Graded Tolerant Inclusion and its Axiomatization

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Abstract

This paper is devoted to an extension of the inclusion operator. The main idea that we suggest is to relax the arguments of the inclusion by means of proximity-based modifiers in order to obtain a more tolerant inclusion. It is shown that the relaxation can be envisaged in order to design a tolerant inclusion whose result is either Boolean, or valued in the unit interval. The focus is particularly made on graded tolerant inclusion. A set of axioms that was previously proposed for the characterization of graded inclusion of fuzzy sets is revisited in order to take into account the specificities of tolerant inclusion.

Keywords: inclusion, graded tolerant inclusion, proximity relation.

1. Introduction

Several types of extensions of inclusion have been proposed in the "fuzzy set research community" in order to either: i) define the inclusion when fuzzy sets come into play, or ii) to make the result of the inclusion more flexible, i.e., valued in [0, 1], or iii) to authorize different types of exceptions. In the first case, Zadeh [17] defined the inclusion of fuzzy sets in the following way: $E \subseteq F \Leftrightarrow \forall u \in U, E(u) \leq F(u)$, where U denotes the universe of discourse. In the second case, the objective is to discriminate between situations significantly different where the usual inclusion does not hold, and a solution consists in using a fuzzy implication to define the graded inclusion [1]. In the third case, two visions of exceptions have been considered so far, which lead to two types of approximate inclusion indicators. A first idea, developed in [6], consists in weakening the universal quantifier underlying the inclusion into "almost all", in the perspective of defining an *approximate* inclusion. The basic idea is to tolerate, in the evaluation of $E \subseteq F$, a certain number of exceptions (i.e., of elements of E which are not totally included in F according to a given implication), and in that sense the corresponding approximate inclusion can be called a *quantitative* one. In [7], another way of defining an approximate inclusion is presented and the idea is rather to give a central role to the intensity of the exceptions in order to define an inclusion indicator that can ignore to a certain extent the "low intensity" ones. In that sense, the operator defined can be called a *qualitative* approximate inclusion.

Here, the idea is to take into account the notion of closeness between the elements of the domain, so as to define a *proximity-based tolerant inclusion*. For instance, one may consider that a set E is included in a set F if, for every element u of E, u is present in F (classical inclusion) or if F contains an element *close* to u. This kind of inclusion can deliver either a Boolean (i.e., 0 or 1) or a gradual (i.e., in [0, 1]) result. Recently in [5], we have considered the Boolean version of tolerant inclusion and have illustrated its concrete usage in the area of databases. In this paper, we are mainly interested in graded tolerant inclusion between fuzzy sets and in its axiomatization.

The rest of the paper is structured as follows. In section 2, we recall some basic notions related to the (Boolean and graded) inclusion of fuzzy sets, as well as the way of defining proximity-based modifiers. In section 3, the principle of the proximity-based tolerant inclusion is introduced. In Section 4, we

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describe the graded version of tolerant inclusion and we revisit the set of axioms proposed by Sinha and Dougherty [8] for the characterization of graded inclusion of fuzzy sets. Other variants of graded tolerant inclusions are considered in section 5. Last, we briefly recall the main contribution of the paper and conclude.

2. Preliminaries and Background

2.1 Inclusion of Fuzzy Sets

2.1.1 Boolean version

If A and B denote two crisp sets built on U, the usual way for defining the inclusion of A in B is:

$$(A \subseteq B) \Leftrightarrow (\forall u \in U, u \in A \Rightarrow U \in B) \quad (1)$$

This definition can be extended in a canonical way to two fuzzy sets A and B, which leads to:

$$(A \subseteq B) \Leftrightarrow (\forall u \in U, A(u) \rightarrow_{\text{R-G}} B(u)) \quad (2)$$

where $\rightarrow_{\text{R-G}}$ stands for Rescher-Gaines implication:

$$p \rightarrow_{\text{R-G}} q = 1 \text{ if } p \le q, 0 \text{ otherwise.}$$

This view does not take into account the proximity between the elements of the universe, therefore it can happen that the result of the inclusion is false while it would be true if that notion was used.

