system's output will be identical to the empirical one:

$$\begin{aligned}\dot{v} &= \mu_{V^{(1)}}(v)\ \dot{v}^{(1)} + \mu_{V^{(2)}}(v)\ \dot{v}^{(2)} \\ &= \left(1 - \frac{v^2}{4}\right) \cdot 0 + \frac{v^2}{4} \cdot (-4) \\ &= -v^2 \end{aligned} \tag{16}$$

The previous idea can be easily generalized to systems with n inputs.

4 Takagi-Sugeno Model

In Takagi-Sugeno model [10, 11], each rule $R^{(i_1 \ldots i_n)}$ is described as follows:

$$IF(x_1\ is\ X_1^{(i_1)})\ AND\ \ldots$$

$$AND\ (x_n\ is\ X_n^{(i_n)})\ THEN$$

$$x_{n+1} = a_0^{(i_1 \ldots i_n)} + a_1^{(i_1 \ldots i_n)} x_1 + \ldots + a_n^{(i_1 \ldots i_n)} x_n$$

The meaning of these affine consequents is to identify the system at different points, one per rule, and interpolate the dynamics of the system among them, in order to compensate identification errors. Then, the output of the system is computed as follows:

$$\begin{aligned} x_{n+1} &= \sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \ldots i_n)}(\mathbf{x}) \left(a_0^{(i_1 \ldots i_n)}\right. \\ &+ \left. a_1^{(i_1 \ldots i_n)} x_1 + \ldots + a_n^{(i_1 \ldots i_n)} x_n\right) \end{aligned} \tag{17}$$

where again we have supposed that the membership functions of the input variables x_l overlap by pairs. As an example, if we identify the vehicle linearizing it at $v^{(1)} = 0$ and $v^{(2)} = 2$ we should obtain that $\dot{v}^{(1)} = 0$ and $\dot{v}^{(2)} = 4 - 4v$. Takagi-Sugeno model is described as follows:

$$R^{(1)}\ :\ IF(v\ is\ V^{(1)})\ THEN\ \dot{v}^{(1)} = 0$$

$$R^{(2)}\ :\ IF(v\ is\ V^{(2)})\ THEN\ \dot{v}^{(2)} = 4 - 4v$$

If we choose triangular membership functions (see figure 11),

$$\mu_{V^{(1)}}(v) = 1 - \frac{v}{2} \tag{18}$$

$$\mu_{V^{(2)}}(v) = \frac{v}{2} \tag{19}$$

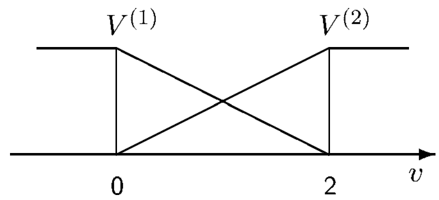

Figure 11: Triangular membership functions for v

the system's output will be given by:

$$\begin{aligned}\dot{v} &= \mu_{V^{(1)}}(v)\ \dot{v}^{(1)} + \mu_{V^{(2)}}(v)\ \dot{v}^{(2)} \\ &= \left(1 - \frac{v}{2}\right) \cdot 0 + \frac{v}{2} \cdot (4 - 4v) \\ &= 2v - 2v^2 \neq -v^2 \end{aligned} \tag{20}$$

Takagi-Sugeno approximation is compared in figure 12 with the empirical model, where we can observe the error between both of them.

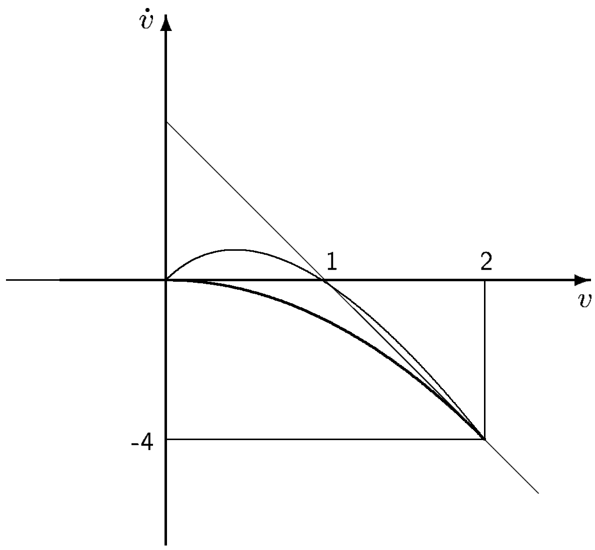

Figure 12: Takagi-Sugeno model

The response of the Takagi-Sugeno model is practically the same than the one of the affine case (see figure 5). The response of the system is much faster than the one of the empirical system, and the final value is 1 instead of 0.

As we can see, the approximation interpolates between the two affine systems, so the function must be always inside both triangles and will never match exactly the empirical model, independently of the membership functions shape. Furthermore, the model matches the function value at the guide points, but not its derivative, which should be the idea of linearizing at those points.

