WOWA Enhancement of the Reference Point Method

Wlodzimierz Ogryczak
Warsaw University of Technology, Institute of Control & Comput. Engg., Warsaw, Poland
W.Ogryczak@ia.pw.edu.pl

Abstract

The Reference Point Method (RPM) is based on the so-called augmented max-min aggregation where the worst individual achievement maximization process is additionally regularized with the average achievement. The regularization by the average achievement is easily implementable but it may disturb the basic max-min model. The OWA regularization applied to the RPM allows one to overcome this flaw since taking into account differences among all ordered achievement values. Further, allowing to introduce importance weights we get the WOWA enhanced RPM. The WOWA regularization is more complicated in implementation but the recent progress made in optimization methods for ordered averages allows one to implement the corresponding RPM method effectively.

Keywords: Multicriteria Decision Making, Aggregation Methods, Reference Point Method, WOWA.

1 Introduction

Consider a decision problem defined as an optimization problem with m criteria (objective functions). In this paper, without loss of generality, it is assumed that all the criteria are maximized (that is, for each outcome 'more is better'). Hence, we consider the following multiple criteria optimization problem:

$$\max \{ \ (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x})) \ : \ \mathbf{x} \in Q \ \} \quad (1)$$

where $\mathbf{x}$ denotes a vector of decision variables to be selected within the feasible set $Q \subset R^n$, and $\mathbf{f}(x) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ is a vector function that maps the feasible set Q into the criterion space R^m. Note that neither any specific form of the feasible set Q is assumed nor any special form of criteria $f_i(\mathbf{x})$ is required. We refer to the elements of the criterion space as outcome vectors. An outcome vector $\mathbf{y}$ is attainable if it expresses outcomes of a feasible solution, i.e. $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for some $\mathbf{x} \in Q$.

Model (1) only specifies that we are interested in maximization of all objective functions f_i for $i \in I = \{1, 2, \ldots, m\}$. Thus it allows only to identify (to eliminate) obviously inefficient solutions leading to dominated outcome vectors, while still leaving the entire efficient set to look for a satisfactory compromise solution. In order to make the multiple criteria model operational for the decision support process, one needs assume some solution concept well adjusted to the DM preferences. This can be achieved with the so-called quasi-satisficing approach to multiple criteria decision problems. The best formalization of the quasi-satisficing approach to multiple criteria optimization was proposed and developed mainly by Wierzbicki [12] as the Reference Point Method (RPM). The reference point method was later extended to permit additional information from the DM and, eventually, led to efficient implementations of the

L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds): Proceedings of IPMU'08, pp. 274–281
Torremolinos (Málaga), June 22–27, 2008

so-called Aspiration/Reservation Based Decision Support (ARBDS) approach with many successful applications [1, 13].

The RPM is an interactive technique. The basic concept of the interactive scheme is as follows. The DM specifies requirements in terms of reference levels, i.e., by introducing reference (target) values for several individual outcomes. Depending on the specified reference levels, a special scalarizing achievement function is built which may be directly interpreted as expressing utility to be maximized. Maximization of the scalarizing achievement function generates an efficient solution to the multiple criteria problem. The computed efficient solution is presented to the DM as the current solution in a form that allows comparison with the previous ones and modification of the reference levels if necessary.

The scalarizing achievement function can be viewed as two-stage transformation of the original outcomes. First, the strictly monotonic partial achievement functions are built to measure individual performance with respect to given reference levels. Having all the outcomes transformed into a uniform scale of individual achievements they are aggregated at the second stage to form a unique scalarization. The RPM is based on the so-called augmented (or regularized) max-min aggregation. Thus, the worst individual achievement is essentially maximized but the optimization process is additionally regularized with the term representing the average achievement. The max-min aggregation guarantees fair treatment of all individual achievements by implementing an approximation to the Rawlsian principle of justice.

