Generating simple, Pareto and absolute special majorities by means of mixture operators

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Abstract

In this paper we consider mixture operators to aggregate individual preferences and we characterize those that allow us to extend some majority rules, such as simple, Pareto and absolute special majorities, to the field of gradual preferences.

Keywords: Mixture operators, Simple majority, Pareto majority, Absolute special majorities.

1 Introduction

Aggregation operators are a fundamental tool in multicriteria decision making procedures. For this reason, they have received a great deal of attention in the literature (see, for instance, Marichal [6], Calvo, Mayor and Mesiar [1] and Xu and Da [15]).

An interesting kind of non-monotonic aggregation is obtained when mixture operators are used. Mixture operators were introduced by Marques Pereira and Pasi [8] in order to consider weighted aggregation operators in which the weights depend on the attribute satisfaction values. Thus, mixture operators are similar to weighted means where the numerical weights have been replaced by weighting functions.

Mixture operators can be used to aggregate individual preferences into a collective preference. Given that some aggregation operators can be seen as extensions of majority rules to the field of gradual preferences, the aim of this paper is to determine the mixture operators that correspond to extensions of some particular classes of majority rules.

We consider individual preferences expressed as pairwise comparisons between alternatives, with preference intensity values in the [0,1] interval. In this way each pairwise comparison is associated with a graded preference profile. Once an aggregation operator is chosen, each graded preference profile produces a collective preference intensity value in the unit interval. On the basis of this value and through a kind of strong α-cut, where $\alpha \in [0.5, 1)$, we can decide if an alternative is chosen or if both alternatives are collectively indifferent. When individuals do not grade their preferences (that is, when they are represented through the values 0, 0.5, and 1), the previous procedure allows us to obtain a majority rule. Hence, once α fixed, it is possible to know what class of majority rule is present in the aggregation process according to the used operator.

We note that this procedure has already been used to characterize several classes of aggregation functions that extend some well-known majority rules. Thus, García-Lapresta and Llamazares [3] generalize two classes of majorities based on difference of votes by using quasiarithmetic means and window OWA operators as aggregation functions. Likewise, Llamazares [4, 5] has characterized the OWA operators that generalize simple, Pareto and absolute special majorities.

In this paper we characterize the mixture op-

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erators that allow us to extend simple, Pareto and absolute special majorities to the field of gradual preferences.

The organization of the paper is as follows. In Section 2 we introduce the used model for extending a majority rule through an aggregation function. In Section 3 we show some characterizations of simple, Pareto and absolute special majorities. In Section 4 we introduce mixture operators and we determine those that satisfy the self-duality property. Finally, in Section 5 we give the main results of the paper.

2 The model

We consider m voters, with $m \geq 3$, and two alternatives x and y. Voters represent their preferences between x and y through variables r_i. If the individuals grade their preferences, then $r_i \in [0,1]$ denotes the *intensity* with which voter i prefers x to y. We also suppose that $1 - r_i$ is the intensity with which voter i prefers y to x. If the individuals do not grade their preferences, then $r_i \in \{0, 0.5, 1\}$ represents that voter i prefers x to y ($r_i = 1$), prefers y to x ($r_i = 0$), or is indifferent between both alternatives ($r_i = 0.5$). The justification of this three-valued representation can be found in García-Lapresta and Llamazares [2].

A *profile of preferences* is a vector $\mathbf{r} = (r_1, \dots, r_m)$ that describes voters' preferences between alternative x and alternative y. Obviously, $\mathbf{1} - \mathbf{r} = (1 - r_1, \dots, 1 - r_m)$ shows voters' preferences between y and x. For each profile of preferences, the collective preference will be obtained by means of an aggregation function.

Definition 1. *An aggregation function is a mapping $F : [0,1]^m \longrightarrow [0,1]$. A discrete aggregation function (DAF) is a mapping $H : \{0, 0.5, 1\}^m \longrightarrow \{0, 0.5, 1\}$.*

The interpretation of collective preference is consistent with the foregoing interpretation for individual preferences. Thus, if F is an aggregation function, then $F(\mathbf{r})$ is the intensity with which x is collectively preferred to y. When H is a DAF, then $H(\mathbf{r})$ shows us if an alternative is collectively preferred to the other ($H(\mathbf{r}) \in \{0,1\}$), or the alternatives are collectively indifferent ($H(\mathbf{r}) = 0.5$).