Example 1. Let A and B be the two fuzzy sets on U :

$$A = \{1/a + 0.6/b\}, B = \{1/a + 0.4/b + 0.9/c\}.$$

According to formula (2), A is not included in B. However, if one has available the knowledge that the element b is *very close* (according to a given proximity relation defined on the universe U) to the element c (which strongly belongs to B), it may make sense to upgrade the membership degree of b to B (which corresponds to modifying B into a fuzzy set B') and then A may be included in B'. In that sense, $A \subseteq B'$ can be viewed as a relaxation of $A \subseteq B$. This is the basic idea that is developed in the following.♦

2.1.2 Graded version

An inclusion whose result values are taken from the unit interval allows to account for set inclusion in a finer way than the original inclusion. The objective of a *graded inclusion* is to extend the notion of inclusion thanks to a degree. Several ways may be envisaged to move from the regular inclusion to a graded one. In the rest of the paper, a logical approach in the spirit of formulas 2 is adopted. From formula 2, the degree of inclusion of E in F is defined as:

$$Inc(E, F) = inf_{u \in U} (E(u) \rightarrow_f F(u)) \quad (3)$$

$\rightarrow_f$ being a *fuzzy implication operator* from $[0, 1]^2$ into [0, 1]. Let us mention that this definition of the graded inclusion has first been proposed by Bandler and Kohout in [1].

There are several families of fuzzy implications. R-implications [14] are defined as follows:

$$p \rightarrow_{R\text{-}i} q = sup\, \{x \in [0, 1] \mid \mathbb{T}(x, p) \le q\}$$

where $\mathbb{T}$ is a triangular norm generalizing the usual conjunction. It is possible to rewrite these implications as:

$$p \rightarrow_{R\text{-}i} q = 1 \text{ if } p \le q, f(p, q) \text{ otherwise}$$

where $f(p, q)$ expresses a degree of satisfaction of the implication when the antecedent (p) exceeds the conclusion (q). The implications of

Gödel: $p \rightarrow_{Gö} q = 1$ if $p \le q$, q otherwise
Goguen : $p \rightarrow_{Gg} q = 1$ if $p \le q$, q/p otherwise
Lukasiewicz : $p \rightarrow_{Lu} q = 1$ if $p \le q$, $1 - p + q$ otherwise

are the three most used R-implications and are obtained respectively with the t-norms: $\mathbb{T}(x, y) = min(x, y)$, $\mathbb{T}(x, y)=xy$ and $\mathbb{T}(x, y)=max(x + y - 1, 0)$.

Example 2. Let us consider the fuzzy sets:

$$E = \{0.1/a1, 0.9/a2, 1/a3, 0.7/a4\},$$

$$F = \{1/a1, 0.6/a2, 0.8/a3, 0.7/a4\}.$$

The regular inclusion of E in F does not hold and the degree of inclusion of E in F is:

$min(1, 0.6, 0.8, 1) = 0.6$ with Gödel implication,

$min(1, 2/3, 0.8, 1) = 2/3$ with Goguen implication,

$min(1, 0.7, 0.8, 1) = 0.7$ if Lukasiewicz implication is used.♦

S-implications [14] generalize the (usual) material implication p → q = (**not** p) **or** q) by:

$$p \rightarrow_{S\text{-}i} q = \mathbb{S}(1 - p, q)$$

where $\mathbb{S}$ denotes a triangular co-norm. It can be noticed that the minimal element of this family, namely Kleene-Dienes implication obtained with $\mathcal{S}(x, y) = max(x, y)$ expresses the inclusion of the support of E in the core of F.

Remark. It must be noticed that the approach based on formula (3) only applies to fuzzy sets since in the presence of regular sets the result of any fuzzy implication is either 0, or 1.

2.1.3 Axiomatization of the graded inclusion

Several researchers aimed at axiomatizing the graded inclusion of fuzzy sets have been proposed. Sinha and Dougherty [8], in particular, defined the following set of axioms:

Let U be a universe and $F(U)$ the class of the fuzzy sets defined over U.

(SD1) $Inc(A, B) = 1 \Leftrightarrow A \subseteq B$ in Zadeh's sense

(SD2) $Inc(A, B) = 0 \Leftrightarrow \exists u \in U$ such that $A(u) = 1$ and $B(u) = 0$

(SD3) *Inc* has increasing second partial mapping: $B \subseteq C \Rightarrow Inc(A, B) \leq Inc(A, C)$

(SD4) *Inc* has decreasing first partial mapping: $B \subseteq C \Rightarrow Inc(C, A) \leq Inc(B, A)$

(SD5) $Inc(A, B) = Inc(S(A), S(B))$ where S is a $F(U) \rightarrow F(U)$ mapping defined by: $\forall u \in U$, $S(A)(u) = A(s(u))$, s denoting an $U \rightarrow U$ mapping.