The max-min aggregation is crucial for allowing the RPM to generate all efficient solutions even for nonconvex (and particularly discrete) problems. On the other hand, the regularization is necessary to guarantee that only efficient solution are generated. The regularization by the average achievement is easily implementable but it may disturb the basic max-min model. Actually, the only consequent regularization of the max-min aggregation is the lex-min order or more practical the OWA aggregation with monotonic weights. The latter combines all the partial achievements allocating the largest weight to the worst achievement, the second largest weight to the second worst achievement, the third largest weight to the third worst achievement, and so on. The recent progress in optimization methods for ordered averages [7] allows one to implement the OWA RPM quite effectively. Further following the concept of Weighted OWA [11] the importance weighting of several achievements may be incorporated into the RPM. Such a WOWA enhancement of the RPM uses importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements rather than straightforward rescaling of achievement values [10]. The paper analyzes both the theoretical and implementation issues of the WOWA enhanced RPM.

2 Scalarizations of the RPM

While building the scalarizing achievement function the following properties of the preference model are assumed. First of all, for any individual outcome y_i more is preferred to less (maximization). To meet this requirement the function must be strictly increasing with respect to each outcome. Second, a solution with all individual outcomes y_i satisfying the corresponding reference levels is preferred to any solution with at least one individual outcome worse (smaller) than its reference level. That means, the scalarizing achievement function maximization must enforce reaching the reference levels prior to further improving of criteria. Thus, similar to the goal programming approaches, the reference levels are treated as the targets but following the quasi-satisficing approach they are interpreted consistently with basic concepts of efficiency in the sense that the optimization is continued even when the target point has been reached already.

The generic scalarizing achievement function takes the following form [12]:

$$S(\mathbf{y}) = \min_{1\leq i\leq m} \{s_i(y_i)\} + \frac{\varepsilon}{m} \sum_{i=1}^{m} s_i(y_i) \qquad (2)$$

where ε is an arbitrary small positive number and $s_i : R \to R$, for $i = 1, 2, \ldots, m$, are the partial achievement functions measuring actual achievement of the individual outcomes y_i with respect to the corresponding reference levels. Let a_i denote the partial achievement for the ith outcome ($a_i = s_i(y_i)$) and $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ represent the achievement vector. The scalarizing achievement function (2) is, essentially, defined by the worst partial (individual) achievement but additionally regularized with the sum of all partial achievements. The regularization term is introduced only to guarantee the solution efficiency in the case when the maximization of the main term (the worst partial achievement) results in a non-unique optimal solution. Due to combining two terms with arbitrarily small parameter ε, formula (2) is easily implementable and it provides a direct interpretation of the scalarizing achievement function as expressing utility.

Various functions s_i provide a wide modeling environment for measuring partial achievements [13]. The basic RPM model is based on a single vector of the reference levels, the aspiration vector $\mathbf{r}^a$ and the piecewise linear functions s_i.

Real-life applications of the RPM methodology usually deal with more complex partial achievement functions defined with more than one reference point [13] which enriches the preference models and simplifies the interactive analysis. In particular, the models taking advantages of two reference vectors: vector of aspiration levels $\mathbf{r}^a$ and vector of reservation levels $\mathbf{r}^r$ [1] are used, thus allowing the DM to specify requirements by introducing acceptable and required values for several outcomes. The partial achievement function s_i can be interpreted then as a measure of the DM's satisfaction with the current value of outcome the ith criterion. It is a strictly increasing function of outcome y_i with value $a_i = 1$ if $y_i = r_i^a$, and $a_i = 0$ for $y_i = r_i^r$. Thus the partial achievement functions map the outcomes values onto a normalized scale of the DM's satisfaction. Various functions can be built meeting those requirements. We use the piece-wise linear partial achievement function introduced in an implementation of the ARBDS system for the multiple criteria transshipment problems with facility location [5]:

