Linear Fuzzy Regression Using Trapezoidal Fuzzy Intervals

A. Bisserier, R. Boukezzoula and S. Galichet

Laboratoire d'Informatique, Systèmes, Traitement de l'Information et de la Connaissance (LISTIC)
Université de Savoie BP 80439 74941 Annecy le Vieux, Cedex France
E.mail : amory.bisserier@univ-savoie.fr

Abstract

In this paper, a revisited approach for possibilistic fuzzy regression methods is proposed. Indeed, a new modified fuzzy linear model form is introduced where the identified model output can envelope all the observed data and ensure a total inclusion property. Moreover, this model output can have any kind of spread tendency. In this framework, the identification problem is reformulated according to a new criterion that assesses the model fuzziness independently of the collected data. The proposed concepts are used in a global identification process in charge of building a piecewise model able to represent every kind of output evolution.

Keywords: fuzzy linear regression

1 Introduction

Fuzzy regression, a fuzzy type of conventional regression analysis, has been proposed to evaluate the functional relationship between input and output variables in a fuzzy environment.
According to [6], fuzzy regression techniques can be classified into two distinct areas. The first proposed by Tanaka which minimizes the total spread of the output is named possibilistic regression. In this case, the problem is viewed as finding fuzzy coefficients of a regression model according to a mathematical programming problem. The second approach developed by Diamond [5], which minimizes the total square error of the output is called the fuzzy least square method.
Since the fuzzy regression has been introduced by Tanaka and al. [12][10], several fuzzy regression approaches have been proposed. In this context, Tanaka, Hayashi and Watada [10] propose different expressions of the criterion to be optimized and different formulations of the constraints to be satisfied for possibility and necessity estimation models. Still in a linear context, Tanaka and Ishibushi [11] extends their approach for dealing with interactive fuzzy parameters. Furthermore, the complete specification of regression problems highly depends on the nature of input-output data [6]. Some works are thus devoted to crisp input - crisp output data [9] while others [8] consider fuzzy input - fuzzy output data. Most commonly, a mixed approach (crisp input - fuzzy output) is chosen [12]. That is the formalism we adopt here in a linear context with the idea of keeping a simple model, possibly invertible [2][3].
From most of these methods, three types of problems emerge:

- The assumption of symmetrical triangular fuzzy parameters is most frequently used. However, such parameters have some limitations, especially when total inclusion of the observed data in the model output must be ensured.
- The identification is made at a chosen level α considered as a degree of the fitting of the obtained model to the observed data. If this allows to simplify the problem by using interval arithmetics to express the inclusion problem, after reconstruction of the parameters, this inclusion isn't guaranteed anymore at any level α.
- The obtained models are not able to represent any tendency of the output spread. In this case, the obtained models become more imprecise than necessary in some situations.

The main objective of this paper is to revisit some theoretical works about fuzzy regression techniques [6] and to propose some slight improvements for the limitations quoted previously.
This paper is organized as follows. In section 2, the concepts of intervals and fuzzy intervals are introduced. Section 3 is devoted to the conventional fuzzy linear regression. A revisited ap-

proach of the latter is detailed in section 4. Section 5 and 6 present the identification process and its application in the identification of a piece-wise model. Applications on several examples are shown in Section 7. Finally, conclusions and perspectives are presented in Section 8.

2 Intervals and Fuzzy Intervals

- *Conventional intervals*

An interval is defined by the set of elements lying between its lower and upper limits as:

$$a = \{x|\ a^- \le x \le a^+, x \in \Re\}$$

Given an interval a, its midpoint $M(a)$ and its radius $R(a)$ are defined by:

$$M(a) = (a^- + a^+)/2 \text{ and } R(a) = (a^+ - a^-)/2 \quad (1)$$

For two intervals [4] a and b, an inclusion relation of a in b is defined as follows:

$$a \subseteq b \Leftrightarrow \begin{cases} b^- \le a^- \\ a^+ \le b^+ \end{cases} \Leftrightarrow \begin{cases} M(b) - R(b) \le M(a) - R(a) \\ M(a) + R(a) \le M(b) + R(b) \end{cases} \quad (2)$$

$$\Leftrightarrow \begin{cases} M(b) - M(a) \le R(b) - R(a) \\ M(a) - M(b) \le R(b) - R(a) \end{cases} \quad (3)$$