It should be noted that the selection of parabolic membership functions, do not provide better results. Suppose that the membership functions are

$$\mu_{V^{(1)}}(v) = 1 - \frac{v^2}{4} \tag{21}$$

$$\mu_{V^{(2)}}(v) = \frac{v^2}{4} \tag{22}$$

then

$$\begin{aligned} \dot{v} &= \mu_{V^{(1)}}(v)\ \dot{v}^{(1)} + \mu_{V^{(2)}}(v)\ \dot{v}^{(2)} \\ &= \left(1 - \frac{v^2}{4}\right) \cdot 0 + \frac{v^2}{4} \cdot (4 - 4v) \\ &= v^2 - v^3 \end{aligned} \tag{23}$$

Takagi-Sugeno affine model always produces a non-convex interpolation. These interpolation difficulties have been widely referred to in the literature [1] but, in most cases, the proposal of other authors is to obtain local affine submodels which do not correspond to the system affine identification.

5 The Affine Fuzzy Dynamic Model

Our goal is to build an affine global model, able to fit the local dynamics without loosing approximation capabilities. First we replace Takagi-Sugeno rules by Mamdani rules with $n+1$ consequents:

$$IF (x_1\ is\ X_1^{(i_1)})\ AND\ \ldots\ AND\ (x_n\ is\ X_n^{(i_n)})$$
$$THEN\ a_0 = a_0^{(i_1 \ldots i_n)}\ AND\ \ldots$$
$$AND\ a_n = a_n^{(i_1 \ldots i_n)}$$

Then, defining

$$a_l(\mathbf{x}) = \sum_{i_1=1}^{r_1} \cdots \sum_{i_n=1}^{r_n} w^{(i_1 \ldots i_n)}(\mathbf{x}) a_l^{(i_1 \ldots i_n)} \tag{24}$$

$\forall l = \{0, \ldots, n\}$, the output is given by

$$x_{n+1} = a_0(\mathbf{x}) + a_1(\mathbf{x}) x_1 + \ldots + a_n(\mathbf{x}) x_n \tag{25}$$

In fact, this equation can be considered as a Mamdani model supervising an affine system with variant coefficients, as shown in figure 13.

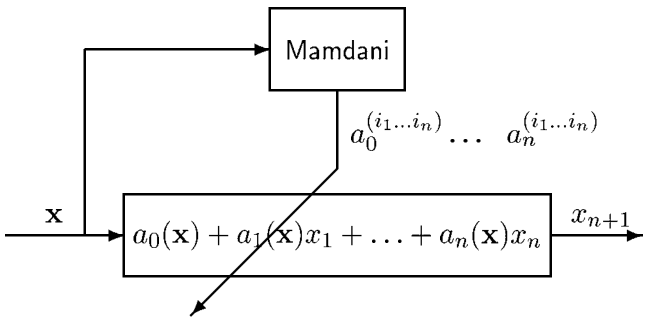

Figure 13: Block representation of Takagi-Sugeno affine global model

$a_0(\mathbf{x})$ is an offset term, while $a_l(\mathbf{x})$ represent the variant dynamics of the system. It should be noted that all these dynamic coefficients are coupled, because they are calculated using the same set of rules.

We propose to use an affine global model with different set of rules for each coefficient $a_l(\mathbf{x})$ (see figure 14).

As we will see now, by using this model we are using the same affine global expression, but maintaining local and global interpretability, and approximating both the function value and its derivative by changing independently the offset and dynamic terms, this means, by decoupling the system dynamics. Furthermore, we keep the number of rules to the minimum. We will see how to build an approximator using this model.

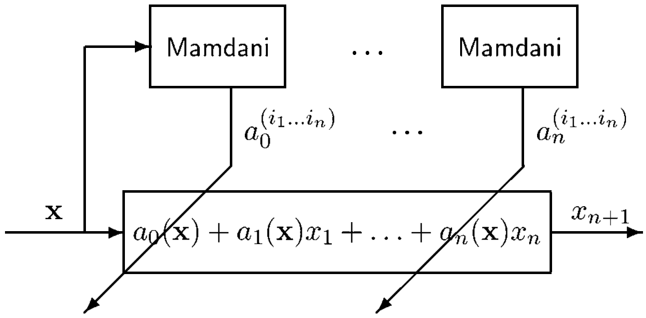

Figure 14: Block representation of the affine global model with decoupled dynamics

Let us suppose the first order case of empirical model $x_2 = f(x_1)$, and define

$$a_1(x_1) = f'(x_1) \tag{26}$$

$$a_0(x_1) = f(x_1) - f'(x_1)x_l \tag{27}$$

with $a_1(x_1)$ and $a_0(x_1)$ strictly monotonous in the range $x_1^{(i_1)} \leq x_1 \leq x_1^{(i_1+1)}$ (otherwise, the membership functions will not belong to $[0,1]$, and another partitioning is required). Then, $f(x_1)$ may be exactly approximated by a Takagi-Sugeno model in this range, with two subsets of rules:

$$R_0^{(i_1)} \;:\; IF(x_1 \; is \; X_{10}^{(i_1)}) \; THEN \; a_0 = a_0^{(i_1)}$$