$$s_i(y_i) = \begin{cases} \gamma \dfrac{y_i - r_i^r}{r_i^a - r_i^r}, & y_i \leq r_i^r \\ \dfrac{y_i - r_i^r}{r_i^a - r_i^r}, & r_i^r < y_i < r_i^a \\ \alpha \dfrac{y_i - r_i^a}{r_i^a - r_i^r} + 1, & y_i \geq r_i^a \end{cases} \quad (3)$$

where α and γ are arbitrarily defined parameters satisfying $0 < \alpha < 1 < \gamma$. Parameter α represents additional increase of the DM's satisfaction over level 1 when a criterion generates outcomes better than the corresponding aspiration level. On the other hand, parameter $\gamma > 1$ represents dissatisfaction connected with outcomes worse than the reservation level.

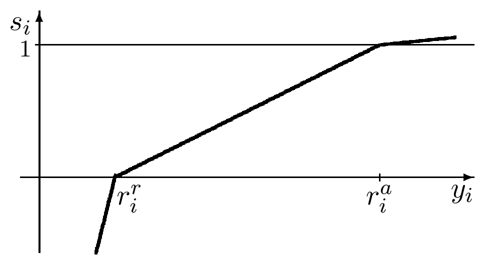

Figure 1: Partial achievement function (3)

For outcomes between the reservation and the aspiration levels, the partial achievement function s_i can be interpreted as a membership function μ_i for a fuzzy target. However, such a membership function remains constant with value 1 for all outcomes greater than the corresponding aspiration level, and with value 0 for all outcomes below the reservation level (Fig. 1). Hence, the fuzzy membership function is neither strictly monotonic nor concave thus not representing typical utility for a maximized outcome. The partial achievement function (3) can be viewed as an extension of the fuzzy membership function to a strictly monotonic and concave utility.

3 OWA refinement of the RPM

The crucial properties of the RPM are related to the max-min aggregation of partial achievements while the regularization is only introduced to guarantee the aggregation monotonicity. Unfortunately, the distribution of achievements may make the max-min criterion partially passive when one specific achievement is relatively very small for all the solutions. Maximization of the worst achievement may then leave all other achievements unoptimized. Nevertheless, the selection is then made according to linear aggregation of the regularization term instead of the max-min aggregation, thus destroying the preference model of the RPM. This can be illustrated with an example of a simple discrete problem of 7 alternative feasible solutions to be selected according to 6 criteria. Table 1 presents six partial achievements for all the solutions where the partial achievements have been defined according to the aspiration/reservation model (3) thus allocating 1 to outcomes reaching the corresponding aspiration level. All the solutions are efficient. Solution S1 to S5 oversteps the aspiration levels (achievement values 1.2) for four of the first five criteria while failing to reach one of them and the aspiration level for the sixth criterion as well (achievement values 0.3). Solution S6 meets the aspiration levels (achievement values 1.0) for the first five criteria while while failing to reach only the aspiration level for the sixth criterion (achievement values 0.3). All the solutions generate the same worst achievement value 0.3 and the final selection of the RPM depends on the total achievement (regularization term). Actually, one of solutions S1 to S5 will be selected as better than S6.

In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the ordered averaging rules [14]. This is mathematically formalized as follows. Within the space of achievement vectors we introduce map $\Theta = (\theta_1, \theta_2, \dots, \theta_m)$ which orders the coordinates of achievements vectors in a non-increasing order, i.e., $\Theta(a_1, a_2, \dots, a_m) = (\theta_1(\mathbf{a}), \theta_2(\mathbf{a}), \dots, \theta_m(\mathbf{a}))$ iff there exists a permutation τ such that $\theta_i(\mathbf{a}) = a_{\tau(i)}$ for all i and $\theta_1(\mathbf{a}) \geq \theta_2(\mathbf{a}) \geq \dots \geq \theta_m(\mathbf{a})$. The standard max-min aggregation depends on maximization of $\theta_m(\mathbf{a})$ and it ignores values of $\theta_i(\mathbf{a})$ for $i \leq m-1$. In order to take into account all the achievement values, one needs to maximize the weighted combination of of the ordered achievements thus representing the so-called Ordered Weighted Averaging (OWA) aggregation [14]. Note that the weights are then assigned to the specific positions within the ordered achievements rather than to the partial achievements themselves. With the OWA aggregation one gets the following RPM model:

Table 1: Sample achievements with passive max-min criterion

Sol.	a_1	a_2	a_3	a_4	a_5	a_6	min	$\sum$
S1	0.3	1.2	1.2	1.2	1.2	0.3	0.3	5.4
S2	1.2	0.3	1.2	1.2	1.2	0.3	0.3	5.4
S3	1.2	1.2	0.3	1.2	1.2	0.3	0.3	5.4
S4	1.2	1.2	1.2	0.3	1.2	0.3	0.3	5.4
S5	1.2	1.2	1.2	1.2	0.3	0.3	0.3	5.4
S6	1.0	1.0	1.0	1.0	1.0	0.3	0.3	5.3
S7	0.3	0.3	0.3	1.0	0.6	1.0	0.3	3.5

$$\max \sum_{i=1}^{m} w_i \theta_i(\mathbf{a}) \qquad (4)$$

where $w_1 < w_2 < \dots < w_m$ are positive and strictly increasing weights. Actually, they should be significantly increasing to represent regularization of the max-min order. When differences among weights tend to infinity, the OWA aggregation approximates the leximin ranking of the ordered outcome vectors [15]. Note that the standard RPM model with the scalarizing achievement function (2) can be expressed as the following OWA model:

$$\max \left((1 + \frac{\varepsilon}{m})\theta_m(\mathbf{a}) + \frac{\varepsilon}{m} \sum_{i=1}^{m-1} \theta_i(\mathbf{a}) \right)$$

Hence, the standard RPM model exactly represents the OWA aggregation (4) with strictly increasing weights in the case of $m = 2$ ($w_1 = \varepsilon/2 < w_2 = 1 + \varepsilon/2$). For $m > 2$ it abandons the differences in weighting of the largest achievement, the second largest one etc ($w_1 = \dots = w_{m-1} = \varepsilon/m$). The OWA RPM model 4 allows one to distinguish all the weights by introducing increasing series (e.g. geometric ones). One may notice in Table 2 that application of increasing weights $\mathbf{w} = (0.02, 0.03, 0.05, 0.15, 0.25, 0.5)$ within the OWA RPM enables selection of solution S6 from Table 1.

Table 2: Ordered achievements values

Sol.	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	$A_{\mathbf{w}}$
S1	1.2	1.2	1.2	1.2	0.3	0.3	0.525
S2	1.2	1.2	1.2	1.2	0.3	0.3	0.525
S3	1.2	1.2	1.2	1.2	0.3	0.3	0.525
S4	1.2	1.2	1.2	1.2	0.3	0.3	0.525
S5	1.2	1.2	1.2	1.2	0.3	0.3	0.525
S6	1.0	1.0	1.0	1.0	1.0	0.3	0.650
S7	1.0	1.0	0.6	0.3	0.3	0.3	0.305
w	0.02	0.03	0.05	0.15	0.25	0.5	

An important advantage of the RPM depends on its easy implementation as an expansion of the original multiple criteria model. Actually, even complicated partial achievement functions of the form (3) are strictly increasing and concave, thus allowing for implementation of the entire RPM model (2) by an LP expansion [5].