From equation (3) it obtains:

$$a \subseteq b \Leftrightarrow |M(b) - M(a)| \le R(b) - R(a) \quad (4)$$

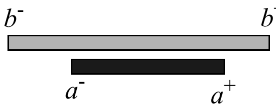

Figure 1: inclusion of two conventional intervals

when a is a scalar value, the relation (4) becomes:

$$a \in b \Leftrightarrow |M(b) - a| \le R(b) \quad (5)$$

- *Fuzzy Intervals*

An interval a can be viewed as a special fuzzy number whose membership function $\mu_a(x)$ takes the value 1 over the interval and 0 anywhere else. A fuzzy interval A is represented by its membership function μ_A. In order to specify the fuzzy interval shape, one has to consider two dimensions. The first one (horizontal dimension) is similar to that used in interval representation, that is the real line $\Re$. The second one (vertical dimension) is related to the handling of the membership degrees and thus restricted to the interval [0, 1]. In this context, two kinds of information are required for completely defining a fuzzy interval. Both pieces of information, called support and kernel intervals, are defined on the horizontal dimension, but are associated to two different levels (level 0 and level 1) on the vertical dimension (see figure 2). For a fuzzy trapezoidal interval A we have:

$$\text{Support}: S_A = [S_A^-, S_A^+],\ \text{Kernel}: K_A = [K_A^-, K_A^+], \quad (1)$$

To completely define the fuzzy interval, two additional functions are used to link the support and the kernel:

$$\begin{cases} (A^-)_\alpha = \mathit{inf}\{x|\ \mu_A(x) \ge \alpha\ ;\ x \ge S_A^-\} \\ (A^+)_\alpha = \mathit{Sup}\{x|\ \mu_A(x) \ge \alpha\ ;\ x \le S_A^+\} \end{cases} \quad (6)$$

where $\alpha \in [0, 1]$ represents the vertical dimension. In this case, for a given α-cut on the fuzzy interval A, a conventional interval is obtained:

$$[A]_\alpha = [(A^-)_\alpha, (A^+)_\alpha] \quad (7)$$

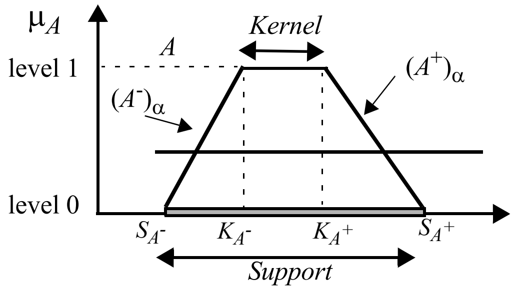

Figure 2 A fuzzy trapezoidal number

Finally, in the same way that the conventional interval a is denoted $[a^-, a^+]$, the fuzzy interval A will be defined by its support and kernel bounds

$$A = (K_A, S_A) = ([K_A^-, K_A^+], [S_A^-, S_A^+]) \quad (8)$$

A particular case of trapezoidal fuzzy intervals are triangular symmetrical ones. In this case, the fuzzy number can be defined by its kernel (modal value) K_A and the radius of its support R_A, i.e. $A = (K_A, R_A)$. In other words:

$$K_A^- = K_A^+ = K_A,\ S_A^+ = K_A + R_A,\ S_A^- = K_A - R_A \quad (9)$$

3 Fuzzy linear regression

Let us consider a set of N observed data samples defined on an interval $D = [x_{min}, x_{max}]$. Let the j^{th} sample be represented by the couple (x_j, Y_j), $j = 1,..., N$ where x_j are crisp inputs stored in an increasing order and the Y_j are the corresponding fuzzy output which is assumed to be triangular and symmetrical fuzzy interval. In this case, this fuzzy interval is completely defined by its modal value K_{Yj} and its radius R_{Yj}, that is:

$$Y_j = (K_{Y_j}, R_{Y_j}) \quad (10)$$

Like any regression technique, the fuzzy regression objective is to determine a predicted functional relationship $\hat{Y} = h(x)$ between inputs x and outputs Y. In this paper, the function h is assumed to be linear and given by the following expression:

$$\hat{Y}(x) = A_0 \oplus A_1 \cdot x \quad (11)$$

defined on the domain D.

In order to consider the fuzziness of the observed outputs, the parameters A_0 and A_1 are fuzzy coefficients. The latter are assumed to be triangular and symmetrical, represented by:

$$A_0 = (K_{A_0}, R_{A_0}), \text{ and } A_1 = (K_{A_1}, R_{A_1}) \quad (12)$$

3.1 Inclusion problem statement

A fuzzy interval is a standard normal fuzzy set defined on the set of real numbers, whose α-cuts, are closed intervals of real numbers with bounded supports. Using an α-cut representation, a fuzzy interval is viewed as a weighted family of nested intervals. By doing so, for a specified α-cut, the fuzzy interval becomes a conventional interval, which states that a fuzzy interval representation is a generalization of a conventional one. Moreover, this strategy has the advantage to reduce the fuzzy computational complexity and makes easier its implementation, especially in optimization and identification problems. That is the approach proposed by Tanaka for the identification of a fuzzy model in the form (11) where a possibilistic regression methodology according to the α-cut representation principle is adopted [10], [12].

Indeed, for a set of observed data, the author tries to identify the fuzzy model parameters A_0 and A_1 such as all the observed data are included in the predicted ones for any α-cut, i.e.,

$$[Y_j]_\alpha \subseteq [\hat{Y}_j]_\alpha \quad (13)$$

Equation (13) is viewed as a constraint in the identification procedure. The latter is based on the minimization of a criterion which exhibits the spreads of the predicted intervals, that is:

$$\min_{K_{A_0}, K_{A_1}, R_{A_0}, R_{A_1}} N \cdot R_{A_0} + R_{A_1} \cdot \Sigma_{j=1}^{N} |x_j| \quad (14)$$

After the optimization method is performed, the obtained parameters computed for a given α-cut are assumed to be defined for all $\alpha \in [0, 1]$.