Next we present some well-known properties of aggregation functions: Symmetry, monotonicity, self-duality and idempotency. Symmetry means that collective intensity of preference depends on only the set of individual intensities of preference, but not on which individuals have these preferences. Monotonicity means that collective intensity of preference does not decrease if no individual intensity decreases. Self-duality means that if everyone reverses their preferences between x and y, then the collective preference is also reversed. Finally, idempotency means that collective intensity of preference coincides with individual intensities when these are the same.

Given $r \in [0,1]$, $\mathbf{r}, \mathbf{s} \in [0,1]^m$ and σ a permutation on $\{1, \dots, m\}$, we will use the following notation: $\mathbf{r}_\sigma = (r_{\sigma(1)}, \dots, r_{\sigma(m)})$; $\mathbf{1} = (1, \dots, 1)$; $r \cdot \mathbf{1} = (r, \dots, r)$; and $\mathbf{r} \geq \mathbf{s}$ will denote $r_i \geq s_i$ for all $i \in \{1, \dots, m\}$.

Definition 2. *Let F be an aggregation function.*

1. *F is symmetric if for every profile $\mathbf{r} \in [0,1]^m$ and for every permutation σ of $\{1, \dots, m\}$ the following holds*
$$F(\mathbf{r}_\sigma) = F(\mathbf{r}).$$
2. *F is monotonic if for all pair of profiles $\mathbf{r}, \mathbf{s} \in [0,1]^m$ the following holds*
$$\mathbf{r} \geq \mathbf{s} \;\Rightarrow\; F(\mathbf{r}) \geq F(\mathbf{s}).$$
3. *F is self-dual if for every profile $\mathbf{r} \in [0,1]^m$ the following holds*
$$F(\mathbf{1} - \mathbf{r}) = 1 - F(\mathbf{r}).$$
4. *F is idempotent if for every $r \in [0,1]$ the following holds*
$$F(r \cdot \mathbf{1}) = r.$$

All the previous properties are also valid for DAFs. Next we show some consequences of the previous properties. The cardinal of a set will be denoted by #.

Remark 1. *If H is a symmetric DAF, then $H(\mathbf{r})$ depends on only the number of 1, 0.5, and 0. Given a profile $\mathbf{r}$, if we consider*

$$\begin{aligned} m_1 &= \#\{i \mid r_i = 1\}, \\ m_2 &= \#\{i \mid r_i = 0.5\}, \\ m_3 &= \#\{i \mid r_i = 0\}, \end{aligned}$$

then $m_1 + m_2 + m_3 = m$.

Definition 3. *Let H be a symmetric DAF and*

$$\begin{aligned} \mathcal{M} = \{&(m_1, m_2, m_3) \in \{0, \ldots, m\}^3 \mid \\ &m_1 + m_2 + m_3 = m\}. \end{aligned}$$

We say that H is represented by the function $h : \mathcal{M} \longrightarrow \{0, 0.5, 1\}$, defined by

$$\begin{aligned} &h(m_1, m_2, m_3) = \\ &\quad = H(1, \overset{(m_1)}{\ldots}, 1, 0.5, \overset{(m_2)}{\ldots}, 0.5, 0, \overset{(m_3)}{\ldots}, 0). \end{aligned}$$

Definition 4. *The binary relation $\succeq$ on $\mathcal{M}$ is defined by*

$$\begin{aligned} &(m_1, m_2, m_3) \succeq (n_1, n_2, n_3) \Leftrightarrow \\ &\qquad \Leftrightarrow \begin{cases} m_1 \geq n_1, \\ m_1 + m_2 \geq n_1 + n_2. \end{cases} \end{aligned}$$

We note that $\succeq$ is a partial order on $\mathcal{M}$ (reflexive, antisymmetric, and transitive binary relation).

