An Average Value-at-Risk Portfolio under Uncertainty

Yuji YOSHIDA
Fac. of Economics, Univ. Kitakyushu, Kitakyushu, Japan
yoshida@kitakyu-u.ac.jp

Abstract

An average value-at-risk portfolio model under randomness and fuzziness is discussed. The randomness and fuzziness are evaluated respectively by the probabilistic expectation and the mean with evaluation weights and λ-mean functions. Extending the average value-at-risk, this paper formulates an average value-at-risk portfolio problem with fuzzy random variables. The analytical solutions of the average value-at-risk portfolio problem are derived. A numerical example is given to explain our idea.

Keywords: Average value-at-risk, risk-sensitive portfolio, fuzzy random variable, possibility-necessity weight, pessimistic-optimistic index.

1 Introduction

The risk allocation is an important topic in asset management under uncertainty, and in finance one of the most important methods for the risk allocation is the portfolio. In classical portfolio theory, *Markowitz's mean-variance model* has brought us fruitful results with mathematical programming ([7, 9, 12, 3, 14, 13]). The variance-minimizing model is also studied as a dual problem of Markowitz's mean-variance model. Recently, *value-at-risk (VaR)* is used widely in real finance to estimate the risk of worst-scenarios, and it is one of the standard criteria in the asset management ([8]). VaR is one of the risk-sensitive criteria based on percentiles and it implies a kind of risk values for the assets price at a specified risk-level probability. We use VaR for portfolio to avoid risky scenarios in investment and VaR is also related to the bankruptcy and the falling rate of asset prices ([22]). Many researchers and financial traders use VaR by numerical approximations since it is not easy to analyze the VaR portfolio mathematically ([24, 25]). Actually Markowitz's mean-variance problem and the variance-minimizing are quadratic programming, however VaR is neither linear nor quadratic. VaR is corresponding directly to parameters in real finance and it is useful in actual risk management. We have a merit to use as a risk allocation tool since it is easy to explain its relation. On the other hand, from the viewpoint of risk theory, it is also known that VaR is not a coherent risk measure ([1]). In this paper, we deal with *average value-at-risk (AVaR)*, which is derived from VaR, and an AVaR-portfolio problem. AVaR is not easy to find the correspondence with parameters in real finance, however it is a coherent risk measure. Thus, VaR and AVaR have a merit and a demerit as a risk allocation tool.

Estimation of uncertain quantities is important in decision making ([23, 16, 17]). To represent uncertainty in this portfolio model, we use *fuzzy random variables* which have two kinds of uncertainties, i.e. randomness and fuzziness. In this paper, randomness is used to represent the uncertainty regarding the belief degree of frequency, and fuzziness

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is applied to linguistic imprecision of data because of a lack of knowledge regarding the current stock market. We extend the AVaR for real random variables to one regarding fuzzy random variables from the viewpoint of perception-based approach in Yoshida [21]. We formulate the AVaR portfolio problem with fuzzy random variables, and we discuss the fundamental properties of the extended AVaR using the results in Yoshida [22]. Recently, Yoshida [18, 20] introduced the mean, the variance and the measurement of fuzziness of fuzzy random variables, using *evaluation weights* and λ-mean functions. This paper estimates fuzzy numbers/fuzzy random variables by the probabilistic expectation and these criteria, which are characterized by *possibility/necessity criteria* for subjective estimation and a *pessimistic-optimistic index* for subjective decision. These parameters are decided by the investor and are based on the degree of his certainty regarding the current information in the market. In this portfolio model, we use triangle-type fuzzy numbers/fuzzy random variables for computation in actual models, and we analyze mathematically the AVaR portfolio problem under some regularity condition.