(SD6) $Inc(A, B) = Inc(B^c, A^c)$ where A^c (resp. B^c) denotes the complement of A (resp. B) in the universe U

(SD7) $Inc(A \cup B, C) = min(Inc(A, C), Inc(B, C))$

(SD8) $Inc(A, B \cap C) = min(Inc(A, B), Inc(A, C))$

Independently of Sinha and Dougherty, Kitainik [13] developed an axiomatic approach to the treatment of fuzzy inclusion indicators. Kitainik's requirements are given hereafter:

(K1) $Inc(A, B) = Inc(B^C, A^C)$ (*contrapositivity*)

(K2) $Inc(A, B \cap C) = min(Inc(A, B), Inc(A, C))$ (*distributivity*)

(K3) $Inc(A, B) = Inc(S(A), S(B))$ where S is defined as in SD5 (*symmetry*)

(K4) When applied to crisp sets, *Inc* coincides with crisp set inclusion.

Kitainik proved that in the Sinha-Dougherty axiom list, the axioms 3, 4 and 7 are a direct consequence of the axioms 1, 2, 5, 6 and 8.

2.2 About Proximity-based Modifiers

Let us first recall the formal definition of the concept of a *proximity relation* [10].

Definition 1. A proximity relation is a fuzzy relation R on a scalar domain U, such that for $u, v \in U$,

$R(u, u) = 1$ (*reflexivity*),

$R(u, v) = R(v, u)$ (*symmetry*).

The quantity $R(u, v)$ can be viewed as a grade of *approximate equality* of u with v.

On a universe U which is a subset of the *real line*, an *absolute proximity* relation is an approximate equality relation E which can be modeled by a fuzzy relation of the form [10]:

$$E: U \times U \rightarrow [0, 1]$$

$$(u, v) \rightarrow E(u, v) = Z(u - v),$$

which only depends on the value of the difference $u - v$, and where Z, called a *tolerance indicator*, is a fuzzy interval centered in 0, such that:

i. $Z(r) = Z(-r)$, i.e., $Z = -Z$;

ii. $Z(0) = 1$;

iii. The support of Z, denoted $\mathcal{S}(Z)$, is bounded and is of the form $[-\Omega, \Omega]$ where Ω is a positive real number.

In terms of trapezoidal membership function (t.m.f.), the parameter Z can be expressed by $(-z, z, \delta, \delta)$ with $\Omega = z + \delta$ and $[-z, z]$ represents the core $\mathcal{C}(Z)$ of Z.

Proposition 1. Let Z_1 and Z_2 be two fuzzy intervals centered in 0 on scalar domain U. The following entailment holds:

$$Z_1 \subseteq Z_2 \Rightarrow E[Z_1] \subseteq E[Z_2].$$

The proof is straightforward. Classical (or crisp) equality is recovered for $Z = \mathbf{0}$ defined as $\mu_0(x - y) = 1$ if $x = y$ and $\mu_0(x - y) = 0$ otherwise. In the following, we will write $E[Z]$ to denote the absolute proximity relation E parameterized by Z. See [10] for other interesting properties of $E[Z]$.

Consider a fuzzy set F on the scalar domain U and an absolute proximity relation $E(Z)$, where Z is a tolerance indicator. The set F can be associated with a nested pair of fuzzy sets when using $E(Z)$ as a *tolerance relation*. Indeed,

i. we can build a fuzzy set $E^Z(F)$ close to F, such that $F \subseteq E^Z(F)$. This is the *dilation operation*. $E^Z(F)$ gathers the elements of F and those outside of F which are somewhat close to an element in F in the sense of $E[Z]$.

ii. we can build a fuzzy set $E_Z(F)$ close to F, such that $E_Z(F) \subseteq F$. This is the *erosion operation*. $E_Z(F)$ gathers the elements of F such that all of their "neighbors" – i.e. those which are somewhat close to them – are in F.

Let $(E[Z])_s$ be the set of elements that are close to s in the sense of $E[Z]$ and defined by $(E[Z])_s(r) = E[Z](s, r)$. These fuzzy sets can be constructed in the following way.

Dilation operation. Dilating the fuzzy set F by Z will provide a fuzzy set $E^Z(F)$ defined by:

$$\begin{aligned} E^Z(F)(s) &= sup_{r\in U}\ \mathbb{T}((E[Z])_s(r), F(r)) \\ &= sup_{r\in U}\ \mathbb{T}(Z(s-r), F(r)), \end{aligned} \qquad (4)$$

where $\mathbb{T}$ is a triangular norm. Formula (4) can be interpreted as the degree to which $(E[Z])_s$ and F overlap. It is easy to check that $F \subseteq E^Z(F)$ and then $E^Z(F)$ can be viewed as a weakened variant of F. Now if $\mathbb{T} = min$, $E^Z(F) = F \oplus Z$ where $\oplus$ is the addition operation extended to fuzzy sets [12].