The OWA aggregation is obviously a piecewise linear function since it remains linear within every area of the fixed order of arguments. The ordered achievements used in the OWA aggregation are, in general, hard to implement due to the pointwise ordering. Its optimization can be implemented by expressing in terms of the cumulated ordered achievements $\bar{\theta}_k(\mathbf{a}) = \sum_{i=1}^{k} \theta_{m-i+1}(\mathbf{a})$ expressing, respectively: the worst (smallest) achievement, the total of the two worst achievements, the total of the three worst achievements, etc. Indeed,

$$\sum_{i=1}^{m} w_i \theta_i(\mathbf{a}) = \sum_{i=1}^{m} w'_i \bar{\theta}_i(\mathbf{a})$$

where $w'_i = w_{m-i+1} - w_{m-i}$ for $i = 1, \ldots, m-1$, and $w'_m = w_1$. This simplifies dramatically the optimization problem since quantities $\bar{\theta}_k(\mathbf{a})$ can be optimized without use of any integer variables [7]. First, let us notice that for any given vector $\mathbf{a}$, the cumulated ordered value $\bar{\theta}_k(\mathbf{a})$ can be found as the optimal value of the following LP problem:

$$\begin{aligned} \bar{\theta}_k(\mathbf{a}) = \min_{u_{ik}} \{\sum_{i=1}^{m} a_i u_{ik} : \\ \sum_{i=1}^{m} u_{ik} = k, 0 \leq u_{ik} \leq 1 \quad \forall i \ \} \end{aligned} \tag{5}$$

The above problem is an LP for a given outcome vector $\mathbf{a}$ while it becomes nonlinear for $\mathbf{a}$ being a vector of variables. This difficulty can be overcome by taking advantage of the LP dual to (5). Introducing dual variable t_k corresponding to the equation $\sum_{i=1}^{m} u_{ik} = k$ and variables d_{ik} corresponding to upper bounds on u_{ik} one gets the following LP dual of problem (5):

$$\begin{aligned} \bar{\theta}_k(\mathbf{a}) = \max_{t_k, d_{ik}} \{k t_k - \sum_{i=1}^{m} d_{ik} : \\ a_i \geq t_k - d_{ik}, \ d_{ik} \geq 0 \quad \forall \ i \ \} \end{aligned} \tag{6}$$

Due the duality theory, for any given vector $\mathbf{a}$ the cumulated ordered coefficient $\bar{\theta}_k(\mathbf{a})$ can be found as the optimal value of the above LP problem.

Taking advantages of the LP expression (6) for $\bar{\theta}_i$ the entire OWA aggregation of the partial achievement functions (4) can be expressed in terms of LP. Moreover, in the case of concave piecewise linear partial achievement functions (as typically used in the RPM approaches), the resulting formulation extends the original constraints and criteria with linear inequalities. In particular, for strictly increasing and concave partial achievement functions (3), it can be expressed in the form:

$$\begin{array}{ll} \max \sum_{k=1}^{m} w'_k z_k & \\ \text{s.t.} & \\ z_k = k t_k - \sum_{i=1}^{m} d_{ik} & \forall \ k \\ \mathbf{x} \in Q, \ y_i = f_i(\mathbf{x}) & \forall \ i \\ a_i \geq t_k - d_{ik}, \ d_{ik} \geq 0 & \forall \ i, k \\ a_i \leq \gamma(y_i - r_i^r)/(r_i^a - r_i^r) & \forall \ i \\ a_i \leq (y_i - r_i^r)/(r_i^a - r_i^r) & \forall \ i \\ a_i \leq \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall \ i \end{array} \tag{7}$$

4 WOWA enhancement

Let $\mathbf{w} = (w_1, \ldots, w_m)$ and $\mathbf{p} = (p_1, \ldots, p_m)$ be weighting vectors of dimension m such that $w_i \geq 0$ and $p_i \geq 0$ for $i = 1, 2, \ldots, m$ as well as $\sum_{i=1}^{m} p_i = 1$ (typically it is also assumed $\sum_{i=1}^{m} w_i = 1$ but it is not necessary in our applications). The corresponding Weighted OWA aggregation of outcomes $\mathbf{a} = (a_1, \ldots, a_m)$ is defined as follows [11]:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{i=1}^{m} \omega_i \theta_i(\mathbf{a}) \tag{8}$$

