A Weights Representation for Fuzzy Constraint-based AHP

Shin-ichi OHNISHI
Hokkai-Gakuen University,
064-0926 Sapporo, JAPAN
ohnishi@eli.hokkai-s-u.ac.jp

Takahiro YAMANOI
Hokkai-Gakuen University,
064-0926 Sapporo, JAPAN
yamanoi@eli.hokkai-s-u.ac.jp

Hideyuki IMAI
Hokkaido University,
060-0814 Sapporo, JAPAN
imai@ist.hokudai.ac.jp

Abstract

Analytic Hierarchy Process (AHP) is very popular method in the field of decision-making today. While one of the most natural extensions of AHP using fuzzy theory is to employ a reciprocal matrix with fuzzy-valued entries, some of the present authors proposed the fuzzy constraint-based approach before. In this paper, we consider about weights for fuzzy constraint-based approach and propose a kind of representation for the weights. It employs not only fuzzy concept but also results of sensitivity analysis. Moreover it is also very useful for investigating data structure and realizing results of AHP.

Keywords: Decision Making; AHP; Fuzzy Sets, Sensitivity Analysis; Weight representation.

1 Introduction

The AHP methodology was proposed by T.L. Saaty in 1977 [10] [11], and it has been widely used in the field of decision making. It elicits weights of criteria and alternatives through ratio judgments of relative importance. And finally the preference for each alternative can be derived. The classical method requires the decision-maker (DM) to express his or her preferences in the form of a precise ratio matrix encoding a valued preference relation. However, it can often be difficult for the DM to express exact estimates of the ratios of importance.

Therefore many kinds of methods employing intervals or fuzzy numbers as elements of a pairwise reciprocal data matrix have been proposed to cope with this problem. This allows for a more flexible specification of pairwise preference intensities accounting for the incomplete knowledge of the DM. A fuzzy representation expresses rich information because the DM can provide i) the core of the fuzzy interval as a rough estimate of his perceived preference and ii) the support set of the fuzzy interval as the range that the DM believes to surely contain the unknown ratio of relative importance. Usually, since components of the pairwise matrix are locally obtained from the DM by pairwise comparisons of criteria or alternatives, its global consistency is not guaranteed. In classical AHP, consistency is usually measured by a consistency index (C.I.) based on the computation of an eigenvalue. And very often, the transitivity of preferences between the elements to be compared is strongly related to the consistency of the matrix. Using fuzzy numbers as elements of the reciprocal matrices, strict transitivity is too hard to preserve in terms of equalities between fuzzy numbers. Therefore the fuzzy constraint-based approach to AHP[13][14] only tries to maintain consistency of precise matrices that fit the imprecise specifications provided by the DM. A new kind of consistency index for fuzzy-valued matrices is computed that corresponds to the degree of satisfaction of the fuzzy specifications by the best fitting consistent reciprocal preference matrices. Importance or priority weights are then derived based on these precise preference matrices.

On the other hand, to analyze how much the components of a pairwise comparison matrix influences weights and consistency of a matrix, we can apply sensitivity analysis to AHP[15]

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[15]. This allows us possible to know how fuzzy their weights are.

Later part of this paper, we propose a kind representation of weights for fuzzy constraint-based approach. They are represented as L-R fuzzy numbers by use of the result from the sensitivity analysis. It can show not only how to represent weights by fuzzy sets, but also a representation of fuzziness of results from AHP.

2 A fuzzy constraint-based approach to the Analytic Hierarchy

In this section we show a fuzzy constraint-based approach [13][14] to the AHP that some of present authors proposed before.

Since using fuzzy numbers as elements of a pairwise matrix is more expressive than using crisp values or intervals, we hope that the fuzzy approach allows a more accurate description of the decision making process. Rather than forcing the DM to provide precise representations of imprecise perceptions, we suggest using an imprecise representation instead. In the traditional method the obtained matrix does not exactly fit the AHP theory and thus must be modified so as to respect mathematical requirements. Here we let the DM be imprecise, and check if this imprecise data encompasses precise preference matrices obeying the AHP requirements.