Erosion Operation. Considering the meaning of $E_Z(F)$ given above, it seems natural to adopt the following definition:

$$E_Z(F)(s) = inf_{r\in U}\ ((E[Z])_s(r) \to_f F(r)) \qquad (5)$$

where $\to_f$ denotes a fuzzy implication. (5) can be interpreted as the degree of inclusion of $(E[Z])_s$ in F. When $\to_f$ is an R-implication induced by $\mathbb{T}$, we have shown in [4] that (5) represents the greatest solution to the equation $Z \oplus X = F$ and writes $E_Z(F) = F \ominus Z$ where $\ominus$ is the extended Minkowski subtraction (see for instance [11][12]).

Remark. Let us mention that similar operations to *dilation* and *erosion* operations have been studied in the fuzzy rough sets setting [16] and in the context of fuzzy mathematical morphologies [3][9][15]. Closer links also exist with closure and interior operators proposed in [2].

E^Z and E_Z are fuzzy modifiers satisfying the following propositions.

Proposition 2. For $F \in \mathcal{P}(U)$, we have:

$$E_Z(F) \subseteq F \subseteq E^Z(F).$$

The proof is straightforward.

Proposition 3. Let $F \in P(U)$ and $G \in P(U)$, we have

(i) $F \subseteq G \Rightarrow E^Z(F) \subseteq E^Z(G)$ (ii) $F \subseteq G \Rightarrow E_Z(F) \subseteq E_Z(G)$

(iii) $(E^Z(F))^c \subseteq E^Z(F^c)$ (iv) $E_Z(F^c) \subseteq (E_Z(F))^c$

(v) $E^Z(F\cap G) \subseteq E^Z(F) \cap E^Z(G)$

(vi) $E_Z(F\cup G) \supseteq E_Z(F) \cup E_Z(G)$.

Proof. See [4]

3. Proximity-based Tolerant Inclusion

The basic idea is to introduce a certain tolerance into the inclusion indicator by taking into account the proximity between the elements of the domain considered. This can be done by replacing $A \subseteq B$ either by:

$A \subseteq_Z^1 B \equiv E_Z(A) \subseteq E^Z(B)$ or by:

$A \subseteq_Z^2 B \equiv (E_Z(A) \subseteq B \vee A \subseteq E^Z(B))$ or by:

$A \subseteq_Z^3 B \equiv (E_Z(A) \subseteq B \wedge A \subseteq E^Z(B))$ or by:

$A \subseteq_Z^4 B \equiv E_Z(A) \subseteq B$ or by:

$A \subseteq_Z^5 B \equiv A \subseteq E^Z(B)$,

Remark. One has:

$A \subseteq_Z^3 B \Rightarrow A \subseteq_Z^2 B \Rightarrow A \subseteq_Z^1 B$.

$A \subseteq_Z^3 B \Rightarrow A \subseteq_Z^4 B$ and $A \subseteq_Z^3 B \Rightarrow A \subseteq_Z^5 B$.

In the following, we focus on indicator $\subseteq_Z^4$, and we use the notation:

$$A \subseteq_Z B \Leftrightarrow E_Z(A) \subseteq B. \qquad (6)$$

To sum up, the principle is to say that A is included (with tolerance) in B iff *very(A)* is included in B where the linguistic modifier *very* is based on the notion of proximity.

Recently in [5] we have tackled the case of Boolean tolerant inclusion when using the indicator $\subseteq_Z^3$. Hereafter, we briefly recall this kind of inclusion but using the indicator of interest $\subseteq_Z^4$.

3.1 Boolean Tolerant Inclusion

3.1.1 Case of Crisp Sets

In the case where crisp sets are dealt with, one must use a Boolean proximity relation $E[Z]$ based on a regular interval $Z = [-\Omega, \Omega]$ centered in 0. One gets: $E[Z](u, v)$ is true if $|u - v| \le \Omega$, false otherwise.

Formulas (4) and (5) rewrite:

$$E^Z(F) \equiv \{s \in U \mid \exists r \in U \text{ such that } r \in F \wedge E[Z](r, s)\} \quad (7)$$

$$E_Z(F) \equiv \{s \in F \mid \forall r \in U, E[Z](r, s) \Rightarrow r \in F\} \quad (8)$$

Example 3. Let us consider the sets

$A = \{46, 47, 48\}$ and $B = \{40, 47, 48, 60\}$,

defined on the interval $[0, 100]$ of the integers, and the interval $Z = [-1, 1]$. We get: $E_Z(A) = \{47\}$. Now, according to (1) A is not included in B but $A \subseteq_Z B$ holds.♦

3.1.2 Case of Fuzzy Sets

Here, the calculus – illustrated by the following example – is based on formulas (4) and (5).