where the weights ω_i are defined as

$$\omega_i = w^*(\sum_{k \leq i} p_{\tau(k)}) - w^*(\sum_{k < i} p_{\tau(k)}) \tag{9}$$

with w^* a monotone increasing function that interpolates points $(\frac{i}{m}, \sum_{k \leq i} w_k)$ together with the point (0.0) and τ representing the ordering permutation for $\mathbf{a}$ (i.e. $a_{\tau(i)} = \theta_i(\mathbf{a})$). Moreover, function w^* is required to be a straight line when the point can be interpolated in this way, thus allowing the WOWA to cover the standard weighted mean with weights p_i as a special case of equal preference weights ($w_i = 1/m$ for $i = 1, 2, \ldots, m$).

The WOWA may be expressed with more direct formula where preferential (OWA) weights w_i are applied to averages of the corresponding portions of ordered achievements (quantile intervals) (according to the distribution defined by importance weights p_i) [8, 9]:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{i=1}^{m} w_i m \int_{\frac{i-1}{m}}^{\frac{i}{m}} \overline{F}_{\mathbf{a}}^{(-1)}(\xi)\, d\xi \tag{10}$$

where $\overline{F}_{\mathbf{a}}^{(-1)}$ is the stepwise function $\overline{F}_{\mathbf{a}}^{(-1)}(\xi) = \theta_i(\mathbf{a})$ for $\beta_{i-1} < \xi \leq \beta_i$. It can also be mathematically formalized as follows. First, we introduce the right-continuous cumulative distribution function (cdf):

$$F_{\mathbf{a}}(d) = \sum_{i=1}^{m} p_i \delta_i(d) \tag{11}$$

where $\delta_i(d) = 1$ if $a_i \leq d$ and 0 otherwise. Next, we introduce the quantile function $F_{\mathbf{a}}^{(-1)} = \inf\{\eta : F_{\mathbf{a}}(\eta) \geq \xi\}$ for $0 < \xi \leq 1$ as the left-continuous inverse of the cumulative distribution function $F_{\mathbf{a}}$, ie., $F_{\mathbf{a}}^{(-1)}(\xi) = \inf\{\eta : F_{\mathbf{a}}(\eta) \geq \xi\}$for $0 < \xi \leq 1$, and finally $\overline{F}_{\mathbf{a}}^{(-1)}(\xi) = F_{\mathbf{a}}^{(-1)}(1 - \xi)$.

For instance applying importance weighting $\mathbf{p} = (\frac{4}{12}, \frac{3}{12}, \frac{2}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$ to solution achievements from Table 1 and using them together with the OWA weights $\mathbf{w}$ from Table 2 one gets the WOWA aggregations from Table 3. The corresponding RPM method selects than solution S6, similarly to the case of equal importance weights. On the other hand, when increasing the importance of the last outcome achievements with $\mathbf{p} = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{12})$ one gets the WOWA values from Table 4.

Formula (10) defines the WOWA value applying preferential weights w_i to importance weighted averages within quantile intervals. It may be reformulated with the tail averages:

$$A_{\mathbf{w},\mathbf{p}}(\mathbf{a}) = \sum_{k=1}^{m} w'_k m L(\mathbf{a}, \mathbf{p}, \frac{k}{m}) \tag{12}$$

where $L(\mathbf{y}, \mathbf{p}, \xi)$ is defined by left-tail integrating of $F_{\mathbf{y}}^{(-1)}$, i.e.

$$L(\mathbf{y}, \mathbf{p}, \xi) = \int_0^{\xi} F_{\mathbf{y}}^{(-1)}(\alpha) d\alpha \tag{13}$$

and weights $w'_k = w_{m-k+1} - w_{m-k}$ for $k = 1, \ldots, m-1$ and $w'_m = w_1$.