$$R_0^{(i_1+1)} \;:\; IF(x_1 \, is \, X_{10}^{(i_1+1)}) \, THEN \, a_0 = a_0^{(i_1+1)}$$

with

$$\mu_{X_{10}^{(i_1)}}(x_1) = \frac{a_0^{(i_1+1)} - a_0(x_1)}{a_0^{(i_1+1)} - a_0^{(i_1)}} \tag{28}$$

$$\mu_{X_{10}^{(i_1+1)}}(x_1) = 1 - \mu_{X_{10}^{(i_1)}}(x_1) \tag{29}$$

and

$$R_1^{(i_1)} \;:\; IF(x_1 \; is \; X_{11}^{(i_1)}) \; THEN \; a_1 = a_1^{(i_1)}$$

$$R_1^{(i_1+1)} \;:\; IF(x_1 \, is \, X_{11}^{(i_1+1)}) \, THEN \, a_1 = a_1^{(i_1+1)}$$

with

$$\mu_{X_{11}^{(i_1)}}(x_1) = \frac{a_1^{(i_1+1)} - a_1(x_1)}{a_1^{(i_1+1)} - a_1^{(i_1)}} \tag{30}$$

$$\mu_{X_{11}^{(i_1+1)}}(x_1) = 1 - \mu_{X_{11}^{(i_1)}}(x_1) \tag{31}$$

The output is calculated as

$$x_2 = a_0(x_1) + a_1(x_1)x_1 \tag{32}$$

For example, in the case of the vehicle, $\dot{v} = -v^2$ in the range $0 \leq v \leq 2$ the affine model with decoupled dynamics is as follows:

$$a_0(v) = -v^2 + 2v^2 = v^2 \tag{33}$$

$$a_1(v) = -2v \tag{34}$$

$$R_0^{(1)} \;:\; IF(v \; is \; V_0^{(1)}) \; THEN \; a_0 = 0$$

$$R_0^{(2)} \;:\; IF(v \; is \; V_0^{(2)}) \; THEN \; a_0 = 4$$

with

$$\mu_{V_0^{(1)}}(v) = \frac{a_0^{(2)} - a_0(v)}{a_0^{(2)} - a_0^{(1)}} = 1 - \frac{v^2}{4} \tag{35}$$

$$\mu_{V_0^{(2)}}(v) = 1 - \mu_{V_0^{(1)}}(v) = \frac{v^2}{4} \tag{36}$$

and

$$R_1^{(1)} \;:\; IF(v \; is \; V_1^{(1)}) \; THEN \; a_1 = 0$$

$$R_1^{(2)} \;:\; IF(v_1 \; is \; V_1^{(2)}) \; THEN \; a_1 = -4$$

with

$$\mu_{V_1^{(1)}}(v) = \frac{a_1^{(2)} - a_1(v)}{a_1^{(2)} - a_1^{(1)}} = 1 - \frac{v}{2} \tag{37}$$

$$\mu_{V_1^{(2)}}(v) = 1 - \mu_{V_1^{(1)}}(v) = \frac{v}{2} \tag{38}$$

So

$$\dot{v} = a_0(v) + a_1(v)v = -v^2 \tag{39}$$

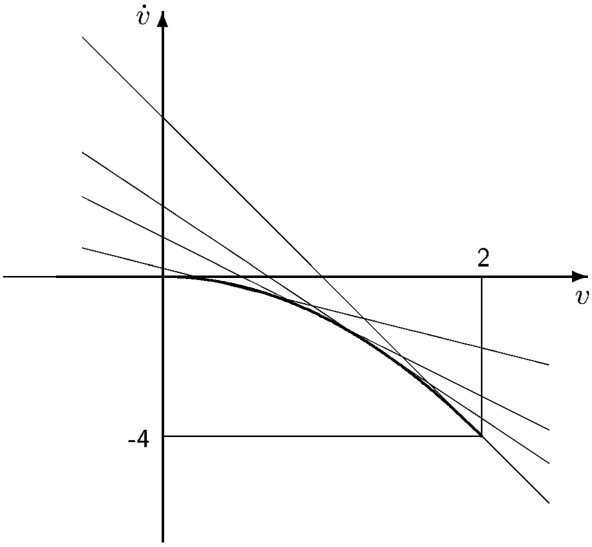

Figure 15: Affine model with decoupled dynamics

Figure 15 interprets this model as an affine local model which moves continuously tangent to the empirical one, by changing independently $a_0(v)$ and $a_1(v)$.

This result can be generalized to n^{th} order systems provided that the empirical model can be expressed as $x_{n+1} = f(\mathbf{x}) = \prod_{l=1}^{n} g_l(x_l)$, or $x_{n+1} = f(\mathbf{x}) = \sum_{l=1}^{n} g_l(x_l)$.

6 Conclusion

This paper has shown how Mamdani and Takagi-Sugeno models can be combined so that local and global interpretations are preserved. A novel affine global fuzzy model has been introduced. This model uses different sets of rules for each coefficient of the affine model, and so decoupling the dynamics of the system. Furhtermore, the model is easy to implement, as it has been shown in the examples.

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