2.1 Fuzzy reciprocal data matrix

At first, we employ a fuzzy pairwise comparison reciprocal $n \times n$ matrix $\tilde{R} = \{\tilde{r}_{ij}\}$ pertaining to n elements (criteria, alternatives). In the AHP model, entry r_{ij} of a preference matrix reflects the ratio of importance weights of element i over element j. In the fuzzy-valued matrix, diagonal elements are singletons (= 1) and the other entries $\tilde{r}_{ij}$ $(i \neq j)$ have membership function μ_{ij} whose support is positive:

$$\tilde{r}_{ii} = 1, \quad \mathrm{supp}(\mu_{ij}) \subseteq (0, \ +\infty), \ i,j=1,\ldots,n$$

Moreover if element i is preferred to element j then supp(μ_{ij}) lies in [1, +∞), while supp(μ_{ij}) lies in (0, 1] if the contrary holds. The DM is supposed to supply the core (modal value) r_{ij} of $\tilde{r}_{ij}$ and its support set $[l_{ij}, u_{ij}]$ for $i<j$. The support set is the range that the DM believes surely contains the unknown ratio of relative importance. The DM may only supply entries above the diagonal like in the classical AHP.

We assume reciprocity $\tilde{r}_{ij} = 1/\tilde{r}_{ji}$ as follows [8]

(1) $$\mu_{ij}(r) = \mu_{ji}(1/r).$$

Therefore

(2) $$\mathrm{core}\left(1/\tilde{r}_{ij}\right) = 1/r_{ij},$$

(3) $$\mathrm{supp}\left(1/\tilde{r}_{ij}\right) = [1/u_{ij}, 1/l_{ij}].$$

We may assume all entries whose core is larger than or equal to 1 form triangular fuzzy sets i.e., if $r_{ij} \geq 1$, we assume $\tilde{r}_{ij}$ is a triangular fuzzy number, denoted as

(4) $$\tilde{r}_{ij} = (l_{ij}, r_{ij}, u_{ij})_{\Delta},$$

but then the symmetric entry $\tilde{r}_{ji}$ is not triangular. Alternatively one may suppose that if $r_{ij} < 1$, $\tilde{r}_{ji}$ is $(1/u_{ij}, 1/r_{ij}, 1/l_{ij})_{\Delta}$. Therefore the following transitivity condition inherited from the AHP theory will not hold

(5) $$\tilde{r}_{ij} \otimes \tilde{r}_{jk} = \tilde{r}_{ik}$$

in particular because multiplication of fuzzy intervals does not preserve triangular membership functions. However, even with intervals, this equality is too demanding (since it corresponds to requesting two usual equalities instead of one) and impossible to satisfy. For instance take $i = k$ in the above equality. On the left hand side is an interval, on the right-hand side is a scalar value (=1). So it makes no sense to consider a fuzzy AHP theory where fuzzy intervals would simply replace scalar entries in the preference ratio matrix.

2.2 Consistency

In this approach, a fuzzy-valued ratio matrix is considered to be a fuzzy set of consistent non-fuzzy ratio matrices. Each fuzzy entry is viewed as a flexible constraint. A ratio matrix is consistent in the sense of AHP (or AHP-consistent) if and only if there exists a set of weights $w_1, w_2, \ldots, w_n$, summing to 1, such that for all i, j, $r_{ij} = w_i / w_j$.

Using a fuzzy reciprocal matrix, some kind of consistency index of the data matrix is necessary[7]. This index will not measure the AHP- consistency of a non-fully consistent matrix, but instead will measure the degree to which an AHP-consistent matrix R exists, that satisfies the fuzzy constraints expressed in the fuzzy reciprocal matrix $\tilde{R}$. More precisely this degree of satisfaction can be attached to a n-tuple of weights $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ since this n-tuple defines an AHP-consistent ratio matrix. This degree is defined as

(6) $$\alpha(\boldsymbol{w}) = \min_{i,j} \mu_{ij}(w_i / w_j).$$

It is the "degree of consistency" of the weight pattern $\boldsymbol{w}$ with the fuzzy ratio matrix $\tilde{R}$ in the sense of fuzzy constraint satisfaction problems (FCSPs) [5]. The coefficient $\alpha(\boldsymbol{w})$ is in some sense an empirical validity coefficient measuring to what extent a weight pattern is close to, or compatible with, the DM revealed preference.

