NOTE ON WEIGHTED AVERAGE STRIKE ASIAN OPTIONS

Masatoshi Miyake
Department of Industrial Administration,
Tokyo University of Science,
Noda-shi Chiba 278-8510, Japan
toshimiya1@yahoo.co.jp

Hiroshi Inoue
School of Management,
Tokyo University of Science,
Kuki-shi Saitama 346-8512, Japan
inoue@ms.kuki.tus.ac.jp

Abstract

Asian options are path-dependent and have payoffs which depend on the average price over a fixed period leading up to the maturity date. This option is of interest and important for thinly-traded assets since price manipulation is prohibited, and both the investor and issuer may enjoy a certain degree of protection from the caprice of the market. There are several nice results for the average options with different approaches. In this paper, we consider to propose a more general weight instead of usual simple average, for which it may be possible to control the weights in the light of unexpected situation incurred. In particular, we focus on strike Asian option with weighted average of asset prices.

Keywords: Strike options, Weighted average Asian options, Option pricing.

1 Introduction

For the study of European option pricing problems F.Black, M.Scholes and R.Merton made a major breakthrough in the past and the Black-Scholes model is still used in various situations, since the idea of options can be easily applied for, in particular, many problems related to finance. Asian options are fully path-dependent and have payoffs which depend on the average price of the underlying asset over a fixed period leading up to the maturity date. This option is of interest and important for thinly-traded assets since price manipulation is prohibited, and both the investor and issuer may enjoy a certain degree of protection from the caprice of the market. Since no general analytical solution for the price of the average option is known, a variety of techniques have been developed to analyze average options. There is historically enormous literature devoted to work of this type of options. Among them Bergman(1981) studies average rate options but only considers options with a zero strike price. Kemna and Vorst(1990,1992) propose a Monte Carlo methodology which employs the corresponding geometric option as a control variable. Carverhill and Clewlow(1990) use the Fourier Transformation to evaluate numerically the necessary convolutions of density functions. Ruttiens(1990) and Vorst(1990) employing the solution to the corresponding geometric average problem improves the speed of calculation. On the other hand, Levy(1992) proposes a more accurate method which relies on the assumption that the distribution of sum of log normal variables is itself well approximated at least to a first order by the log normal. Also, the author assumes that the valuation of average option becomes possible for typical range of volatility experienced. The idea of such options are of particular interest and importance for thinly-traded assets since price manipulation is inhibited. This option is in general considered path dependent options

for which the payoff at maturity date depends on the history of the prices the underlying asset takes.

In this paper, we consider to propose a more general weight instead of usual simple average, for which it may be possible to control the weights in the light of unexpected situation incurred. In particular, we focus on strike option for weighted average of asset prices, providing numerical examples.

2 SIMPLE AVERAGE OPTION

We first assume that a perfect security market which is open continuously, offers a constant riskless interest rate to borrowers and lenders, in which no transaction costs and /or taxes are incurred. Let the underlying asset price at time follow the geometric Brownian motion process

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),$$

where $dW(t)$ stands for a Wiener process which has a normal distribution with the mean 0 the variance dt, μ is the drift parameter and σ is the volatility parameter.

The standard partial differential equation for the option price C can be derived by hedging arguments([10],[11]):

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{ss} + r(SC_s - C) = 0$$

where C_t, C_s and C_{ss} are the first order partial derivatives with respect to t and S and C_{ss} a second order partial derivative with respect to S.

For $t \in [T_0, T]$ consider the variable $A(t)$ as

$$A(t) = \frac{1}{t - T_0} \int_{T_0}^{t} S(\tau)d\tau$$

where T is the maturity date. Then $A(t)$ is defined as the part of the final average up to time t and the payoff on the call option can be expressed as $\text{Max}\,(A(t) - K, 0)$, where K is the exercise price of the average option. Kemna and Vorst derive the expression for the value of the average option

$$C(S(t), A(t), t) = e^{-r(T-t)} \times$$
$$E^{\#}\left[\text{Max}\,(A(t) - K, 0)\right]$$

with appropriate boundary conditions, where $E^{\#}$ indicates the conditional expectation with respect to $S(t)$, $A(t)$ and t.

