Generalization of IGOWA operators

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Abstract

Generalization of induced generalized OWA - IGOWA operators is discussed. Following [3], [5] [7], [8], we focus on a connection of two approaches - a generalization of IGOWA operators and minimization based aggregation operators $A_{\mathbf{w},D}$. Special attention is paid to dissimilarity function $D(x,y) = (f(x) - f(y))^2$. We introduce next generalization of IGOWA operators - induced generalized quasi-OWA - IGQOWA operators. Finally, we introduce a generalization of IGQOWA operators by using some positive inducing function for an order inducing variable. We will call them function induced generalized quasi-OWA - FIGQOWA operators.

Keywords: Aggregation operator, Dissimilarity function, Induced generalized OWA - IGOWA operator, Inducing variable, Inducing function.

1 Introduction

One of distinguished aggregation operators is the order weighted averaging OWA operator introduced in [11]. Yager and Filev in [14] developed an extension to the OWA operator called induced ordered weighted averaging - IOWA operator. In this case the reordering step is induced by their associated order inducing variables. Recently Yager in [13] introduced a generalized OWA operator where was introduced a reordering step in the generalized mean with respect to the values of the arguments. More recently, Beliakov in [1] has suggested a further generalization to the GOWA operator by using quasi-arithmetic mean. Quasi-OWA operator was studied in different works, for example [2], [4], [9] and [10].

In this paper, we introduce the generalization of induced generalized OWA operators - IGOWA operators introduced by Merigó and Gil-Lafuente in [5] by using minimization based aggregation operator $A_{\mathbf{w},D}$. We will call them induced generalized quasi-OWA - IGQOWA operators. In this case for special weights we get well known operators - the arithmetic mean AM, the weighted arithmetic mean WAM, the ordered weighted averaging operator OWA, the induced OWA - IOWA operator, the generalized OWA - GOWA operator and the induced generalized OWA - IGOWA operator. We develop a generalization of IGOWA by using so called the order inducing function for the order inducing variable, and we will call it the function induced generalized quasi-OWA - FIGQOWA operator and the function induced generalized OWA - FIGOWA operator.

This paper is organized as follows. In Section 2 we recall basic definition for a dissimilarity function $D(x,y)$, and we recall some properties of minimization based aggregation operators. In Section 3 we describe definitions of the OWA, IOWA, GOWA, and IGOWA operators. In Section 4 we search a generalization of IGOWA operators by using of minimization based aggregation operators - IGQOWA operators. In Section 5 we introduce further generalization of IGQOWA operators by using of so called an order inducing function, and we get FIGQOWA and FIGOWA operators. Finally, in Section 6 some conclusions and indication of further research of this topic is included.

2 Dissimilarity function and minimization problem

Let $I \subset R$ be a closed interval, $I = [a, b]$. Following [3] we recall the definition of dissimilarity function on I.

Definition 2.1. *Let $K : R \to R$ be a convex function with unique minimum $K(0) = 0$ and let $f : I \to R$ be a continuous strictly monotone function. Then the function $D : I^2 \to R$ given by $D(x, y) = K(f(x) - f(y))$ is called a dissimilarity function (on I).*

Based on dissimilarity functions, minimization problem $A_D(a_1, \ldots, a_n) = s$ such that

$$\sum_{i=1}^{n} D(a_i, s) = min \left\{ \sum_{i=1}^{n} D(a_i, r) | r \in I \right\}$$

was discussed in [3], [7].

Introducing of fixed weights in this approach was easy and it leads to the modified minimization problem related to $\sum_{i=1}^{n} w_i \cdot D(a_i, r)$. For special choice of D we have recognized here common idempotent aggregation operator like the (weighted) quasi-arithmetic mean linked to $D(x, y) = (f(x) - f(y))^2$.

In what follows, we will deal with weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$, $\mathbf{w} \neq \mathbf{0}$ and $w_i \geq 0$, $i = 1, \ldots, n$.