Remark 2. *If H is a symmetric DAF represented by h, then it is monotonic if and only if $h(m_1, m_2, m_3) \geq h(n_1, n_2, n_3)$ for all $(m_1, m_2, m_3), (n_1, n_2, n_3) \in \mathcal{M}$ such that $(m_1, m_2, m_3) \succeq (n_1, n_2, n_3)$.*

Remark 3. *If H is a symmetric DAF represented by h, then it is self-dual if and only if $h(m_3, m_2, m_1) = 1 - h(m_1, m_2, m_3)$ for all $(m_1, m_2, m_3) \in \mathcal{M}$. In this case, H is characterized by the set $h^{-1}(1)$, since*

$$\begin{aligned} h^{-1}(0) &= \{(m_1, m_2, m_3) \in \mathcal{M} \mid \\ &\qquad h(m_3, m_2, m_1) = 1\}, \\ h^{-1}(0.5) &= \mathcal{M} \setminus (h^{-1}(1) \cup h^{-1}(0)). \end{aligned}$$

When a DAF is self-dual, both alternatives have an egalitarian treatment. Therefore, if the DAF is also symmetric and the number of voters who prefer x to y coincides with the number of voters who prefer y to x, then x and y are collectively indifferent.

Remark 4. *If H is a symmetric and self-dual DAF represented by h, then $h(m_1, m_2, m_3) = 0.5$ for all $(m_1, m_2, m_3) \in \mathcal{M}$ such that $m_1 = m_3$.*

By Remark 3, it is possible to define a symmetric and self-dual DAF H by means of the elements $(m_1, m_2, m_3) \in \mathcal{M}$ where the mapping that represents H takes the value 1. Based on this, we now show some DAFs widely used in real decisions.

Definition 5.

1. *The simple majority, H_S, is the symmetric and self-dual DAF defined by*
$$h(m_1, m_2, m_3) = 1 \Leftrightarrow m_1 > m_3.$$
2. *The absolute majority, H_A, is the symmetric and self-dual DAF defined by*
$$h(m_1, m_2, m_3) = 1 \Leftrightarrow m_1 > \frac{m}{2}.$$
3. *The Pareto majority, H_P, is the symmetric and self-dual DAF defined by*
$$h(m_1, m_2, m_3) = 1 \Leftrightarrow \begin{cases} m_1 > 0, \\ m_3 = 0. \end{cases}$$
4. *The unanimous majority, H_U, is the symmetric and self-dual DAF defined by*
$$h(m_1, m_2, m_3) = 1 \Leftrightarrow m_1 = m.$$
5. *Given $\beta \in [0.5, 1)$, the absolute special majority Q_β is the symmetric and self-dual DAF defined by*
$$h(m_1, m_2, m_3) = 1 \Leftrightarrow m_1 > \beta m.$$

It should be noted that absolute and unanimous majorities are specific cases of absolute special majorities.

Given an aggregation function, we can generate different DAFs by means of a parameter $\alpha \in [0.5, 1)$. The procedure employed is based on strong α-cuts. Moreover, it is easy to check that the DAFs obtained are symmetric, monotonic, and self-dual when the original aggregation function satisfies these properties.

Definition 6. *Let F be an aggregation function and $\alpha \in [0.5, 1)$. Then the α–DAF associated with F is the DAF F_α defined by*

$$F_\alpha(\mathbf{r}) = \begin{cases} 1, & \text{if } F(\mathbf{r}) > \alpha, \\ 0.5, & \text{if } 1-\alpha \le F(\mathbf{r}) \le \alpha, \\ 0, & \text{if } F(\mathbf{r}) < 1-\alpha. \end{cases}$$

Remark 5. *Given an aggregation function F and $\alpha \in [0.5, 1)$, the following statements hold:*

1. *If F is symmetric, then F_α is also symmetric.*
2. *If F is monotonic, then F_α is also monotonic.*
3. *If F is self-dual, then F_α is also self-dual.*

Similar to the case of symmetric DAFs, when F is a symmetric aggregation function, the restriction $F|_{\{0,0.5,1\}^m}$ can be represented by $f : \mathcal{M} \longrightarrow [0, 1]$, where

$$f(m_1, m_2, m_3) = \\ = F(1, \overset{(m_1)}{\ldots}, 1, 0.5, \overset{(m_2)}{\ldots}, 0.5, 0, \overset{(m_3)}{\ldots}, 0).$$

Now we show the relationship between f and the family of mappings f_α that represent the α–DAFs associated with F.