Example 4. Let us consider the fuzzy sets:

$A = \{0.7/47, 0.9/48, 0.6/49, 1/50\}$,

$B = \{0.6/41, 0.7/48, 1/49\}$,

defined on the interval $[0, 100]$ of the integers, and the fuzzy set Z represented by the t.m.f. $(-1, 1, 2, 2)$. According to (2), A is not included in B.

Using the triangular norm minimum in (4), and thus Kleene-Dienes implication in (5), we get: $E_Z(A) = \{0.5/48, 0.5/49\}$. Then, $A \subseteq_Z B$ holds as well.♦

Let us recall the three axioms valid for Boolean inclusion: (i) $A \subseteq B \Leftrightarrow B^c \subseteq A^c$; (ii) $A \subseteq (B \cap C) \Leftrightarrow (A \subseteq B) \wedge (A \subseteq C)$ and (iii) $A \subseteq B \Leftrightarrow S(A) \subseteq S(B)$ where the set $S(A)$ is defined as $S(A)(u) = A(S(u))$ with a one-to-one mapping $s: U \to U$. It is easy to check that these axioms remain valid when the regular inclusion is replaced by a tolerant one (i.e., $A \subseteq_Z B$). The proof is similar to the one given in [5].

4. Graded Tolerant Inclusion (GTI)

The aim of this section is to investigate the graded version of the tolerant inclusion introduced in (6) (i.e., whose result is valued in the unit interval).

In spirit of formula (3), the degree of tolerant inclusion of A in B can be defined as follows:

$$\begin{aligned} Tol_Z\text{-}Inc(A, B) &= \inf_{u \in U} (A \subseteq_Z B) \\ &= \inf_{u \in U} (E_Z(A) \subseteq B) \\ &= \inf_{u \in U} (E_Z(A)(u) \to_f B(u)), \end{aligned} \quad (9)$$

where $\to_f$ denotes a fuzzy implication. This definition is a relaxation of formula (3) (corresponding to the graded inclusion in Bandler-Kohout's sense). It is easy to see that when $E[Z]$ is the classical equality (i.e., $Z = \mathbf{0}$), formula (9) boils down to formula (3) and then recovers the non-tolerant inclusion. Let us note that, as defined above, GTI writes also:

$$Tol_Z\text{-}Inc(A, B) = Inc(A_Z, B),$$

where A_Z stands for $E_Z(A)$.

In the following, we consider the complete set of Sinha-Dougherty axioms and we check whether these axioms remain valid (or must be relaxed, or do not hold anymore) when a tolerant inclusion in the sense of formula (9) is considered. Moreover, we complete this set with some new axioms describing the specificities of a tolerant inclusion.

First, we have to determine which of the SD axioms have to be modified in order to fit the notion of a tolerant inclusion. This is clearly the case of axioms SD1 and SD2. Axiom SD1 expresses that:

$Inc(A, B) = 1 \Leftrightarrow \forall u \in U, A(u) \le B(u)$.

It must be modified into

(A1) $Tol_Z\text{-}Inc(A, B) = 1 \Leftrightarrow \forall u \in U, A_Z(u) \le B(u)$.

The right part of the above equivalence means that the eroded variant of A is included in B. Let us note that the implication in the reverse sense ($\Leftarrow$) is also valid when $A \subseteq B$ holds (i.e., $A \subseteq B \Rightarrow Tol_Z\text{-}Inc(A, B) = 1$). Indeed, $Tol_Z\text{-}Inc(A, B) = Inc(A_Z, B) = 1$ since $B \supseteq A \supseteq A_Z$ and according to (SD1). Axiom (SD2) expresses

$Inc(A, B) = 0 \Leftrightarrow \exists u \in U$ such that $A(u) = 1$ and $B(u) = 0$.

It must modified into (where $\mathcal{C}(A)$ and $\mathcal{S}(A)$ denote the core and the support of A respectively)

(A2) $Tol_Z\text{-}Inc(A, B) = 0 \Leftrightarrow \mathcal{C}(A) \cap (\mathcal{S}(B))^c \neq \varnothing$, if $Z = (0, 0, \delta, \delta)$.

This form of Z allows for preserving the core of A when applying the erosion operation, i.e., $\mathcal{C}(A_Z) = \mathcal{C}(A)$.

We now have to check whether the remaining axioms are still valid for the tolerant inclusion. Concerning axiom SD3, which expresses the monotonicity (under inclusion) of the indicator with respect to the second argument, it appears that it has a counterpart in the tolerant inclusion case and we have:

(A3) $B \subseteq C \Rightarrow Tol_Z\text{-}Inc(A, B) \leq Tol_Z\text{-}Inc(A, C)$.