In this paper, we are concerned with an average strike option. Average strike call options may guarantee that the average price paid for an asset in frequent trading over a period of time is not greater than the final price. Thus, we have

$$C(S(t), A(t), t) = e^{-r(T-t)} \times$$
$$E^{\#}\left[\text{Max}\,(S(t) - A(t), 0)\right]$$

Also, log value of $S(\tau)$ after τ period passed is expressed as

$$\ln S(\tau) = \ln S(T_0) + (r - 1/2\sigma^2)\tau + \sigma W(\tau)$$

where $\sigma W(\tau) \sim N(0, \sigma^2\tau)$.

If asset prices are taken as just simple average its variation the asset possesses are almost lost. Therefore, we consider a more general weighted average rather than the simple average.

3 WEIGHTED AVERAGE STRIKE ASIAN OPTION

We want to propose a more general way, in which weighted average rather than the simple average to be able to control its variation is considered. In this study, we find a way to obtain the weighted sums of prices which depend on each asset price $S(T_i)$ at time T_i. Let the weight at T_i be W_i. Denote by $T_i = ih$ for $i = 1, 2, \cdots, N$ where $h = (T - T_0)/N$.

Then define the weighted average, for $1 < m < N$

$$A(T_m) = \sum_{i=1}^{m} W_i S(T_i) \ \text{ with } \ \sum_{i=1}^{m} W_i = 1,$$

$W_i > 0$, $i = 1, 2, \cdots, N$.

Thus, the weighted average strike Asian options are characterized by the payoff function at time T_m, which are given by $\text{Max}(S(T_m) - A(T_m), 0)$ for a call and $\text{Max}(A(T_m) - S(T_m), 0)$ for a put option. Over a fixed contract date to the maturity date T_m we consider the following weighted average defined

below

$$A(T_m) = \sum_{i=1}^{m} \frac{(1+ah)^i}{\sum_{k=1}^{N}(1+ah)^k} S\left(T_i\right), \quad (1)$$

$-\infty < a < \infty$

In this expression a variable a to control the weights is incorporated and hence it may cope with more different situations incurred by underlying asset. But the expression above is not suitable since each asset price $S(T_i)$ is assumed to be log normally distributed and hence $A(T_m)$ is a weighted sum of lognormally distributed. Then, $A(T_m)$ is no longer lognormally distributed. Therefore, for a possible way to develop an approximation another type of weighted average option needs to be considered.

The final payoff at maturity of a call option on the arithmetic average is, denoting by C_A the values of the option,

$$C_A = e^{-r(T-T_0)} \times$$
$$E^{\#}\left[\text{Max}\left(S(T_m) - A(T_m), 0\right)\right].$$

Finally, the value of arithmetic weighted average of (1) for continuous case is defined as

$$A(T) = \int_{T_0}^{T} \frac{e^{a\tau}}{\int_{T_0}^{T} e^{as}ds} S\left(\tau\right) d\tau \qquad (2)$$

4 A PRICING FORMULA BASED ON MOMENTS

If the conventional assumption of a geometric diffusion is specified for the underlying price process, as we have seen above, options involving the arithmetic average may not have closed-form solutions. Levy[6] proposes a more accurate method which relies on the assumption that the distribution of sum of log normal variables is itself well approximated at least to a first order by the lognormal. He mentions that the valuation of average option becomes possible for typical range of volatility experienced. Turnbull and Wakeman[15] also recognize the suitability of the log normal as a first-order approximation. Thus, since first and second moments for arithmetic weighted average are possible to obtain we first define

a new variable which follows lognormal distribution and corresponds up to first and second moments of the arithmetic weighted average. Then, option price obtained for such variable defined is nothing but that for plain option and hence Black-Scholes formula may be applicable.