2 A portfolio model under stochastic and fuzzy environment

In this paper, we consider a portfolio model with n stocks as risky assets, where n is a positive integer. We assume *small investors hypothesis* such that an investor's actions do not have any impact on the stock market ([9]). Let a positive integer T denote an expiration date, and let $\mathbb{R}$ denote the set of all real numbers. Let (Ω, P) be a probability space, where P is a non-atomic probability measure on a sample space Ω. For an asset $i = 1, 2, \cdots, n$, a *stock price process* $\{S_t^i\}_{t=0}^T$ is given by *rates of return* R_t^i at time t as follows. Let a stock price $S_t^i := S_{t-1}^i(1 + R_t^i)$ for time $t = 1, 2, \cdots, T$, where $\{R_t^i\}_{t=1}^T$ is assumed to be a sequence of integrable real random variables. In this paper, we discuss a portfolio model where stock prices S_t^i take fuzzy values using fuzzy random variables, taking into account from linguistic imprecision of data because of a lack of knowledge regarding the current stock market. Mathematical notations of fuzzy random variables are introduced later. Hence, we deal with a portfolio with *portfolios given by portfolio weight vectors* $w = (w^1, w^2, \cdots, w^n)$ such that $w^1 + w^2 + \cdots + w^n = 1$ and $w^i \geq 0\,(i = 1, 2, \cdots, n)$. The rate of return for the portfolio $w = (w^1, w^2, \cdots, w^n)$ is given by

$$R_t := w^1 R_t^1 + w^2 R_t^2 + \cdots + w^n R_t^n. \qquad (1)$$

This paper assumes that $R_t^i\,(i = 1, 2, \cdots, n)$ has a normal distribution ([8, 24, 25]).

Next, we introduce fuzzy numbers/fuzzy random variable and we give a portfolio model under uncertainty. A fuzzy number is denoted by its membership function $\tilde{a} : \mathbb{R} \mapsto [0, 1]$ which is normal, upper-semicontinuous and quasi-concave and has a compact support ([16, 17, 26]). $\mathcal{R}$ denotes the set of all fuzzy numbers. In this paper, we identify fuzzy numbers with their corresponding membership functions. The α-cut of a fuzzy number $\tilde{a}(\in \mathcal{R})$ is given by $\tilde{a}_\alpha := \{x \in \mathbb{R} \mid \tilde{a}(x) \geq \alpha\}$ $(\alpha \in (0, 1])$ and $\tilde{a}_0 := \mathrm{cl}\{x \in \mathbb{R} \mid \tilde{a}(x) > 0\}$, where cl denotes the closure of an interval. We write the closed intervals as $\tilde{a}_\alpha := [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ for $\alpha \in [0, 1]$. Hence we also introduce a partial order $\succeq$, so called the *fuzzy max order*, on fuzzy numbers $\mathcal{R}$ ([4]). An addition, a subtraction and a scalar multiplication for fuzzy numbers are defined by Zadeh's extension principle ([16, 17, 26]).

A fuzzy-number-valued map $\tilde{X} : \Omega \mapsto \mathcal{R}$ is called a *fuzzy random variable* if the maps $\omega \mapsto \tilde{X}_\alpha^\pm(\omega)$ are measurable for all $\alpha \in (0, 1]$, where $\tilde{X}_\alpha(\omega) = [\tilde{X}_\alpha^-(\omega), \tilde{X}_\alpha^+(\omega)] = \{x \in \mathbb{R} \mid \tilde{X}(\omega)(x) \geq \alpha\}$ ([6, 10]). We need to introduce expectations of fuzzy random variables in order to describe a portfolio model. A fuzzy random variable $\tilde{X}$ is said to be integrably bounded if $\omega \mapsto \tilde{X}_\alpha^\pm(\omega)$ are integrable for all $\alpha \in (0, 1]$. Let $\tilde{X}$ be an integrably bounded fuzzy random variable. The expectation $E(\tilde{X})$ of the fuzzy random variable $\tilde{X}$ is defined by a fuzzy number $E(\tilde{X})(x) := \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{E(\tilde{X})_\alpha}(x)$, where $E(\tilde{X})_\alpha := [\int_\Omega \tilde{X}_\alpha^-(\omega)\,\mathrm{d}P(\omega), \int_\Omega \tilde{X}_\alpha^+(\omega)\,\mathrm{d}P(\omega)]$ for $\alpha \in$

(0, 1] ([5, 10, 15]).