Proof. We have
$Tol_Z\text{-}Inc(A, B) = inf_{u \in U}\, (A_Z(u) \rightarrow_f B(u))$

$\leq inf_{u \in U}\, (A_Z(u) \rightarrow_f C(u))$ since $\forall u \in U, B(u) \leq C(u)$ due to the monotonicity of '$\rightarrow_f$' (i.e., $(a \rightarrow_f b) \leq (a \rightarrow_f b')$ if $b \leq b'$). Then, $Tol_Z\text{-}Inc(A, B) \leq Tol_Z\text{-}Inc(A, C)$ holds.

Concerning axiom SD4, which expresses the monotonicity (under inclusion) of the indicator with respect to the first argument, the question is: do we have:

(A4) $B \subseteq C \Rightarrow Tol_Z\text{-}Inc(C, A) \leq Tol_Z\text{-}Inc(B, A)$?

Due to the monotony of the implication:

$$B \subseteq C \Rightarrow inf_{u \in U}\, (B(u) \rightarrow_f A(u)) \geq inf_{u \in U}\, (C(u) \rightarrow_f A(u)),$$

and to proposition 3-(ii), we can easily check that (A4) holds.

Let us now consider the counterpart of axiom SD7 which writes in a tolerant inclusion context as follows

(A7) $Tol_Z\text{-}Inc(B \cup C, A) \leq min(Tol_Z\text{-}Inc(B, A), Tol_Z\text{-}Inc(C, A))$.

Indeed, it is easy to check that $(B \cup C)_Z \supseteq B_Z \cup C_Z$. Now, by SD4 we have $Inc((B \cup C)_Z, A) \leq Inc(B_Z \cup C_Z, A) = min(Inc(B_Z, A), Inc(C_Z, A))$ using SD7 as well. Then, (A7) is true.

Concerning axiom SD8, it still hold in a tolerant inclusion case and we have:

(A8) $\mathrm{Tol}_Z\text{-}\mathrm{Inc}(A, B \cap C) = \min(\mathrm{Tol}_Z\text{-}\mathrm{Inc}(A, B), \mathrm{Tol}_Z\text{-}\mathrm{Inc}(A, C))$.

Proof. We have

$Tol_Z\text{-}Inc(A, B \cap C) = Inc(A_Z, B \cap C) = min(Inc(A_Z, B), Inc(A_Z, C)) = min(Tol_Z\text{-}Inc(A, B), Tol_Z\text{-}Inc(A, C))$ using SD8.

Axiom SD6 which states that $\mathrm{Inc}(A, B) = \mathrm{Inc}(B^c, A^c)$ does not hold when using a graded tolerant inclusion. We have $Tol_Z\text{-}Inc(A, B) = Inc(A_Z, B)$ and $Tol_Z\text{-}Inc(B^c, A^c) = Inc((B^c)_Z, A^c) = Inc(A, ((B^c)_Z)^c)$ by SD6. Now, for the S-implication, it easy to check that $((B^c)_Z)^c = B^Z$. Let us now compare the two indexes $Inc(A_Z, B)$ and $Inc(A, B^Z)$ in the case of the following example:

Example 5. Let A = {0.2/41, 1/59}, B = {0.3/48, 0.6/59, 1/60} and Z = (1, 1, 0, 0). It is easy to see that $A_Z = \varnothing$ and B^Z = {0.3/47, 0.3/48, 0.3/49, 0.6/58, 1/59, 1/60, 1/61}. Using the Kleene-Dienes implication, we obtain $Inc(A_Z, B) = 1 \neq Inc(A, B^Z) = 0.8$. Then, $Tol_Z\text{-}Inc(A, B) = Tol_Z\text{-}Inc(B^c, A^c)$ does not hold in general.