Letting T_0 be 0, the value of $S(\tau)$ after τ period passed is expressed as

$$S\left(\tau\right) = S(0)e^{\left(r-\frac{1}{2}\sigma^2\right)\tau + \sigma W(\tau)}$$

where $\sigma W\left(\tau\right) \sim N\left(0, \sigma^2\tau\right)$. We also found the first moment of $A(T)$ as (2), and the expected value is easily obtained

$$E[A(T)] = \frac{aS(0)}{e^{aT}-1}\left(\frac{e^{(r+a)T}-1}{r+a}\right). \quad (3)$$

Note: As $a \to 0$ $E[A(T)] \to \frac{S(0)e^{rT}-1}{rT}$.

Then, the second moment becomes

$$E\left[A\left(T\right)^2\right] = 2\left(\frac{aS\left(0\right)}{e^{aT}-1}\right)^2$$
$$\left(\frac{e^{\left(2(r+a)+\sigma^2\right)T}-1}{\left(2\left(r+a\right)+\sigma^2\right)\left(r+a+\sigma^2\right)}\right.$$
$$\left.-\frac{e^{(r+a)T}-1}{\left(r+a\right)\left(r+a+\sigma^2\right)}\right). \quad (4)$$

Also, the expected value of the product of $S\left(T\right), A\left(T\right)$ is

$$E[S(T)A(T)] = \frac{aS(0)^2 e^{rT}(e^{(r+a+\sigma^2)T}-1)}{(e^{aT}-1)(r+a+\sigma^2)}. \quad (5)$$

On the other hand, a new variable, $X\left(T\right)$ we now introduce is defined to have the same price as that of underlying asset after T period passed. Assume $\sigma W\left(\tau\right) \sim N\left(0, \sigma^2\tau\right)$. Denoting r_x, σ_x by drift rate and volatility for the variable $X\left(\tau\right)$, respectively, define

$$X\left(\tau\right) = S(0)e^{\left(r_x - \frac{1}{2}\sigma_x^2\right)\tau + \sigma_x W(\tau)}$$

Then, the first and second moments for $X\left(T\right)$ can be obtained as

$$E\left[X\left(T\right)\right] = S(0)e^{r_x T}$$
$$E\left[X\left(T\right)^2\right] = S(0)^2 e^{\left(2r_x + \sigma_x^2\right)T}.$$

Also, the expected value of $X(T), Y(T)$ is

$$E[X(T)Y(T)] = S(0)^2 e^{(r+r_x+\rho_{sx}\sigma\sigma_x)T}.$$

These results along with (3),(4),(5) give the expression

$$S(0)e^{r_xT} = \frac{aS(0)}{e^{aT}-1}\left(\frac{e^{(r+a)T}-1}{r+a}\right).$$

From the above we obtain

$$r_x = \frac{1}{T}\ln\left(\frac{a}{e^{aT}-1}\left(\frac{e^{(r+a)T}-1}{r+a}\right)\right),$$

and for σ_x we have

$$S(0)^2 e^{(2r_x+\sigma_x^2)T} = 2\left(\frac{aS(0)}{e^{aT}-1}\right)^2$$
$$\left(\frac{e^{(2(r+a)+\sigma^2)T}-1}{(2(r+a)+\sigma^2)(r+a+\sigma^2)}\right.$$
$$\left.-\frac{e^{(r+a)T}-1}{(r+a)(r+a+\sigma^2)}\right).$$

Hence, we have

$$\sigma_x = \sqrt{\frac{1}{T}\ln A - 2r_x}$$

where

$$A = 2\left(\frac{a}{e^{aT}-1}\right)^2$$
$$\left(\frac{e^{(2(r+a)+\sigma^2)T}-1}{(2(r+a)+\sigma^2)(r+a+\sigma^2)}\right.$$
$$\left.-\frac{e^{(r+a)T}-1}{(r+a)(r+a+\sigma^2)}\right)$$

For ρ_{sx}

$$\rho_{sx} = \frac{1}{\sigma\sigma_x}\left(\frac{1}{T}\ln\left(\frac{ae^{rT}\left(e^{(r+a+\sigma^2)T}-1\right)}{(e^{aT}-1)(r+a+\sigma^2)}\right)\right.$$
$$\left.-r-r_x\right)$$