Now we deal with a case where the rate of return $\{R_t^i\}_{t=1}^T$ has some imprecision ([20]). In this paper, we use triangle-type fuzzy random variables for computation, however we can apply the similar approach to general fuzzy random variables. We define a *rate of return process with imprecision* $\{\tilde{R}_t^i\}_{t=0}^T$ by a sequence of triangle-type fuzzy random variables

$$\tilde{R}_t^i(\cdot)(x) = \begin{cases} 0 & \text{if } x < R_t^i - c_t^i \\ \frac{x - R_t^i + c_t^i}{c_t^i} & \text{if } R_t^i - c_t^i \le x < R_t^i \\ \frac{x - R_t^i - c_t^i}{-c_t^i} & \text{if } R_t^i \le x < R_t^i + c_t^i \\ 0 & \text{if } x \ge R_t^i + c_t^i, \end{cases} \tag{2}$$

where c_t^i is a positive number. We call c_t^i a *fuzzy factor* for asset i at time t. Hence we can represent $\tilde{R}_t^i$ by the sum of the real random variable R_t^i and a fuzzy number $\tilde{a}_t^i$:

$$\tilde{R}_t^i(\omega)(\cdot) := 1_{\{R_t^i(\omega)\}}(\cdot) + \tilde{a}_t^i(\cdot) \tag{3}$$

for $\omega \in \Omega$, where $1_{\{\cdot\}}$ denotes the characteristic function of a singleton and $\tilde{a}_t^i$ is a triangle-type fuzzy number defined by

$$\tilde{a}_t^i(x) = \begin{cases} 0 & \text{if } x < -c_t^i \\ \frac{x + c_t^i}{c_t^i} & \text{if } -c_t^i \le x < 0 \\ \frac{x - c_t^i}{-c_t^i} & \text{if } 0 \le x < c_t^i \\ 0 & \text{if } x \ge c_t^i. \end{cases} \tag{4}$$

For assets $i = 1, 2, \cdots, n$, we define *stock price processes* $\{\tilde{S}_t^i\}_{t=0}^T$ by the *rates of return with imprecision* $\tilde{R}_t^i$ as follows: $\tilde{S}_0^i := S_0^i$ is a positive number and

$$\tilde{S}_t^i = \tilde{S}_0^i \prod_{s=1}^t (1 + \tilde{R}_s^i) \tag{5}$$

for $t = 1, 2, \cdots, T$ ([17]). Hence, we present a portfolio with *trading strategies given by portfolio weight vectors* $w = (w^1, w^2, \cdots, w^n)$ such that $w^1 + w^2 + \cdots + w^n = 1$ and $w^i \ge 0\,(i = 1, 2, \cdots, n)$. For the portfolio $w = (w^1, w^2, \cdots, w^n)$, the rate of return with imprecision for the portfolio is given by a linear combination of fuzzy random variables

$$\tilde{R}_t := w^1 \tilde{R}_t^1 + w^2 \tilde{R}_t^2 + \cdots + w^n \tilde{R}_t^n. \tag{6}$$

In Section 4, we discuss an AVaR model regarding (6).

3 An extension of AVaR for fuzzy random variables

In this section, we introduce an average value-at-risk for fuzzy random variables and we apply it to the rate of return (6). Let $\mathcal{X}$ be the set of all integrable real random variables X on Ω with a continuous distribution function $x \mapsto F_X(x) := P(X < x)$ for which there exists a non-empty open interval I such that $F_X(\cdot) : I \mapsto (0, 1)$ is a strictly increasing and onto. Then there exists a strictly increasing and continuous inverse function $F_X^{-1} : (0, 1) \mapsto I$. We put $F_X(\inf I) := \lim_{x \downarrow \inf I} F_X(x) = 0$ and $F_X(\sup I) := \lim_{x \uparrow \sup I} F_X(x) = 1$. Then, the *value-at-risk (VaR)* at a risk probability p is given by the percentile of the distribution function F_X. Define

$$\mathrm{VaR}_p(X) := \sup\{x \in I \mid F_X(x) \le p\} \tag{7}$$

if $0 < p < 1$, $\mathrm{VaR}_p(X) := 0$ if $p = 0$ and $\mathrm{VaR}_p(X) := 1$ if $p = 1$. Then we have $\mathrm{VaR}_p(X) = F_X^{-1}(p)$ for $0 < p < 1$. The *average value-at-risk (AVaR)* at a probability level p (*Expected Shortfall* with at a confidence probability level $1 - p$) is given by

$$\mathrm{AVaR}_p(X) := \frac{1}{p} \int_0^p \mathrm{VaR}_q(X)\, dq \tag{8}$$

if $0 < p \le 1$ and $\mathrm{AVaR}_p(X) := \inf I$ if $p = 0$ ([11]) It is known that AVaR has the following properties, which implies AVaR is a *coherent risk measure.*