The best fitting weight patterns can thus be found by solving the following FCSP:

$$\text{maximize } \alpha \equiv \min_{i,j}\left\{\mu_{ij}\left(\frac{w_i}{w_j}\right)\right\}$$

$$0 \le w_i \le 1, \ \sum_i^n w_i = 1, \ i=1,\ldots,n,$$

where w_i is the weight of alternative i, and n is the total number of alternatives. Maximizing α corresponds to getting as close as possible to the ideal preference patterns of the DM (in the sense of the Chebychev norm). Let

(7) $$\alpha^* \equiv \max_{w_1,\ldots,w_n} \min_{i,j}\left\{\mu_{ij}\left(\frac{w_i}{w_j}\right)\right\}.$$

α^* is a degree of consistency different from Saaty's index, but which can be used as a natural substitute to the AHP-consistency index for evaluating the DM's degree of rationality when expressing his or her preferences.

Solving this flexible constraint satisfaction problem in terms of α enables the fuzzy ratio matrix to be turned into an interval-valued matrix defining crisp constraints for the main problem of calculating local weights, as shown in the next subsection.

As usual, the FCSP problem can be re-stated as follows

maximize α

$$\text{s.t.} \quad \mu_{ij}\left(\frac{w_i}{w_j}\right) \ge \alpha,$$

$$\sum_i^n w_i = 1, \ i,j = 1,\ldots,n,$$

and we can express the first constraint as follows

(8) $$w_i / w_j \in [\underline{\mu_{ij}^{-1}(\alpha)}, \overline{\mu_{ij}^{-1}(\alpha)}],$$

where $\underline{\mu_{ij}^{-1}(\alpha)}$ and $\overline{\mu_{ij}^{-1}(\alpha)}$ are the lower and upper bound of the α-cut of $\mu_{ij}(w_i / w_j)$, respectively. This becomes

(9) $$w_j \underline{\mu_{ij}^{-1}(\alpha)} \le w_i \le w_j \overline{\mu_{ij}^{-1}(\alpha)}.$$

Here, if all μ_{ij} are triangular fuzzy numbers $(l_{ij}, r_{ij}, u_{ij})_\Delta$, the problem becomes a non-linear programming problem as follows,

[NLP]

maximize α

$$w_j\{l_{ij} + \alpha(r_{ij} - l_{ij})\} \le w_i \le w_j\{u_{ij} + \alpha(r_{ij} - u_{ij})\}$$

$$\sum_i^n w_i = 1, \ i,j=1,\ldots,n.$$

The problem is one of finding a solution to a set of linear inequalities if we fix the value of α. Hence it can be solve by the dichotomy method.

2.3 Unicity of the optimal weight pattern

Results obtained by Dubois and Fortemps [6] on best solutions to maxmin optimization problems with convex domains can be applied here. Indeed, it is obvious that the set of weight patterns obeying (9), for all i, j is convex. Call this domain D_α. If $\boldsymbol{w}^1$ and $\boldsymbol{w}^2$ are in D_α, so is their convex combination $\boldsymbol{w} = \lambda \boldsymbol{w}^1 + (1-\lambda)\boldsymbol{w}^2$. Note that the ratios w_i/w_j lie between w^1_i/w^1_j and w^2_i/w^2_j. Hence, if for all i, j, w^1_i/w^1_j differs from w^2_i/w^2_j, it is clear that

(10) $$\mu_{ij}\left(\frac{w_i}{w_j}\right) > \min\left\{\mu_{ij}\left(\frac{w_i^1}{w_j^1}\right), \mu_{ij}\left(\frac{w_i^2}{w_j^2}\right)\right\}$$

So in particular, suppose there are two optimal weight patterns $\boldsymbol{w}^1$ and $\boldsymbol{w}^2$ in the optimal

α*-cuts, whose ratio matrices differ for all non-diagonal components. It implies that for $\boldsymbol{w} = (\boldsymbol{w}^1 + \boldsymbol{w}^2)/2$,

$$\mu_{ij}\left(\frac{w_i}{w_j}\right) > \alpha^*, \quad i,j = 1,\ldots,n, \tag{11}$$

which is contradictory. In this case, there is only one weight pattern $\boldsymbol{w}$ coherent with the interval-valued matrix whose entries are intervals $(\tilde{r}_{ij})_{\alpha^*}$.