Graphs of functions $L(\mathbf{a}, \mathbf{p}, \xi)$ (with respect to ξ) take the form of convex piecewise linear curves, the so-called absolute Lorenz curves [6] connected to the relation of the second order stochastic dominance (SSD). Therefore, formula (12) relates the WOWA average to the SSD consistent risk measures based on the tail means provided that the importance weights are treated as scenario probabilities.

According to (13), values of function $L(\mathbf{a}, \mathbf{p}, \xi)$ for any $0 \leq \xi \leq 1$ can be given by optimization:

$$L(\mathbf{a}, \mathbf{p}, \xi) = \min_{s_i} \{\sum_{i=1}^{m} a_i s_i : \sum_{i=1}^{m} s_i = \xi, \quad 0 \leq s_i \leq p_i \quad \forall\ i\} \tag{14}$$

Introducing dual variable t corresponding to the equation $\sum_{i=1}^{m} s_i = \xi$ and variables d_i

Table 3: WOWA selection with $\mathbf{p} = (\frac{4}{12}, \frac{3}{12}, \frac{2}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12})$

w	0.02		0.03		0.05		0.15		0.25		0.5		$A_{\mathbf{w},\mathbf{p}}(\mathbf{a})$
S1	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.4575
S2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.525
S3	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.6375
S4	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.3	0.3	0.75
S5	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2	0.3	0.3	0.75
S6	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.3	0.825
S7	1.0	1.0	0.6	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3185

Table 4: WOWA selection with $\mathbf{p} = (\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{12})$

w	0.02		0.03		0.05		0.15		0.25		0.5		$A_{\mathbf{w},\mathbf{p}}(\mathbf{a})$
S1	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.345
S2	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.345
S3	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.345
S4	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.345
S5	1.2	1.2	1.2	1.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.345
S6	1.0	1.0	1.0	1.0	1.0	0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3525
S7	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.6	0.3	0.3	0.3	0.5125

corresponding to upper bounds on s_i one gets the following LP dual expression of $L(\mathbf{a},\mathbf{p},\xi)$

$$L(\mathbf{a},\mathbf{p},\xi) = \max_{t,d_i} \{\xi t - \sum_{i=1}^{m} p_i d_i : \quad t - d_i \le a_i,\ d_i \ge 0 \quad \forall\, i\} \tag{15}$$

Following (12) and (15) taking into account piecewise linear partial achievement functions (3) one gets finally the following model for the WOWA Reference Point Method with piecewise linear partial achievement functions (3):

$$\begin{array}{ll} \max \sum_{k=1}^{m} w'_k z_k & \\ \text{s.t.} & \\ z_k = k t_k - m \sum_{i=1}^{m} p_i d_{ik} & \forall\, k \\ \mathbf{x} \in Q,\ y_i = f_i(\mathbf{x}) & \forall\, i \\ a_i \ge t_k - d_{ik},\ d_{ik} \ge 0 & \forall\, i,k \\ a_i \le \gamma(y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ a_i \le (y_i - r_i^r)/(r_i^a - r_i^r) & \forall\, i \\ a_i \le \alpha(y_i - r_i^a)/(r_i^a - r_i^r) + 1 & \forall\, i \end{array} \tag{16}$$

Conclusions

The reference point method is a very convenient technique for interactive analysis of the multiple criteria optimization problems. It provides the DM with a tool for an open analysis of the efficient frontier. The interactive analysis is navigated with the commonly accepted control parameters expressing reference levels for the individual objective functions. The partial achievement functions quantify the DM satisfaction from the individual outcomes with respect to the given reference levels. The final scalarizing function is built as the augmented max-min aggregation of partial achievements which means that the worst individual achievement is essentially maximized but the optimization process is additionally regularized with the term representing the average achievement. The regularization by the average achievement is easily implementable but it may disturb the basic max-min aggregation. In order to avoid inconsistencies caused by the regularization, the max-min solution may be regularized according to the Rawlsian principle of justice leading to the nucleolar RPM model. The nucleolar RPM implements a consequent max-min aggregation taking into account also the second worst achievement, the third worse and so on, thus resulting in much better modeling of the reference levels concept.