Lemma 1. *Let $X, Y \in \mathcal{X}$ and let p be a positive probability. Then the average value-at-risk* AVaR_p *defined by (8) has the following properties:*

(i) *If $X \le Y$, then* $\mathrm{AVaR}_p(X) \le \mathrm{AVaR}_p(Y)$. *(monotonicity)*

(ii) $\mathrm{AVaR}_p(\zeta X) = \zeta\, \mathrm{AVaR}_p(X)$ *for $\zeta > 0$. (positively homogeneity)*

(iii) $\mathrm{AVaR}_p(X + \theta) = \mathrm{AVaR}_p(X) + \theta$ *for $\theta \in \mathbb{R}$. (translation invariance)*

(iv) $\mathrm{AVaR}_p(X + Y) \ge \mathrm{AVaR}_p(X) + \mathrm{AVaR}_p(Y)$. *(super-additivity)*

Remark. Regarding Lemma 1(iv), we note that the super-additivity for the value-at-risk

$$\mathrm{VaR}_p(X+Y) \geq \mathrm{VaR}_p(X) + \mathrm{VaR}_p(Y),$$

$(X, Y \in \mathcal{X})$ does *not hold in general* ([1]).

Let $\tilde{\mathcal{X}}$ be the set of all fuzzy random variables $\tilde{X}$ on Ω such that their α-cuts $\tilde{X}_\alpha^\pm$ are integrable and $\lambda \tilde{X}_\alpha^- + (1-\lambda)\tilde{X}_\alpha^+ \in \mathcal{X}$ for all $\lambda \in [0,1]$ and $\alpha \in [0,1]$. Hence, from (8) we introduce an AVaR for a fuzzy random variable $\tilde{X}(\in \tilde{\mathcal{X}})$ at a positive risk probability p as follows.

$$\begin{aligned}&\mathrm{AVaR}_p(\tilde{X})(x)\\ &:= \sup_{X \in \mathcal{X}\,:\,\mathrm{AVaR}_p(X)=x} \ \inf_{\omega \in \Omega} \tilde{X}(\omega)(X(\omega)),\end{aligned}$$

$x \in \mathbb{R}$. Yoshida [21] has studied *perception-based estimations* extending the concept of the expectations in Kruce and Meyer [5]. This definition is an extension from the AVaR on real random variables to the AVaR on fuzzy random variables. Hence, the AVaR on fuzzy random variables is characterized by the following representation ([21]).

Theorem 1. *Let $\tilde{X} \in \tilde{\mathcal{X}}$ be a fuzzy random variable and let p be a positive probability. Then the average value-at-risk $\mathrm{AVaR}_p(\tilde{X})$ is a fuzzy number whose α-cuts are*

$$\mathrm{AVaR}_p(\tilde{X})_\alpha = [\mathrm{AVaR}_p(\tilde{X}_\alpha^-), \mathrm{AVaR}_p(\tilde{X}_\alpha^+)], \tag{9}$$

for $\alpha \in (0,1]$.

The AVaR on fuzzy random variables has the following properties similar to Lemma 1 for the AVaR (8). Theorem 2 shows that AVaR is a *coherent risk measure on the fuzzy random variables.*

Theorem 2. *Let $\tilde{X}, \tilde{Y} \in \tilde{\mathcal{X}}$ be fuzzy random variables and let p be a positive probability. Then the average value-at-risk AVaR_p on fuzzy random variables has the following properties:*

(i) *If $\tilde{X} \preceq \tilde{Y}$, then $\mathrm{AVaR}_p(\tilde{X}) \preceq \mathrm{AVaR}_p(\tilde{Y})$. (monotonicity)*

(ii) $\mathrm{AVaR}_p(\zeta\tilde{X}) = \zeta\,\mathrm{AVaR}_p(\tilde{X})$ *for $\zeta > 0$. (positively homogeneity)*