Therefore, the approximation formulae for strike option of arithmetic weighted average are described below,

$$\tilde{C}_A = S(0)N(d_1) - S(0)e^{(r_x-r)T}N(d_2)$$
$$\tilde{P}_A = -S(0)N(-d_1) + S(0)e^{(r_x-r)T}N(-d_2)$$
$$(6)$$

where

$$d_1 = \frac{\left(r-r_x+\bar{\sigma}^2/2\right)T}{\bar{\sigma}\sqrt{T}}$$
$$d_2 = \frac{\left(r-r_x-\bar{\sigma}^2/2\right)T}{\bar{\sigma}\sqrt{T}}$$
$$\bar{\sigma} = \sqrt{\sigma^2 + \sigma_x^2 - 2\rho_{sx}\sigma\sigma_x}$$

5 NUMERICAL EXAMPLE

Approximation forms for pricing of arithmetic weighted average are showed when the weight is allowed to vary. Let $S(0)=100$, non-risk rate=0.5% volatility=20% and T=1year. It is observed in Figure 1 that as $a \to +\infty$ the premium values for both call and put approach to 0 while the premium values become close to the values of plain options, call=8.19,put=7.699 respectively, with exercise price $S(0)$ as $a \to -\infty$.

As $a \to 0$ the premium values approach to the values of usual average strike option, call=4.706, put=4.456, respectively.
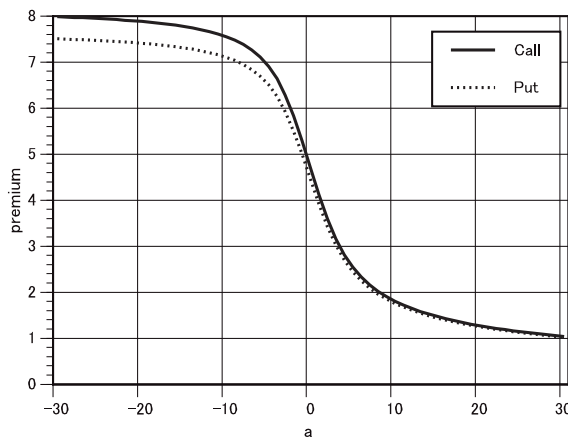

Figure 1: Premiums for different weights

It is also observed from Figure 2 that the premium value for each different weight increases linearly with the asset prices. The reason is understood from the fact that figure are characterized as average strike option having payoff of two variables, which is different from the plain option with fixed exercise price. Under the influence of the weights a the premium values always become high as more weights are placed on around contract date, while the

premium values become small if relative importance is moved to the date of maturity date.
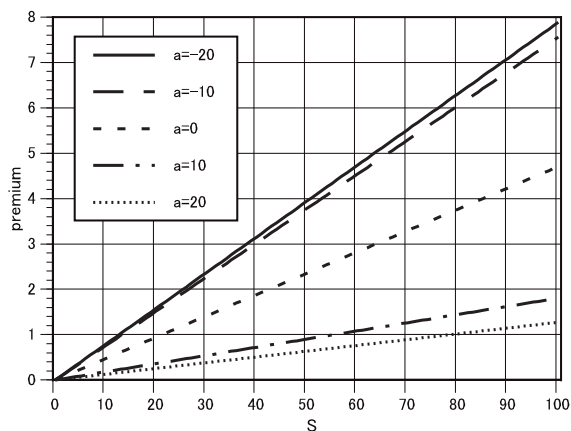

Figure 2: Premiums of call for different asset prices

6 NUMERICAL COMPUTATIONS

The performance of the analytical method is examined and compared with the approximation method by Monte Carlo simulations.