Proposition 2.1. *For a given interval $I \subset R$, let $D : I^2 \to R$ be given by $D(x, y) = (f(x) - f(y))^2$. Then for any weighting vector $\boldsymbol{w}$ the operator $A_{\boldsymbol{w},D} : \bigcup_{n \in N} I^n \to I$ is a quasi-arithmetic mean given by*

$$A_{\boldsymbol{w},D}(a_1, \ldots, a_n) = f^{-1} \left(\frac{\sum_{i=1}^{n} w_i f(a_i)}{\sum_{i=1}^{n} w_i} \right). \qquad (1)$$

Proof. We have to minimize the expression

$$k(r) = \sum_{i=1}^{n} w_i \cdot (f(a_i) - f(r))^2,$$

$r \in I$. The function k is differentiable and $k'(r) = -2 \sum_{i=1}^{n} w_i \cdot (f(a_i) - f(r)) \cdot f'(r)$ implying (1). □

3 Aggregation operators

In this section we recall some definitions of ordered weighted averaging operators, consequently OWA, IOWA, GOWA, IGOWA operators.

3.1 OWA operator

The OWA operator was introduced by Yager in [11]. It can be defined as follows.

Definition 3.1. *For a given interval $I \subset R$ an OWA operator of dimension n, is a mapping $OWA : I^n \to I$ that has an associated weighting vector $\boldsymbol{w}$, then*

$$OWA(a_1, \ldots, a_n) = \frac{\sum_{j=1}^{n} w_j b_j}{\sum_{j=1}^{n} w_j}, \qquad (2)$$

where b_j is the jth lowest of the a_i.

We distinguish with respect to the reordering step between the descending OWA - DOWA and the ascending OWA - AOWA operator, in general. The AOWA operator is related to DOWA operator by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DOWA operator and w^*_{n+1-j} is the jth weight of the AOWA operator. For more details see also [5], [12]. The AOWA operator has the same definition than the OWA operator in Definition 3.1.. The OWA operator is commutative, monotonic, bounded and idempotent operator.

3.2 IOWA operator

The IOWA operator represents an extension of the OWA operator. The IOWA operator was introduced by Yager and Filev in [14]. Main difference is that the reordering step is not developed with the values of the arguments a_i, but with order inducing variables u_i (see [5]).

Definition 3.2. *For a given interval $I \subset R$ an IOWA operator of dimension n is a mapping $IOWA : I^{2n} \to I$ that has an associated weighting vector $\boldsymbol{w}$, then*

$$IOWA(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) = \frac{\sum_{j=1}^{n} w_j b_j}{\sum_{j=1}^{n} w_j} \tag{3}$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i\rangle$ having the jth lowest u_i, u_i is the order inducing variable and a_i is the argument variable.

In this case also with respect to the reordering step we distinguish between the descending DIOWA operator and the ascending AIOWA operator, [6]. Definition 3.2. of the IOWA operator is equivalent to the definition of AIOWA operator.

3.3 GOWA operator

The generalized OWA - GOWA operator was introduced by Yager in [13].

Definition 3.3. *For a given interval $I \subset [0, \infty[$ an GOWA operator of dimension n is a mapping $GOWA : I^n \to I$ that has an associated weighting vector $\boldsymbol{w}$, then*

$$GOWA(a_1, \ldots, a_n) = \left(\frac{\sum_{j=1}^{n} w_j b_j^{\lambda}}{\sum_{j=1}^{n} w_j} \right)^{1/\lambda} \tag{4}$$

where b_j is the jth lowest of the a_i, and λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, \infty)$.

Remark 3.1. For different parameters λ we obtain different types of GOWA operators. Note that if λ approaches 0 we obtain the ordered weighted geometric average operator.

In this case with respect to the reordering step we distinguish also between descending GOWA operator - DGOWA and ascending GOWA operator - AGOWA operator. The AGOWA operator is related to DGOWA operator by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DGOWA operator and w^*_{n+1-j} is the jth weight of the AGOWA operator. The Definition 3.3. of the GOWA operator is equivalent to the AGOWA operator.

3.4 IGOWA operator

The induced generalized OWA - IGOWA operator represents an extension of the GOWA operator. The reordering step is induced by another mechanism represented as u_i, where ordered position of the arguments a_i depends upon the values of the order inducing variable u_i.

Definition 3.4. *For a given interval $I \subset [0, \infty[$ an IGOWA operator of dimension n is a mapping $IGOWA : I^{2n} \to I$ that has an associated weighting vector $\boldsymbol{w}$, then*

$$IGOWA(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) =$$

$$= \left(\frac{\sum_{j=1}^{n} w_j b_j^{\lambda}}{\sum_{j=1}^{n} w_j} \right)^{1/\lambda} \tag{5}$$

where b_j is the a_i value of the IGOWA pair $\langle u_i, a_i\rangle$ having the jth lowest u_i, u_i is the order inducing variable, a_i is the argument variable, and λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, \infty)$.