(iii) $\mathrm{AVaR}_p(\tilde{X} + \tilde{a}) = \mathrm{AVaR}_p(\tilde{X}) + \tilde{a}$ *for a fuzzy number $\tilde{a} \in \mathcal{R}$. (translation invariance)*

(iv) $\mathrm{AVaR}_p(\tilde{X} + \tilde{Y}) \succeq \mathrm{AVaR}_p(\tilde{X}) + \mathrm{AVaR}_p(\tilde{Y})$. *(super-additivity)*

Next we need to evaluate the fuzziness of fuzzy numbers/fuzzy random variables since the average value-at-risk $\mathrm{AVaR}_p(\tilde{R}_t)$ for the rate of return (6) with portfolio is a fuzzy number. There are many studies regarding the defuzzification of fuzzy numbers. Here we adopt the evaluation method of fuzzy numbers/fuzzy random variables, which is given by possibility/necessity criteria ([3, 16, 17]). In the rest of this section we introduce the definitions from [18, 19, 20], and in the next section we estimate the AVaR regarding the rate of return (6) by the evaluation method. Yoshida [18, 20] has studied an evaluation of fuzzy numbers by *evaluation weights* which are induced from fuzzy measures to evaluate a confidence degree that a fuzzy number takes values in an interval. With respect to fuzzy random variables, the randomness is evaluated by the probabilistic expectation and the fuzziness is estimated by the evaluation weights and the following function. Let $g^\lambda : \mathcal{I} \mapsto \mathbb{R}$ be a map such that

$$g^\lambda([x,y]) := \lambda x + (1-\lambda)y \tag{10}$$

for $[x,y] \in \mathcal{I}$, where λ is a constant satisfying $0 \leq \lambda \leq 1$ and $\mathcal{I}$ denotes the set of all bounded closed intervals. This scalarization is used for the estimation of fuzzy numbers to give a mean value of the interval $[x,y]$ with a weight λ, and g^λ is called a λ-*mean function* and λ is called a *pessimistic-optimistic index* which indicates the pessimistic degree of attitude in decision making ([2]). Let a fuzzy number $\tilde{a} \in \mathcal{R}$. A mean value of the fuzzy number $\tilde{a}$ with respect to λ-mean functions g^λ and an evaluation weight $w(\alpha)$, which depends only on $\tilde{a}$ and α, is given as follows

([18, 19]):

$$\tilde{E}(\tilde{a}) := \int_0^1 g^\lambda(\tilde{a}_\alpha)\, w(\alpha)\, \mathrm{d}\alpha \Big/ \int_0^1 w(\alpha)\, \mathrm{d}\alpha, \tag{11}$$

where $\tilde{a}_\alpha = [\tilde{a}_\alpha^-, \tilde{a}_\alpha^+]$ is the α-cut of the fuzzy number $\tilde{a}$. In (11), $w(\alpha)$ indicates a *confidence degree that the fuzzy number $\tilde{a}$ takes values in the interval $\tilde{a}_\alpha$ at each level α*. Hence, an evaluation weight $w(\alpha)$ is called the *possibility evaluation weight* $w^P(\alpha)$ if $w^P(\alpha) := 1$ for $\alpha \in [0,1]$, and $w(\alpha)$ is called the *necessity evaluation weight* $w^N(\alpha)$ if $w^N(\alpha) := 1 - \alpha$ for $\alpha \in [0,1]$. Especially, for a fuzzy number $\tilde{a} \in \mathcal{R}$, the mean $\tilde{E}^P(\tilde{a})$ in the possibility case and the mean $\tilde{E}^N(\tilde{a})$ in the necessity case are represented as follows ([18, 19]):

$$\tilde{E}^P(\tilde{a}) = \int_0^1 g^\lambda(\tilde{a}_\alpha)\, \mathrm{d}\alpha, \tag{12}$$

$$\tilde{E}^N(\tilde{a}) = \int_0^1 g^\lambda(\tilde{a}_\alpha)\, (2 - 2\alpha)\, \mathrm{d}\alpha. \tag{13}$$

The mean $\tilde{E}$ has the following natural properties of the linearity and the monotonicity regarding the fuzzy max order.