Let $S(0){=}100$, volatility$=10\%,20\%,30\%$, non-risk rate$=0.5\%$, T$=1$year and N$=50$. Simulation was performed 10000 times.
To perform numerical computation the period $[0, T]$ is divided into n subintervals. Then, it is distributed as a normal distribution with the mean $(r - \sigma^2/2)T/n$, variance $\sigma^2 T/n$. Therefore, the random sequence $S(T_1), \cdots, S(T_n)$ can be generated as follows

$$\ln S(T_i) = \ln S(T_{i-1}) + \left(r - \frac{1}{2}\sigma^2\right)\frac{T}{n} + \sigma\sqrt{\frac{T}{n}}z_i,$$

where z_i is assumed to be drawn from the standard normal distribution.

Then, with the form (1) the premium values for the arithmetic weighted sums are obtained by $\tilde{C}_A$ and $\tilde{P}_A$.

$$\tilde{C}_A = e^{-rT}E^{\#}[\text{Max}\,(S(T_m) - A(T_m), 0)]$$
$$\tilde{P}_A = e^{-rT}E^{\#}[\text{Max}\,(A(T_m) - S(T_m), 0)]$$

Thus, by Monte Carlo simulation the values of realizations of $\tilde{C}_A$ are calculated.

Table 1: Premium values ($\sigma{=}10\%$)

a	Monte Carlo	Approximation
-30	4.17569	4.12331
-20	4.13079	4.06785
-10	3.92563	3.89721
0	2.37631	2.42370
10	0.86805	0.91642
20	0.57390	0.64308
30	0.44578	0.52325

Table 2: Premium values ($\sigma{=}20\%$)

a	Monte Carlo	Approximation
-30	8.09412	7.98907
-20	7.99626	7.88266
-10	7.65032	7.55446
0	4.68927	4.70597
10	1.70827	1.80543
20	1.14697	1.27279
30	0.90823	1.037703

Table 3: Premium values ($\sigma{=}30\%$)

a	Monte Carlo	Approximation
-30	12.01262	11.83785
-20	11.78060	11.68088
-10	11.36930	11.19539
0	6.90673	6.96221
10	2.53635	2.69032
20	1.71177	1.90104
30	1.33115	1.55136

First, Monte Carlo shows higher premium values than those of Approximation method as long as the weights are negative and the opposite situation is observed for non-negative weights.

It is observed from Table1 through Table3 that the differences of premium values between Monte Carlo and Approximation method occur in most cases in the first decimal place with $\sigma{=}30\%$ while they occur only in the second decimal with $\sigma{=}10\%$.

7 HEDGE RATIO

Issuers of options will not only be interested in the price of options but they must also

develop a fair understanding of the risks involved. In particular, the hedge ratio including other sensitivity parameters of call and put options are of important and interest to them. The analyses are able to assess how a contract fit into the existing portfolio and to what extent additional hedging is necessary. We calculate the hedge ratio, Δ for which the approximation formulae of weighted average are used. The derivatives of formulae developed in the previous sections are based on (6). As expected, the delta for call and put options are described, respectively, as

$$\frac{\partial \tilde{C}_A}{\partial S} = N\left(d_1\right) - e^{(r_x - r)T} N\left(d_2\right)$$

$$\frac{\partial \tilde{P}_A}{\partial S} = -N\left(-d_1\right) + e^{(r_x - r)T} N\left(-d_2\right)$$
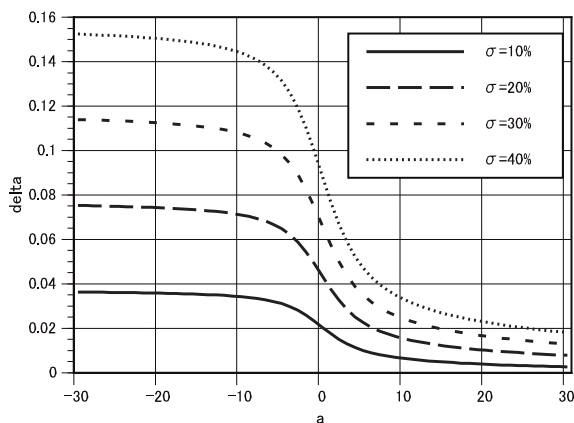