Remark 3.2. For different parameters λ we obtain different types of IOWA operators. Note that if λ approaches 0 we obtain induced ordered weighted geometric average operator.

Again, we distinguish between the descending induced generalized DIGOWA operator and the ascending induced generalized AIGOWA operator. The Definition 3.4. of the IGOWA operator is equivalent to the AIGOWA operator. More information about families of

IGOWA operators for different parameters λ can be found in [5].

4 Generalization of IGOWA operators

4.1 Induced generalized quasi-OWA operator

In this section we will deal with an arbitrary interval I and $D : I^2 \to R$ given by $D(x, y) = (f(x) - f(y))^2$.

Definition 4.1. *Let $D : I \to R$ be a given dissimilarity function and let $\boldsymbol{w}$ be a weighting vector. For any fixed $n \in N$, the operator $A_{\boldsymbol{w},D} : I^{2n} \to I$ is given by $A_{\boldsymbol{w},D}(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) = s$, such that*

$$\sum_{j=1}^{n} w_j \cdot D(b_j, s) =$$

$$= min \left\{ \sum_{j=1}^{n} w_j \cdot D(b_j, r) | r \in I \right\}$$

where b_j is the a_i value of the pair $\langle u_i, a_i\rangle$ having the jth lowest u_i, u_i is the order inducing variable and a_i is the argument variable. If there are more minimization solutions, then $s = \frac{(\underline{s} + \overline{s})}{2}$ where

$$[\underline{s}, \overline{s}] =$$

$$= \left\{ t \in I | \sum_{j=1}^{n} w_j \cdot D(b_j, t) = min\{\sum_{j=1}^{n} w_j \cdot D(b_j, r) | r \in I\} \right\}.$$

Definition 4.2. *An IGQOWA operator of dimension n is a mapping $IGQOWA : I^{2n} \to I$ that has an associated weighting vector $\boldsymbol{w}$, $f : I \to R$ is strictly monotone function, then*

$$IGQOWA(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) =$$

$$= f^{-1} \left(\frac{\sum_{j=1}^{n} w_j \cdot f(b_j)}{\sum_{j=1}^{n} w_j} \right) \qquad (6)$$

where b_j is the a_i value of the IGQOWA pair $\langle u_i, a_i\rangle$ having the jth lowest u_i, u_i is the order inducing variable and a_i is the argument variable.

Proposition 4.1. *For a given interval $I \subset [0, \infty[$, let $D : I^2 \to R$ be given by $D(x, y) = (f(x) - f(y))^2$. Then for an associated weighting vector $\boldsymbol{w}$ the operator $A_{\boldsymbol{w},D} : \bigcup_{n \in N} I^{2n} \to I$ is an induced generalized quasi-OWA operator*

$$IGQOWA(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) =$$

$$= A_{\boldsymbol{w},D}(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) =$$

$$= f^{-1} \left(\frac{\sum_{j=1}^{n} w_j \cdot f(b_j)}{\sum_{j=1}^{n} w_j} \right) \qquad (7)$$

where b_j is the a_i value of the pair $\langle u_i, a_i\rangle$, having the jth lowest u_i, u_i is the order inducing variable, and a_i is the argument variable.

Proof. We have to minimize the expression $k(r) = \sum_{j=1}^{n} w_j (f(b_j) - f(r))^2$. The function k is differentiable and $k'(r) = -\sum_{j=1}^{n} w_j \cdot 2 \cdot (f(b_j) - f(r))$ implying (7). □

If in the equation (7) we assume $f = x^\lambda$, $\lambda \in (-\infty, 0) \cup (0, \infty)$, and

- $\mathbf{u} \neq \mathbf{a}$, we get the induced generalized OWA - IGOWA operator (5),
- $\mathbf{u} = \mathbf{a}$, we get the generalized OWA - GOWA operator (4).

If in the equation (7) we assume $f = id$, and

- $w_j = \frac{1}{n}$, for all a_i the IGOWA operator is the arithmetic mean AM,
- b_j is the a_i value of the pair $\langle u_i, a_i\rangle$ having the ith u_i for all a_i the IGOWA operator is the weighted arithmentic mean WAM,
- $\mathbf{u} = \mathbf{a}$, the IGOWA operator is the OWA operator (2),
- $\mathbf{u} \neq \mathbf{a}$, the IGOWA operator is the IOWA operator (3).