In the case when there are at least two optimal weight patterns $\boldsymbol{w}^1$ and $\boldsymbol{w}^2$ in the optimal α*-cuts, their ratio matrices coincide for at least one non-diagonal component ($w^1_i/w^1_j = w^2_i/w^2_j$). In [6], it is shown that in this case,

$$\mu_{ij}\left(\frac{w^1_i}{w^1_j}\right) = \mu_{ij}\left(\frac{w^2_i}{w^2_j}\right) = \alpha^*. \tag{12}$$

So the procedure is then iterated: For such entries of the matrix, the fuzzy numbers in place (i, j) must be replaced by the α^*-cut of $\tilde{r}_{ij}$. It can be done using the best weight pattern $\boldsymbol{w}^*$ obtained from the dichotomy method, checking for (i, j) such that

$$\mu_{ij}\left(\frac{w^*_i}{w^*_j}\right) = \alpha^*. \tag{13}$$

The problem [NLP] is solved again with the new fuzzy matrix. It yields an optimal consistency degree $\beta^* > \alpha^*$ (by construction). If the same lack of unicity phenomenon reappears, some fuzzy matrix entries are again turned into intervals, and so on, until all entries are interval. The obtained solution is called "discrimin"-optimal solution in [6] and is provably unique from theorem 5 in the paper.

The global (aggregated) evaluation of a decision f is given by means of the unique (discrimin) optimal weight pattern $\boldsymbol{w}^*$ in D_α:

$$V_f = \sum_i w^*_i u_i(f), \tag{14}$$

where $u_i(f)$ is the utility of decision f under criterion i.

3 Sensitivity Analysis of AHP

When we actually use AHP, it often occurs that a comparison matrix is not consistent or that there is not great difference among the overall weights of the alternatives. Thus, it is very important to investigate how components of a pairwise comparison matrix influence on its consistency or on weights. So as to analyze how results are influenced when a certain variable has changed, we use the sensitivity analysis. On the basis of the reasons mentioned above, we also proposed sensitivity analysis of AHP [15][16].

It evaluates a fluctuation of the consistency index and weights, when a comparison matrix is perturbed, and it is useful because it does not change a structure of the data.

The evaluations of consistency index and the weights of a perturbed comparison matrix are performed as follows.

1. Perturbations $\varepsilon\, a_{ij}d_{ij}$ are imparted to component a_{ij} of a comparison matrix, and the fluctuation of the consistency index and the weight are expressed by the power series of ε.
2. Fluctuations of the consistency index and the weights are represented by the linear combination of d_{ij}.
3. From the coefficient of d_{ij}, it found that how the component of the comparison matrix gives influence on the consistency index and the weight.

Since a pairwise comparison matrix A is a positive square matrix, Perron- Frobenius theorem holds. From this theorem, the following theorem about a perturbed comparison matrix holds.

Theorem 1 *Let $A = (a_{ij})$, $i,j = 1,\ldots,n$ be a comparison matrix and let $A(\varepsilon) = A + \varepsilon D_A$, $D_A = (a_{ij}d_{ij})$ be a matrix that has been perturbed.*

Moreover, let λ_A be the Frobenius root of A, $\boldsymbol{w}_1$ be the eigenvector corresponding to it, and $\boldsymbol{w}_2$ be the eigenvector corresponding to the Frobenius root of A', then, the Frobenius root $\lambda(\varepsilon)$ of $A(\varepsilon)$ and the corresponding eigenvector $\boldsymbol{w}_1(\varepsilon)$ can be expressed as follows

$$\lambda(\varepsilon) = \lambda_A + \varepsilon\lambda^{(1)} + o(\varepsilon), \tag{15}$$

$$\boldsymbol{w}_1(\varepsilon) = \boldsymbol{w}_1 + \varepsilon\boldsymbol{w}^{(1)} + \boldsymbol{o}(\varepsilon), \tag{16}$$

where

(17) $$\lambda^{(1)} = \frac{\boldsymbol{w}_2' D_A \boldsymbol{w}_1}{\boldsymbol{w}_2' \boldsymbol{w}_1},$$

$\boldsymbol{w}^{(1)}$ is an n-dimension vector that satisfies

(18) $$(A - \lambda_A I)\boldsymbol{w}^{(1)} = -(D_A - \lambda^{(1)} I)\boldsymbol{w}_1,$$

where $\boldsymbol{o}(\varepsilon)$ denotes an n-dimension vector in which all components are $o(\varepsilon)$.