The OWA aggregation with monotonic weights combines all the partial achievements allocating the largest weight to the worst achievement, the second largest weight to the second worst achievement, the third largest weight to the third worst achievement, and so on. It approximates nucleolar RPM introducing explicit scalarizing achievement function to be interpreted as utility. Further following the concept of Weighted OWA [11] the importance weighting of several achievements may be incorporated into the RPM. Such a WOWA enhancement of the RPM uses importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements rather than straightforward rescaling of achievement values [10]. The ordered regularizations are more complicated in implementation due to the requirement of pointwise ordering of partial achievements. However, the recent progress in optimization methods for ordered averages [7] allows one to implement the OWA RPM quite effectively by taking advantages of piecewise linear expression of the cumulated ordered achievements. Similar, model can be achieved for the WOWA RPM. Actually, in the case of concave piecewise linear partial achievement functions (typically used in the RPM), the resulting formulation extends the original constraints and criteria with simple linear inequalities thus allowing for a quite efficient implementation.

References

[1] A.Lewandowski, A.P.Wierzbicki (1989). *Aspiration Based Decision Support Systems – Theory, Software and Applications*. Berlin, Springer.

[2] X.Liu (2006). Some properties of the weighted OWA operator. *IEEE Trans. Systems, Man Cyber. B*, vol. 368, pp. 118–127.

[3] W.Ogryczak (2001). On Goal Programming Formulations of the Reference Point Method. *J. Opnl. Res. Soc.*, vol. 52, pp. 691–698.

[4] W.Ogryczak, S.Lahoda (1992). Aspiration/Reservation Decision Support – A Step Beyond Goal Programming. *J. MCDA*, vol. 1, pp. 101–117.

[5] W.Ogryczak, K.Studziński, K.Zorychta (1992). DINAS: A Computer-Assisted Analysis System for Multiobjective Transshipment Pproblems with Facility Location. *Comp. Opns. Res.*, vol. 19, pp. 637–647.

[6] W.Ogryczak, A.Ruszczyński (2002). Dual stochastic dominance and related mean-risk models. *SIAM J. Optimization*, vol. 13, pp. 60–78.

[7] W.Ogryczak, T.Śliwiński (2003). On solving linear programs with the ordered weighted averaging objective. *Eur. J. Opnl. Res.*, vol. 148, pp. 80–91.

[8] W.Ogryczak, T.Śliwiński (2007). On Optimization of the Importance Weighted OWA Aggregation of Multiple Criteria. In *ICCSA 2007*, *LNCS*, vol. 4705, pp. 804-817.

[9] W.Ogryczak, T.Śliwiński (2007). On decision Support Under Risk by the WOWA Optimization. In *ECSQARU 2007*, *LNAI*, vol. 4724, pp. 779-790.

[10] F.Ruiz, M.Luque, J.M.Cabello (2008). A classification of the weighting schemes in reference point procedures formultiobjective programming. *J. Opnl. Res. Soc.*, forthcoming.

[11] V.Torra (1997). The weighted OWA operator. *Int. J. Intell. Syst.*, vol. 12, pp. 153–166.

[12] A.P.Wierzbicki (1982). A Mathematical Basis for Satisficing Decision Making. *Math. Modelling*, vol. 3, pp. 391–405.

[13] A.P.Wierzbicki, M.Makowski, J.Wessels (2000). *Model Based Decision Support Methodology with Environmental Applications*. Dordrecht: Kluwer.

[14] R.R.Yager (1988). On ordered weighted averaging aggregation operators in multicriteria decision making. *IEEE Trans. Sys., Man Cyber.*, vol. 18, pp. 183–190.

[15] R.R.Yager (1997). On the analytic representation of the Leximin ordering and its application to flexible constraint propagation. *Eur. J. Opnl. Res.*, vol. 102, pp. 176–192.