Let us now provide some new axioms which are specific to the tolerant inclusion notion:

(N-A1) *Monotonicity w.r.t. the parameter Z*
If $Z_1 \subseteq Z_2$, $\mathrm{Tol}_{Z_1}\text{-}\mathrm{Inc}(A, B) \leq \mathrm{Tol}_{Z_2}\text{-}\mathrm{Inc}(A, B)$,

(N-A2) *Behavior w.r.t.* $\cup$
If $Z = Z_1 \cup Z_2$,
$\mathrm{Tol}_Z\text{-}\mathrm{Inc}(A, B) \geq \max(\mathrm{Tol}_{Z_1}\text{-}\mathrm{Inc}(A, B), \mathrm{Tol}_{Z_2}\text{-}\mathrm{Inc}(A, B))$

(N-A3) *Behavior w.r.t.* $\cap$
If $Z = Z_1 \cap Z_2$,
$\mathrm{Tol}_Z\text{-}\mathrm{Inc}(A, B) \leq \min(\mathrm{Tol}_{Z_1}\text{-}\mathrm{Inc}(A, B), \mathrm{Tol}_{Z_2}\text{-}\mathrm{Inc}(A, B))$

Proof. For axiom (N-A1), we have $Z_1 \subseteq Z_2$ which implies that $A_{Z_1} \supseteq A_{Z_2}$. Then, we deduce that $Inc(A_{Z_1}, B) \leq Inc(A_{Z_2}, B)$ using (SD4). Now in order to prove (N-A2), we have to check that $\forall (a, b, c) \in [0, 1]^3$, $max(a, b) \rightarrow_f c = min(a \rightarrow_f c, b \rightarrow_f c)$, which holds since if $a \geq b$, $a \rightarrow_f c \leq b \rightarrow_f c$ due to the monotonicity of $\rightarrow_f$ w.r.t. the first argument. From this equality, it is easy to check that $A_Z = A_{Z_1} \cap A_{Z_2}$. Then we use (SD4). Axiom (N-A3) can be proved in a similar way.

5. GTI based on other Indicators

5.1 Indicator $\subseteq_Z^5$

Let us now consider the semantics of GTI based on the indicator $\subseteq_Z^5$. Formula (9) writes then:

$$Tol_Z\text{-}Inc(A, B) = inf_{u \in U} (A(u) \to_f E^Z(B(u)) = Inc(A, B^Z), \quad (10)$$

where $B^Z = E^Z(B)$. Now, it is interesting to investigate the behavior of this variant of GTI w.r.t. Sinha-Dougherty axioms. Let us first look at axioms (A1) and (A2) which write now in the following way:

(A'1) $Tol_Z\text{-}Inc(A, B) = 1 \Leftrightarrow \forall u \in U, A(u) \leq B^Z(u)$.

The right part of the above equivalence means that A is included in the dilated variant of B. Let us note that the implication in the reverse sense ($\Leftarrow$) is also valid when $A \subseteq B$ holds (i.e., $A \subseteq B \Rightarrow Tol_Z\text{-}Inc(A, B) = 1$). Indeed, $Tol_Z\text{-}Inc(A, B) = Inc(A, B^Z) = 1$ since $B^Z \supseteq B \supseteq A$ and according to (SD1).

(A'2) $Tol_Z\text{-}Inc(A, B) = 0 \Leftrightarrow \mathcal{C}(A) \cap (\mathcal{S}(B))^c \neq \emptyset$, if Z is such that $\mathcal{S}(B^Z) = \mathcal{S}(B)$.

Here the form of Z required must preserve the support and affects only the core of B when applying the dilation operation[1].

Let us now come back to axioms (A3) and (A4) which are preserved:

(A'3) $B \subseteq C \Rightarrow Tol_Z\text{-}Inc(A, B) \leq Tol_Z\text{-}Inc(A, C)$

(A'4) $B \subseteq C \Rightarrow Tol_Z\text{-}Inc(C, A) \leq Tol_Z\text{-}Inc(B, A)$

Proof. Let us first prove (A'3). We have $Tol_Z\text{-}Inc(A, B) = Inc(A, B^Z)$. Now from $(B \subseteq C \Rightarrow B^Z \subseteq C^Z)$ and by SD3, we deduce that $Inc(A, B^Z) \leq Inc(A, C^Z)$. This implies that (A'3) holds. (A'4) can be proved in a similar way.
As to axiom (A7), it still hold but in its original version (i.e., SD7). Namely,

(A'7) $Tol_Z\text{-}Inc(B \cup C, A) = min(Tol_Z\text{-}Inc(B, A), Tol_Z\text{-}Inc(C, A))$.

Indeed, we have $Tol_Z\text{-}Inc(B \cup C, A) = Inc(B \cup C, A^Z) = min(Inc(B, A^Z), Inc(C, A^Z)) = min(Tol_Z\text{-}Inc(B, A), Tol_Z\text{-}Inc(C, A))$ using SD7.

Axiom (A8) is preserved in a weakened form and writes as follows:

(A'8) $Tol_Z\text{-}Inc(A, B \cap C) \leq min(Tol_Z\text{-}Inc(A, B), Tol_Z\text{-}Inc(A, C))$.