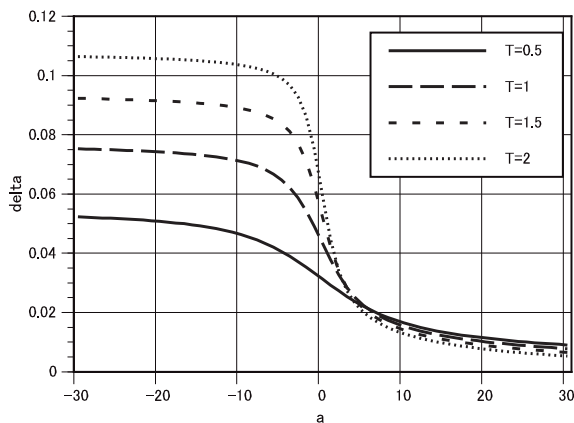Figure 3: Delta for different volatilities

Figure 4: Delta for different time periods

With increase of the relative weights on around maturity date the values of delta become small. Also, it is observed that delta receives the most influence of σ at σ=40% and the influence becomes low as σ becomes small. For remaining period T when more weights are placed on around contract date the longer the period remains the higher the delta value will be. Contrary to the above, when more weights are placed on around maturity date the longer the period remains the less the value of delta will be.

8   CONCLUSION

A simple average option like Asian options, in general, may cause to excessively lose its nature of variation the underlying asset possess. In the light of losing its variation, we proposed a weighted average strike Asian option in which more weights are placed on around maturity date or contract date, making enable for dealing of variation of the underlying asset. As $a \rightarrow +\infty$ weights are placed more on around maturity date, and finally the payoff becomes 0, while as $a \rightarrow -\infty$ the weights on around contract date become large so that call option is reduced to plain option, $\text{Max}(S(T) - S(0), 0)$ with exercise price $S(0)$. Also, concerning the valuation of strike Asian option with weighted average Levy's method was applied for derivation of pricing formulae. Since strike option involves two variables for payoff the methods developed by Kemna & Vorst and Vorst, which are often used for rate options, were not used here.

References

[1] Kemna,A.G.D and A.C.F.(1990). Vorst, A price method for options based on average asset values, Journal of Banking and Finance, 14, 113-129.

[2] Vorst,T.(1992). Price and hedge ratios of average exchange rate options, International Review of Financial Analysis, 1, 179-193.

[3] Vorst,T.(2001). New pricing of asian options, Working Paper.

[4] Henderson(2004). Bounds for in-progress floating-strike asian options using symmetry, JPrinceton University, ORFE and Bendheim Center for Finance.

[5] Wilmott,P.,S.Howison and J. Dewynn (1995). The mathematics of financial derivatives, Cambridge University Press.

[6] Levy,E(1992). Pricing european average rate currency options, Journal of International Money and Finance, 11,474-491.

[7] Cox,J.C and S.A.Ross(1976). The valuation of options for alternative stochastic process, Journal Financial Economics,3, 145-166.

[8] Cox,J.C and M. Rubinstein(1985). Options markets, Prentice Hall.

[9] Jarrow,R.A and A.Rudd(1983). Option pricing, Homewood, Illinois, Dow Jones-Irwin.

[10] Black,F. and M.Scholes(1973). The pricing of options and corporate liabilities, Journal of Political Economics, 81, 637-659.

[11] Merton. R. C. (1973). Theory of rational option pricing, Bell Journal of Economics and Management Science, 4, 141-183.

[12] Carverhill,A.P and L.J.Clewlow(1990). Flexible convolution, Risk, 3, 25-29.

[13] Bergman,Y.Z.(1981). Pricing path-dependent european options, Working Paper, University of California, Berkeley,CA.

[14] Beckenbach,E.F and R.Bellman(1971). Inequalities, Springer, Berlin.

[15] Turnbull and Wakeman(1991). A quick algorithm for pricing european average, Journal of Financial and Quantitative Analysis, 26, 377-389.