Lemma 2 ([18, 19, 20]). *Let $\lambda \in [0,1]$. For fuzzy numbers $\tilde{a}, \tilde{b} \in \mathcal{R}$ and real numbers θ, ζ, the following (i) – (iv) hold.*

(i) $\tilde{E}(\tilde{a} + 1_{\{\theta\}}) = \tilde{E}(\tilde{a}) + \theta$, *where $1_{\{\cdot\}}$ is the characteristic function of a set.*

(ii) $\tilde{E}(\zeta\tilde{a}) = \zeta\tilde{E}(\tilde{a})$ *if* $\zeta \geq 0$.

(iii) $\tilde{E}(\tilde{a} + \tilde{b}) = \tilde{E}(\tilde{a}) + \tilde{E}(\tilde{b})$.

(iv) *If $\tilde{a} \succeq \tilde{b}$, then $\tilde{E}(\tilde{a}) \geq \tilde{E}(\tilde{b})$.*

For a fuzzy random variable $\tilde{X}$, *the mean of the expectation* $E(\tilde{X})$ is a real number

$$E(\tilde{E}(\tilde{X})) := E\left(\int_0^1 g^\lambda(\tilde{X}_\alpha)\, w(\alpha)\, \mathrm{d}\alpha \Big/ \int_0^1 w(\alpha)\, \mathrm{d}\alpha \right).$$

From Lemma 2, we obtain the following results regarding fuzzy random variables.

Lemma 3 ([18, 19, 20]). *Let $\lambda \in [0,1]$. For a fuzzy number $\tilde{a} \in \mathcal{R}$, integrable fuzzy random variables $\tilde{X}, \tilde{Y}$, an integrable real random variable Z and a nonnegative real number ζ, the following (i) – (v) hold.*

(i) $E(\tilde{E}(\tilde{X})) = \tilde{E}(E(\tilde{X}))$.

(ii) $E(\tilde{E}(\tilde{a})) = \tilde{E}(\tilde{a})$ *and* $E(\tilde{E}(Z)) = E(Z)$.

(iii) $E(\tilde{E}(\zeta\tilde{X})) = \zeta E(\tilde{E}(\tilde{X}))$.

(iv) $E(\tilde{E}(\tilde{X} + \tilde{Y})) = E(\tilde{E}(\tilde{X})) + E(\tilde{E}(\tilde{Y}))$.

(v) *If $\tilde{X} \succeq \tilde{Y}$, then $E(\tilde{E}(\tilde{X})) \geq E(\tilde{E}(\tilde{Y}))$.*

4 An AVaR portfolio model under stochastic and fuzzy environment

In this section, we discuss portfolio problems under uncertainty. First we estimate the rate of return with imprecision for a portfolio. Let the mean, the variance and the covariance of the rate of return R_t^i by

$$\begin{aligned} \mu_t^i &:= E(R_t^i), \\ (\sigma_t^i)^2 &:= E((R_t^i - \mu_t^i)^2), \\ \sigma_t^i &:= E((R_t^i - \mu_t^i)(R_t^j - \mu_t^j)) \end{aligned}$$

for $i, j = 1, 2, \cdots, n$. We assume that the determinant of the variance-covariance matrix $[\sigma_t^{ij}]$ is not zero and there exists its inverse matrix. For a portfolio $w = (w^1, w^2, \cdots, w^n)$ satisfying $w^1 + w^2 + \cdots + w^n = 1$ and $w^i \geq 0$ $(i = 1, 2, \cdots, n)$, we calculate the expectation and the variance regarding $\tilde{R}_t = w^1\tilde{R}_t^1 + w^2\tilde{R}_t^2 + \cdots + w^n\tilde{R}_t^n$. From Lemma 3, the expectation $\tilde{\mu}_t := E(\tilde{E}(\tilde{R}_t))$ follows

$$\tilde{\mu}_t = \sum_{i=1}^n w^i \tilde{\mu}_t^i, \tag{14}$$

where $\tilde{\mu}_t^i := E(\tilde{E}(\tilde{R}_t^i))$ for $i = 1, 2, \cdots, n$. On the other hand, regarding this model, in Yoshida [20] we can find that the variance $(\tilde{\sigma}_t)^2 := E((\tilde{E}(\tilde{R}_t) - \tilde{\mu}_t)^2)$ of $\tilde{R}_t$ equals to the variance $(\sigma_t)^2 := E((R_t - \mu_t)^2)$ of R_t:

$$(\tilde{\sigma}_t)^2 = (\sigma_t)^2 = \sum_{i=1}^n \sum_{j=1}^n w^i w^j \sigma_t^i. \tag{15}$$