(Proof)

From Perron-Frobenius theorem, the Frobenius root λ_A is the simple root. Thus, expansions (15) and (16) are valid. Therefore, characteristic equations become

$$\begin{aligned}&(A + \varepsilon D_A)(\boldsymbol{w}_1 + \varepsilon \boldsymbol{w}^{(1)} + \boldsymbol{o}(\varepsilon)) \\ &\quad = (\lambda_A + \varepsilon \lambda^{(1)} + o(\varepsilon))(\boldsymbol{w}_1 + \varepsilon \boldsymbol{w}^{(1)} + \boldsymbol{o}(\varepsilon)),\end{aligned}$$

$$A\boldsymbol{w}_1 = \lambda_A \boldsymbol{w}_1.$$

From these two equations, (18) can be obtained. Further, by Perron-Frobenius theorem, $\boldsymbol{w}_2'A = \lambda_A \boldsymbol{w}_2'$ holds, and it becomes

$$\boldsymbol{w}_2' \boldsymbol{w}_1 \lambda^{(1)} = \boldsymbol{w}_2' D_A \boldsymbol{w}_1.$$

Thus, equation (17) holds. (Q.E.D)

3.1 Sensitivity analysis of the consistency index

About a fluctuation of the consistency index, the following corollary can be obtained from Theorem1.

Corollary1 *Using an appropriate g_{ij}, we can represent the consistency index* C.I.(ε) *of the perturbed comparison matrix as follows*

(19) $$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon \sum_i^n \sum_j^n g_{ij} d_{ij} + o(\varepsilon).$$

(Proof)

From the definition of the consistency index and (15),

$$\text{C.I.}(\varepsilon) = \text{C.I.} + \varepsilon \frac{\lambda^{(1)}}{n-1} + o(\varepsilon)$$

holds. Here, let $\boldsymbol{w}_1 = (w_{1i})$ and $\boldsymbol{w}_2 = (w_{2i})$, from (17), $\lambda^{(1)}$ is represented as

$$\lambda^{(1)} = \frac{1}{\boldsymbol{w}_2' \boldsymbol{w}_1} \sum_i^n \sum_j^n w_{2i} a_{ij} w_{1j} d_{ij},$$

so the second part of the right side is expressed by a linear combination of d_{ij}. (Q.E.D)

To see g_{ij} in the equation (19) in Corollary 1, how the components of a comparison matrix impart influence on its consistency can be found.

On the other hand, since the comparison matrix $A(\varepsilon) = (a_{ij}(\varepsilon))$ is reciprocal; $a_{ji}(\varepsilon) = 1/a_{ij}(\varepsilon)$ holds, and it becomes

(20) $$a_{ji} + \varepsilon a_{ji} d_{ji} = \frac{1}{a_{ij}} - \varepsilon \frac{d_{ij}}{a_{ij}} + o(\varepsilon).$$

Here, since $a_{ji} = 1/a_{ij}$,

(21) $$d_{ji} = -d_{ij}$$

is obtained. In fact, we can see influence more easily if we use this property.

3.2 Sensitivity analysis of weights

About the fluctuation of the weight, the following corollary also can be obtained from Theorem 1.

Corollary 2 *Using an appropriate $h_{ij}^{(k)}$, we can represent the fluctuation $\boldsymbol{w}^{(1)} = (w_k^{(1)})$ of the weight (i.e. the eigenvector corresponding to the Frobenius root) as follows*

(22) $$w_k^{(1)} = \sum_i^n \sum_j^n h_{ij}^{(k)} d_{ij}.$$

(Proof)