4.2 Function induced generalized quasi-OWA operator

In this section we introduce an extension to the IGOWA operators by using a transformation function h for an order inducing variable u_i. The function induced generalized quasi-OWA operator - FIGQOWA operator represents an extension of the IGOWA operator.

Definition 4.3. *An FIGQOWA operator of dimension n is a mapping $FIGQOWA : I^{2n} \to I$ that has an associated weighting vector $\boldsymbol{w}$, $h : I \to 0, \infty[$ is some positive continuous function, $f : I \to R$ is strictly monotone function, then*

$$FIGQOWA((\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) =$$

$$= f^{-1}\left(\frac{\sum_{j=1}^{n} w_j \cdot f(b_j)}{\sum_{j=1}^{n} w_j}\right) \qquad (8)$$

where b_j is the a_i value of the pair $\langle u_i, a_i\rangle$ having the jth lowest value $h(u_i)$, u_i is the order inducing variable and a_i is the argument variable.

Proposition 4.2. *For a given interval $I \subset [0, \infty[$, let $D : I^2 \to R$ be given by $D(x, y) = (f(x) - f(y))^2$. Then for an associated weighting vector $\boldsymbol{w}$ the operator $A_{\boldsymbol{w},D} : \bigcup_{n \in N} I^{2n} \to I$ is a function induced generalized quasi-OWA operator*

$$FIGQOWA(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) =$$

$$= A_{w,D}(\langle h(u_1), a_1\rangle, \ldots, \langle h(u_n), a_n\rangle) =$$

$$= f^{-1}\left(\frac{\sum_{j=1}^{n} w_j \cdot f(b_j)}{\sum_{j=1}^{n} w_j}\right) \qquad (9)$$

where b_j is the a_i value of the pair $\langle u_i, a_i\rangle$ having the jth lowest value $h(u_i)$, u_i is the order inducing variable and a_i is the argument variable.

4.3 Function induced generalized OWA operator

If we assume in (8) $f = x^\lambda$, $\lambda \in (-\infty, 0) \cup (0, \infty)$ we introduce the function induced generalized OWA - FIGOWA operator. The function induced generalized OWA operator - FIGOWA operator represents an extension of the IGOWA operator.

Definition 4.4. *An FIGOWA operator of dimension n is a mapping $FIGOWA : I^{2n} \to I$ that has an associated weighting vector $\boldsymbol{w}$, $h : I \to 0, \infty[$ is some positive continuous function, then*

$$FIGOWA(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) =$$

$$= \left(\frac{\sum_{j=1}^{n} w_j \cdot b_j^\lambda}{\sum_{j=1}^{n} w_j}\right)^{1/\lambda} \qquad (10)$$

where b_j is the a_i value of the pair $\langle u_i, a_i\rangle$ having the jth lowest value $h(u_i)$, u_i is the order inducing variable, a_i is the argument variable, and λ is a parameter such that $\lambda \in (-\infty, 0) \cup (0, \infty)$.

Remark 4.1. For different parameters λ we obtain different types of FIGOWA operators. Note that if λ approaches 0 we obtain function induced ordered weighted geometric average operator.

Proposition 4.3. *For a given interval $I \subset [0, \infty[$, let $D : I^2 \to R$ be given by $D(x, y) = (f(x) - f(y))^2$. Then for an associated weighting vector $\boldsymbol{w}$ the operator $A_{\boldsymbol{w},D} : \bigcup_{n \in N} I^{2n} \to I$ is a function induced generalized OWA operator*

$$FIGOWA(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) =$$

$$A_{w,D}(\langle h(u_1), a_1\rangle, \ldots, \langle h(u_n), a_n\rangle) =$$

$$= \left(\frac{\sum_{j=1}^{n} w_j \cdot b_j^\lambda}{\sum_{j=1}^{n} w_j}\right)^{1/\lambda} \qquad (11)$$

where b_j is the a_i value of the pair $\langle u_i, a_i\rangle$ having the jth lowest value $h(u_i)$, u_i is the order inducing variable and a_i is the argument variable.