Hence, applying Lemmas 2 and 3 to (3), we obtain the following lemma regarding AVaR of the rates of return $\tilde{R}_t^i$.

Lemma 4. *Let p be a positive probability. The following (i) and (ii) hold:*

(i) $\tilde{\mu}_t^i = \mu_t^i + \tilde{E}(\tilde{a}_t^i)$ *for* $i = 1, 2, \cdots, n$.

(ii) *The mean of* $\mathrm{AVaR}_p(\tilde{R}_t)$ *is evaluated by*

$$\tilde{E}(\mathrm{AVaR}_p(\tilde{R}_t)) = \sum_{i=1}^{n} w^i \tilde{\mu}_t^i - \kappa \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w^i w^j \sigma_t^i}$$

where $\kappa := \frac{1}{p}\int_0^p \kappa(q)\,dq$ *for* $\kappa(q)$ *defined by* $\tilde{E}(\mathrm{VaR}_q(\tilde{R}_t)) = \sum_{i=1}^{n} w^i \tilde{\mu}_t^i - \kappa(q)\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w^i w^j \sigma_t^i}$.

Now we discuss the following AVaR portfolio without allowance for short selling. The following form (16) comes from the average value-at-risk $\tilde{E}(\mathrm{AVaR}_p(\tilde{R}_t))$ in Lemma 4.

AVaR-portfolio problem (P): Maximize the average value-at-risk

$$\sum_{i=1}^{n} w^i \tilde{\mu}^i - \kappa \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w^i w^j \sigma_t^{ij}} \qquad (16)$$

with respect to portfolios $w = (w^1, w^2, \cdots, w^n)$ satisfying $w^1 + w^2 + \cdots + w^n = 1$ and $w^i \geq 0$ for $i = 1, 2, \cdots, n$.

Let $\tilde{\mu}$ be the vector whose elements are $\tilde{\mu}^i = \mu_t^i + \tilde{E}(\tilde{a}_t^i)$ $(i = 1, 2, \cdots)$, and let $\mathbf{1}$ be the vector whose elements are 1. Let

$$\Sigma := \begin{bmatrix} \sigma^{11} & \sigma^{12} & \cdots & \sigma^{1n} \\ \sigma^{21} & \sigma^{22} & \cdots & \sigma^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma^{n1} & \sigma^{n2} & \cdots & \sigma^{nn} \end{bmatrix},$$

$A := \mathbf{1}^{\mathrm{T}}\Sigma^{-1}\mathbf{1}$, $B := \mathbf{1}^{\mathrm{T}}\Sigma^{-1}\tilde{\mu}$, $C := \tilde{\mu}^{\mathrm{T}}\Sigma^{-1}\tilde{\mu}$ and $\Delta := AC - B^2$.

Hence, in a similar way as the proof in Yoshida [22, Theorem 4.2], we arrive at the following analytical solutions regarding AVaR-portfolio problem for $\kappa = \frac{1}{p}\int_0^p \kappa(q)\,dq$.

Theorem 3. *Let A and Δ be positive. Let κ satisfy $\kappa^2 > C$. Then the following (i) and (ii) hold.*

(i) *The solution of AVaR-portfolio problem (P) is given by*

$$w^* := \xi\Sigma^{-1}\mathbf{1} + \eta\Sigma^{-1}\tilde{\mu} \qquad (17)$$

and then the corresponding AVaR is

$$v^* := \frac{B - \sqrt{A\kappa^2 - \Delta}}{A} \qquad (18)$$

at the expected rate of return

$$\gamma^* := \frac{B}{A} + \frac{\Delta}{A\sqrt{A\kappa^2 - \Delta}}, \qquad (19)$$

where $\xi := \frac{C - B\gamma^*}{\Delta}$ *and* $\eta := \frac{A\gamma^* - B}{\Delta}$.