Remark 6. *Let F be a symmetric aggregation function and $\alpha \in [0.5, 1)$. Then F_α and $F|_{\{0,0.5,1\}^m}$ can be represented by the mappings f_α and f, respectively. The following relationship between these mappings exists:*

$$f_\alpha(m_1, m_2, m_3) = \\ = \begin{cases} 1, & \text{if } f(m_1, m_2, m_3) > \alpha, \\ 0.5, & \text{if } 1-\alpha \le f(m_1, m_2, m_3) \le \alpha, \\ 0, & \text{if } f(m_1, m_2, m_3) < 1-\alpha. \end{cases}$$

3 Characterization of simple, Pareto and absolute special majorities

In order to generalize simple, Pareto and absolute special majorities by means of mixture operators, we show in this section some characterizations of these majority rules. It is worth noting that these characterizations, given in Llamazares [4, 5], are based on the monotonicity of these rules.

Simple majority is characterized through the elements $(m_1, m_2, m_3) \in \mathcal{M}$ such that $m_1 = m_3 + 1$.

Proposition 1. *Let H be a symmetric, monotonic, and self-dual DAF represented by h. Then the following statements are equivalent:*

1. *$H = H_S$.*
2. *$h(m_3+1, m-(2m_3+1), m_3) = 1$ for all $m_3 \in \{0, \ldots, [\frac{m-1}{2}]\}$.*

Pareto and absolute special majorities are both characterized through two elements of $\mathcal{M}$. The first one corresponds to the minimum support that alternative x needs to be selected. The second one corresponds to the maximum support that alternative x can obtain without being selected.

Proposition 2. *Let H be a symmetric, monotonic, and self-dual DAF represented by h. Then the following statements are equivalent:*

1. *$H = H_P$.*
2. *$h(1, m-1, 0) = 1$ and $h(m-1, 0, 1) < 1$.*

Proposition 3. *Let H be a symmetric, monotonic, and self-dual DAF represented by h and $\beta \in [0.5, 1)$. Then the following statements are equivalent:*

1. *$H = Q_\beta$.*
2. *$h([\beta m]+1, 0, m-[\beta m]-1) = 1$ and $h([\beta m], m-[\beta m], 0) < 1$.*

4 Mixture operators

Mixture operators were introduced by Marques Pereira and Pasi [8] in order to consider weighted aggregation operators in which the weights depend on the attribute satisfaction values.

Definition 7. *Let $\varphi : [0,1] \longrightarrow]0,\infty[$ be a continuous function. The mixture operator $W^\varphi : [0,1]^m \longrightarrow [0,1]$ generated by φ is the aggregation function defined by*

$$W^\varphi(\mathbf{r}) = \frac{\sum_{i=1}^{m} \varphi(r_i) r_i}{\sum_{j=1}^{m} \varphi(r_j)}.$$

The mixture operator $W^\varphi(\mathbf{r})$ can be written as a weighted average of the variables r_i,

$$W^\varphi(\mathbf{r}) = \sum_{i=1}^{m} w_i(\mathbf{r}) r_i,$$

where the classical constant weights w_i are replaced by the weighting functions

$$w_i(\mathbf{r}) = \frac{\varphi(r_i)}{\sum_{j=1}^{m} \varphi(r_j)}.$$

Mixture operators are symmetric and idempotent aggregation functions. The monotonicity of mixture operators has been studied by Marques Pereira and Pasi [8], Marques Pereira [7], Ribeiro and Marques Pereira [11, 12], Marques Pereira and Ribeiro [9], Mesiar and Špirková [10] and Špirková [13, 14].

With regard to self-duality property, it is easy to check that the mixture operator W^φ is self-dual if and only if for every $\mathbf{r} \in [0,1]^m$ the following holds

$$\sum_{i=1}^{m} \Big(w_i(\mathbf{r}) - w_i(\mathbf{1} - \mathbf{r}) \Big) r_i = 0.$$

From this relationship, it is possible to obtain a characterization of self-dual mixture operators based on the fulfillment of a similar property by the function φ.

Proposition 4. *Let W^φ be the mixture operator generated by φ. W^φ is self-dual if and only if $\varphi(r) = \varphi(1-r)$ for all $r \in [0,1]$.*

5 Majority rules obtained through mixture operators

In this section we establish the main results of the paper. Simple, Pareto and absolute special majorities are generated through α-DAFs associated with self-dual mixture operators. In this way, the outcomes of this section allow us to extend these majority rules to the framework of gradual preferences by means of mixture operators.