Proof. From $(B \cap C)^Z \subseteq B^Z \cap C^Z$, we have $Inc(A, (B \cap C)^Z) \leq Inc(A, B^Z \cap C^Z) = min(Inc(B_Z, A), Inc(C_Z, A))$ using SD3 then SD8. This means that (A'8) is true.

Now axiom (A'6) which consists to compare $Tol_Z\text{-}Inc(A, B))$ and $Tol_Z\text{-}Inc(B^c, A^c)$ reduces to axiom (A6) by considering the same assumptions. Indeed, we have $Tol_Z\text{-}Inc(A, B)) = Inc(A, B^Z)$ and $Tol_Z\text{-}Inc(B^c, A^c) = Inc(B^c, (A^c)^Z) = Inc(A_Z, B)$. Then, it does not hold as well.

Concerning the axioms describing the specific features of the tolerant inclusion. In this case, they are similar to (N-A1)-(N-A3). Namely:

(N-A'1) *Monotonicity w.r.t. the parameter Z*
If $Z_1 \subseteq Z_2$, $Tol_{Z_1}\text{-Inc}(A, B) \leq Tol_{Z_2}\text{-Inc}(A, B)$,

(N-A'2) *Behavior w.r.t.* $\cup$
If $Z = Z_1 \cup Z_2$,
$Tol_Z\text{-Inc}(A, B) \geq max(Tol_{Z_1}\text{-Inc}(A, B), Tol_{Z_2}\text{-Inc}(A, B))$

(N-A'3) *Behavior w.r.t.* $\cap$
If $Z = Z_1 \cap Z_2$,
$Tol_Z\text{-Inc}(A, B) \leq min(Tol_{Z_1}\text{-Inc}(A, B), Tol_{Z_2}\text{-Inc}(A, B))$

Proof. Let us just prove (N-A'2). To do this, we have to check that $\forall (a, b, c) \in [0, 1]^3$, $\mathbb{T}[max(a, b), c] = max[\mathbb{T}(a, c), \mathbb{T}(b, c)]$, which holds since if $a \geq b$, $\mathbb{T}(a, c) \geq \mathbb{T}(a, b)$ due to the non-decreasingness of $\mathbb{T}$. From this equality, it is easy to check that $A^Z = A^{Z_1} \cup A^{Z_2}$. Then we use (SD4).

5.2 Indicator $\subseteq_Z^3$

In this case, the graded tolerant inclusion between A and B writes:

$$Tol_Z\text{-}Inc(A, B) = Inc(A_Z, B) \wedge Inc(A, B^Z), \quad (11)$$

where the symbol '$\wedge$' stands for the conjunction and is interpreted by the '*min*' operator. Obviously, this variant of GTI is more demanding than the two previous ones given in (9) and (10). Now, our aim is to check which axioms are preserved (resp. missed) in this context. First, it is easy to see that the pair of axioms (A1, A'1) is preserved but in a modified form:

(A''1) $Tol_Z\text{-}Inc(A, B) = 1 \Leftrightarrow \forall u \in U, A_Z(u) \leq B(u) \wedge A(u) \leq B^Z(u)$.

[1] At first glance, one can assign to Z the following t.m.f. $(-z, z, 0, 0)$. Unfortunately, this form of Z does not ensure the preservation of the support, see [4] for more details.

It is worth noticing that to have $Tol_Z\text{-}Inc(A, B) = 1$, it suffices only that $A \subseteq B$ holds. Concerning (A3) and (A4), they are preserved in their original forms.

Now, we can easily check that (A7) (resp. (A'8)) is preserved but not (A'7) (resp. (A8)):

(A"7) $Tol_Z\text{-}Inc(B \cup C, A) \leq \min(Tol_Z\text{-}Inc(B, A), Tol_Z\text{-}Inc(C, A))$

(A"8) $Tol_Z\text{-}Inc(A, B \cap C) \leq \min(Tol_Z\text{-}Inc(A, B), Tol_Z\text{-}Inc(A, C))$.

Axiom (A2) is missed in this context since we can not find a form of Z that simultaneously preserves the core when applying the erosion operation and the support when applying the dilation operation. Obviously, axiom (A6) is also missed.

Finally, let us emphasize that the newly introduced axioms describing the specific properties of the tolerant inclusion notion, are also preserved in this context.

6. Conclusion

Various extensions of set inclusion have been proposed in the framework of fuzzy sets. In this paper, the novelty is to consider a proximity-based inclusion indicator, in order to take into account the closeness between the elements of the domain. Such an operator, based on a tolerance indicator defined over the domain considered, has been defined and its axiomatization has been provided for the graded version. One of the perspectives of this work is to illustrate its practical use, for instance, in databases field.

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