(ii) *Further, if* $\Sigma^{-1}\mathbf{1} \geq \mathbf{0}$ *and* $\Sigma^{-1}\tilde{\mu} \geq \mathbf{0}$, *then the portfolio (17) satisfies* $w^* \geq \mathbf{0}$, *i.e. the portfolio* w^* *is a portfolio without allowance for short selling. Here,* $\mathbf{0}$ *denotes the zero vector.*

5 A numerical example

In this session, we give a simple example to illustrate our idea. For the numerical computation, we need to evaluate fuzzy numbers representing the rates of return (2). Hence we consider a combination of the both cases with a parameter $\nu(\in [0,1])$, which is called a *possibility-necessity weight* ([20]). Then the combination of the possibility case and the necessity case is $\tilde{E}^{\nu}(\tilde{a}_t^i) := \nu\tilde{E}^P(\tilde{a}_t^i) + (1-\nu)\tilde{E}^N(\tilde{a}_t^i) = \frac{(1-2\lambda)(4-\nu)}{6}c_t^i$. Therefore from (14) we obtain the expected rate of return $\tilde{\mu}_t = \sum_{i=1}^{n} w^i(\mu_t^i + \tilde{E}(\tilde{a}_t^i)) = \sum_{i=1}^{n} w^i\left(\mu_t^i + \frac{(1-2\lambda)(4-\nu)}{6}c_t^i\right)$ for the possibility-necessity weight $\nu(\in [0,1])$ and the pessimistic-optimistic index $\lambda(\in [0,1])$. In (5.3), the decision maker may choose the parameters $\lambda(\in [0,1])$ and $\nu(\in [0,1])$. The pessimistic-optimistic index is taken as $\lambda = 1$ if he has pessimistic personal forecast in the market and he takes careful decision, and $\lambda = 0$ if he has optimistic personal forecast and he is not nervous. The possibility-necessity weight is taken as $\nu = 1$ when he has enough confidential information about the market, and $\nu = 0$ when he does not have confidential information. In this model, $\nu = 0$ is reasonable since our objective function is AVaR, which is a kind of risk, and we need to take into account of the fuzziness of information in the market. While λ depends on the decision maker's attitude in his investment. In this example, we compute the pessimistic case $\lambda = 1$. Let $n = 4$ be the number of assets. Take the expected rate of return, a variance-covariance matrix and fuzzy factors as Table 1. It is assumed that the rate of return R_t^i has the normal distributions. We discuss a risk probability 1% in the normal distribution, and then the corresponding constant is $\kappa = \frac{1}{p}\int_0^p \kappa(q)\,dq = 2.66521$. Then, the conditions in Theorem 3 are satisfied and by formulae of Theorem 3 we easily obtain the optimal portfolio $w^* = (w^1, w^2, w^3, w^4) = (0.191723, 0.28305, 0.262884, 0.262343)$ for AVaR-portfolio problem (P), and then the corresponding AVaR is $v^* = -0.638258$ and the expected rate of return is $\gamma^* = 0.0521709$.

Table 1. Expected rates of return, a variance-covariance matrix and fuzzy factors.

Asset i	μ_t^i
1	0.04
2	0.06
3	0.07
4	0.05

Asset i	c_t^i
1	0.006
2	0.008
3	0.007
4	0.005

σ^{ij}	1	2	3	4
1	0.31	0.04	0.05	−0.07
2	0.04	0.23	−0.08	0.06
3	0.05	−0.08	0.34	−0.03
4	−0.07	0.06	−0.03	0.27

6 Conclusion

In this paper, we have discussed the following terms:

- Extension of AVaR for fuzzy random variable, and its coherence as a risk measure.
- An VaR-portfolio model under randomness and fuzziness.
- An optimality portfolio for this model.

VaR is directly related to the falling rate of the asset prices, and it is used widely in real finance. On the other hand, AVaR is not easy to find a direct relation with parameters in real finance, however AVaR is a coherence risk measure. The coherence is a necessary property as a criterion from the viewpoint of axiomatic approach for risk measures.

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