First of all, we give a necessary and sufficient condition in order to obtain simple majority through α-DAFs associated with self-dual mixture operators.

Theorem 1. *Let W^φ be a self-dual mixture operator and $\gamma = \dfrac{\varphi(0.5)}{\varphi(1)}$. The following statements hold:*

1. *If $\gamma \geq 1$:*
$$W^\varphi_\alpha = H_S \;\Leftrightarrow\; \alpha < \frac{2 + (m-1)\gamma}{2(1 + (m-1)\gamma)}.$$
2. *If $\gamma < 1$:*
   (a) *If m is odd:*
$$W^\varphi_\alpha = H_S \;\Leftrightarrow\; \alpha < \frac{m+1}{2m}.$$
   (b) *If m is even:*
$$W^\varphi_\alpha = H_S \;\Leftrightarrow\; \alpha < \frac{m+\gamma}{2(m+\gamma-1)}.$$

From the previous result it is straightforward to obtain the values of α for which simple majority can be generated through the α-DAF associated with a self-dual mixture operator.

Corollary 1.

1. *If m is odd, then there exists a self-dual mixture operator W^φ such that $W^\varphi_\alpha = H_S$ if and only if $\alpha < \dfrac{m+1}{2m}$.*

2. *If m is even, then there exists a self-dual mixture operator W^φ such that $W^\varphi_\alpha = H_S$ if and only if* $\alpha < \dfrac{m}{2(m-1)}$.

In the following theorem we characterize the self-dual mixture operators for which the α-DAFs associated are Pareto majority.

Theorem 2. *Let W^φ be a self-dual mixture operator and* $\gamma = \dfrac{\varphi(0.5)}{\varphi(1)}$. *The following statement holds:*

$$W^\varphi_\alpha = H_P \;\Leftrightarrow\; \gamma < \frac{2}{(m-1)(m-2)} \;\; \textit{and}$$
$$1 - \frac{1}{m} \leq \alpha < \frac{1}{2} + \frac{1}{2(1+(m-1)\gamma)}.$$

In the next theorem we give a necessary and sufficient condition in order to obtain absolute special majorities through α-DAFs associated with self-dual mixture operators.

Theorem 3. *Let W^φ be a self-dual mixture operator and* $\gamma = \dfrac{\varphi(0.5)}{\varphi(1)}$. *The following statement holds:*

$$W^\varphi_\alpha = Q_\beta \;\Leftrightarrow$$
$$\gamma > 2[\beta m]\frac{m-[\beta m]-1}{(m-[\beta m])(2[\beta m]+2-m)} \;\; \textit{and}$$
$$\frac{1}{2} + \frac{1}{2\left(1 + \dfrac{m-[\beta m]}{[\beta m]}\gamma\right)} \leq \alpha < \frac{1+[\beta m]}{m}.$$

As particular cases of this theorem it is straightforward to give necessary and sufficient conditions to obtain absolute and unanimous majorities through α-DAFs associated with self-dual mixture operators.

Corollary 2. *Let W^φ be a self-dual mixture operator and* $\gamma = \dfrac{\varphi(0.5)}{\varphi(1)}$. *The following statements hold:*

1. (a) *If m is odd:*

$$W^\varphi_\alpha = H_A \;\Leftrightarrow\; \gamma > \frac{(m-1)^2}{m+1} \;\; \textit{and}$$
$$\frac{1}{2} + \frac{1}{2\left(1 + \dfrac{m+1}{m-1}\gamma\right)} \leq \alpha < \frac{1}{2} + \frac{1}{2m}.$$

   (b) *If m is even:*

$$W^\varphi_\alpha = H_A \;\Leftrightarrow\; \gamma > \frac{m}{2} - 1 \;\; \textit{and}$$
$$\frac{1}{2} + \frac{1}{2(1+\gamma)} \leq \alpha < \frac{1}{2} + \frac{1}{m}.$$

2. $W^\varphi_\alpha = H_U \;\Leftrightarrow\; \gamma > 0$ *and*

$$\alpha \geq \frac{1}{2} + \frac{m-1}{2(m+\gamma-1)}.$$

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