The k-th row component of the right side of (18) in Theorem 1 is represented as

$$\sum_i^n \sum_j^n \left\{ \frac{w_{1k} w_{2i} a_{ij} w_{1j}}{\boldsymbol{w}_2' \boldsymbol{w}_1} - \delta(i,k) a_{ij} w_{1j} \right\} d_{ij},$$

and is expressed by a linear combination of d_{ij}. Here, $\delta(i,k)$ is Kronecker's symbol

$$\delta(i,k) = \begin{cases} 1 & (i = k), \\ 0 & (i \neq k). \end{cases}$$

On the other hand, since λ_A is a simple root, Rank(A- $\lambda_A I$) = n-1. Accordingly, the weight vector is normalized as

$$\sum_k^n (w_{1k} + \varepsilon w_k^{(1)}) = \sum_k^n w_{1k} = 1,$$

then the following condition follows.

$$\sum_k^n w_k^{(1)} = 0. \tag{23}$$

Using elementary transformation to formula (19) in the condition above, we also can represent $w_k^{(1)}$ by linear combinations of d_{ij}. (Q.E.D)

From the equation (16) in Theorem 1, the component that has a great influence on weight $\boldsymbol{w}_1(\varepsilon)$ is the component which has the greatest influence on $\boldsymbol{w}^{(1)}$. Accordingly, from Corollary 2, how components of a comparison matrix impart influence on the weights, can be found, to see $h_{ij}^{(k)}$ in the equation (22).

Of course, we can also see influence more easily by use of equation (21).

4 A kind of weight representation

With the fuzzy component matrix data, we consider that components of a comparison matrix are results from fuzzy judgment of human. Therefore weights could be treated as fuzzy numbers.

The multiple of coefficients $g_{ij}h_{ij}^{(k)}$ in Corollary 1 and 2 is considered as the influence concerned with a_{ij}, from the fluctuation of the consistency index.

Since g_{ij} is always positive, if the coefficient $h_{ij}^{(k)}$ is positive, the real weight of criterion k is considered as bigger than crisp weights w_k^* from section 2, and if $h_{ij}^{(k)}$ is negative, the real weight of criterion k is considered as smaller. Therefore, a sign of $h_{ij}^{(k)}$ represents a direction the spread of the fuzzy number. Of course the absolute value $g_{ij}|h_{ij}^{(k)}|$ represents the amount of the influence.

On the other hand, α^* becomes bigger then the judgment becomes more fuzziness.

Consequently, multiple α^* $g_{ij}|h_{ij}^{(k)}|$ can be regarded as a spread of $\tilde{v}_k$, a fuzzy weight of criterion k, concerned with a_{ij}.

Definition (fuzzy weight) *Let w_k^* be a crisp weight of criterion k in fuzzy constraint-based AHP, and g_{ij}, $|h_{ij}^{(k)}|$ denote the coefficients found in Corollary* 1 *and* 2*, a fuzzy weight of criterion k, $\tilde{v}_k$ is defined by*

$$\tilde{v}_k = (w_k^*, p_k, q_k)_{LR}, \tag{24}$$

where

$$p_k = \alpha^* \sum_i^n \sum_j^n s(-, h_{ij}^{(k)}) g_{ij} |h_{ij}^{(k)}|, \tag{25}$$

$$q_k = \alpha^* \sum_i^n \sum_j^n s(+, h_{ij}^{(k)}) g_{ij} |h_{ij}^{(k)}|, \tag{26}$$

$$s(+,h) = \begin{cases} 1 & (h > 0), \\ 0 & (h \leq 0), \end{cases}$$

$$s(-,h) = \begin{cases} 0 & (h > 0), \\ 1 & (h \leq 0). \end{cases}$$

We can calculate the fuzzy weights of activities using definition above. Then, the fuzzy weights of alternatives are calculated by operations of fuzzy numbers. They show how the result from AHP has fuzziness.

5 Conclusions

In classical AHP it is often difficult for the DM to provide an exact pairwise data matrix because it is hard to estimate ratios of importance in a precise way. Therefore we use fuzzy reciprocal matrices, and propose a new kind of consistency index and weights. This index is considered as an empirical validity coefficient evaluating to what extent a weight pattern is close to the DM revealed preference.

Then, from these and results from sensitivity analysis, we propose a kind of weights representation for the fuzzy constraint-based approach to AHP.

In the next step, we will be able to show examples of AHP entirely. Moreover we plan to implement these results on actual data. We will also try to refine the search for appropriate weights that employs the DM's subjective distance.

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