If we assume in the equation (8) $f = x^\lambda$, $\lambda \in (-\infty, 0) \cup (0, \infty)$, and

- $h = id$ and $\mathbf{a} \neq \mathbf{u}$ the FIGQOWA operator is the IGOWA operator (5),

- $h = id$ and $\mathbf{a} = \mathbf{u}$ the FIGQOWA operator is the GOWA operator (4),
- $h \neq id$ the FIGQOWA operator is the FIGOWA operator (10). If h is strictly increasing function and $\mathbf{a} \neq \mathbf{u}$ we get ascending FIGOWA - AFIGOWA operator, and if h is strictly decreasing function and $\mathbf{a} \neq \mathbf{u}$ we get descending FIGOWA - DFIGOWA operator. Definition of the AFIGOWA operator is equivalent to the Definition 4.4.. In the case, when h is strictly increasing function and $h(\mathbf{u}) = \mathbf{a}$ we get ascending GOWA - AGOWA operator, and in the case, when h is strictly decreasing function and $h(\mathbf{u}) = \mathbf{a}$ we get descending GOWA - DGOWA operator,
- h is not monotone function, we introduce new special type of FIGQOWA operator.

If in the equation (8) is $f = id$, and

- $h = id$, $\mathbf{a} = \mathbf{u}$ the FIGQOWA operator is the OWA operator (2),
- $h \neq id$ is strictly increasing function and $h(\mathbf{u}) = \mathbf{a}$ the FIGQOWA operator is ascending OWA - AOWA operator. In the case, when h is strictly decreasing function the FIGQOWA operator is descending OWA - DOWA operator,
- $h = id$, $\mathbf{a} \neq \mathbf{u}$ the FIGQOWA operator is the IOWA operator (3),
- $w_j = \dfrac{1}{n}$ the FIGQOWA operator is the arithmetic mean AM,
- b_j is the a_i value of the pair $\langle u_i, a_i \rangle$ having the ith $h(u_i)$ for all a_i the FIGQOWA operator is the weighted arithmetic mean WAM.

Example 4.1. Let $h : I \to I$, $I \subset [0, \infty[$ be the function $h(x) = x + 1$, and let $f : I \to R$ be $f(x) = x^3$. We assume an input vector (a_1, a_2), an order inducing vector (u_1, u_2) and a weighting vector (w_1, w_2).

If $u_1 > u_2$, the FIGOWA operator is

$$FIGOWA\left(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle\right) =$$

$$= \sqrt[3]{\frac{w_1 a_2^3 + w_2 a_1^3}{w_1 + w_2}}.$$

If $u_1 < u_2$, the FIGOWA operator is

$$FIGOWA\left(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle\right) =$$

$$= \sqrt[3]{\frac{w_1 a_1^3 + w_2 a_2^3}{w_1 + w_2}}.$$

The FIGOWA operator is non-decreasing in both cases. If $\sum_{j=1}^{2} w_j = 1$, the FIGOWA operator is idempotent operator.

However, FIGOWA operator need not be monotone, in general.

Example 4.2. Let $\mathbf{a} = \mathbf{u}$ be a input vector, and $h(u) = 1 + |2u - 1|$ be the positive function, $f = id$ and $\mathbf{w} = (0.4; 0.6)$ be the weighting vector.

For the input vector $\mathbf{a} = (0.05; 0.9)$

$$FIGOWA(\langle 0.05; 0.05\rangle, \langle 0.9; 0.9\rangle) =$$

$$= 0.9 \cdot 0.4 + 0.05 \cdot 0.6 = 0.66,$$

and for $\mathbf{a} = (0.2; 0.9)$

$$FIGOWA(\langle 0.2; 0.2\rangle, \langle 0.9; 0.9\rangle) =$$

$$= 0.2 \cdot 0.4 + 0.9 \cdot 0.6 = 0.62.$$

However, $FIGOWA(\langle 0.05; 0.05\rangle, \langle 0.9; 0.9\rangle)$ is greater than

$FIGOWA(\langle 0.2; 0.2\rangle, \langle 0.9; 0.9\rangle)$.

5 Conclusion

We have studied a connection of two approaches to the aggregation operators - minimization based aggregation operators and a generalization of the IGOWA operators. First, we have recalled a definition of dissimilarity function, and a minimization based aggregation operators. Second, we have reviewed some basic operators such as the OWA operator, IOWA operator, GOWA operator. Special attention was paid to the IGOWA operator introduced by Merigó and Gil-Lafuente. On the basis of this information we have introduced an extension to the

IGOWA operator - induced generalized quasi-IGOWA - IGQOWA, function induced generalized quasi-IGOWA - FIGQOWA, and function induced generalized IGOWA - FIGOWA operator. Study of some properties of IGQOWA, FIGQOWA and FIGOWA operators with special stress on their monotonicity will be the topic of our next investigations.

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