Let us give a simple example used by Tanaka in [10] to illustrate this method (see Table 1). In this example, the pessimistic case (maximum of uncertainty) is adopted, i.e. α = 0. In this case, the constraints are the following ones:

$$\begin{cases} K_{A_0} + K_{A_1} \cdot x_j + R_{A_0} + R_{A_1} \cdot |x_j| \geq K_{Y_j} + R_{Y_j} \\ K_{A_0} + K_{A_1} \cdot x_j - (R_{A_0} + R_{A_1} \cdot |x_j|) \leq K_{Y_j} - R_{Y_j} \end{cases} \quad (15)$$

The identification method gives the fuzzy symmetrical triangular coefficients A_0=(3.85, 3.85), A_1 = (2.1, 0), and the predicted intervals represented in Table 1.

Table 1: Observed and predicted intervals.

j	x_j	observed intervals	predicted intervals
1	1	[6.2 , 9.8]	[2.1 , 9.8]
2	2	[4.2 , 8.6]	[4.2 , 11.9]
3	3	[6.9 , 12.1]	[6.3 , 14]
4	4	[10.9 , 16.1]	[8.4 , 16.1]
5	5	[10.6 , 15.4]	[10.5 , 18.2]

For example, when j = 1, the observed and the predicted output are respectively Y_1 = [6.2, 9.8] and $\hat{Y}_1$ = [2.1, 9.8]. It can be stated the inclusion constraint is respected for α = 0, i.e.,

$$[Y_1]_{\alpha=0} \subseteq [\hat{Y}_1]_{\alpha=0} \cdot$$

According to Figure 3, it's obvious that although the inclusion is respected for α = 0, it is not respected for any α ∈[0,1].

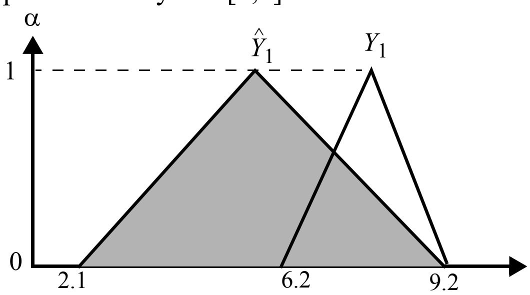

Figure 3: Observed and predicted outputs for j = 1

From a general point of view, if the fuzzy model parameters are identified for a chosen α-cut level under the constraint (13), the inclusion of all observed outputs in the predicted ones is not guaranted. Indeed, the inclusion relation between intervals at α-cut level is not sufficient to guarantee the total inclusion of the fuzzy intervals. For example, when the inclusion is ensured for α = 0, according to the kernel value positions in the fuzzy interval, three cases have to be obtained (see Figure 4 for two fuzzy intervals A and B).

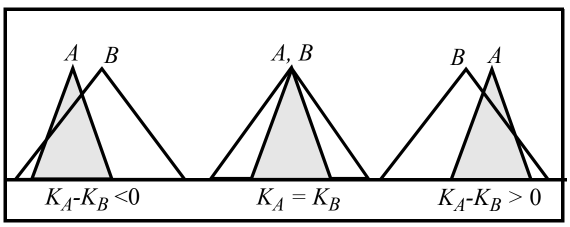

Figure 4: Three cases of support inclusion

Indeed, when the supports are included, the total inclusion of fuzzy intervals is respected if and only if the modal values are equal. In this case, the identification at $\alpha = 1$ is impossible if the observed modal values are not strictly lined up [6]. Moreover, the higher the α considered for identification is, the wider the support of the predicted fuzzy number is [8]. These drawbacks weaken the potential use of this method, especially in real identification problems.

3.2 Tendency problem statement

Let us now apply the Tanaka identification method for another example where the intervals resulting for a α-cut equal to 0 are presented in Table 2. In this case, it can be stated that the observed outputs have a spread which is globally decreasing.

The identified fuzzy parameters are $A_0 = (2.574, 4)$ and $A_1 = (2.43,0)$. A representation of the model output is given in Figure 5.

Table 2: Observed and predicted intervals.

j	x_j	observed intervals	predicted intervals
1	1	[1 , 9]	[1 , 9]
2	2	[5.4 , 10.6]	[3.43 , 11.43]
3	3	[8 , 12]	[5.85 , 13.85]
4	4	[10 , 12]	[8.28 , 16.28]
5	5	[13.5 , 14.5]	[10.71 , 18.71]

According to Table 2 and Figure 5, it can be observed that the identified model output spread is constant. Obviously, it should be better if it was decreasing, i.e. if the identified model presented the same spread variation than the observed data.

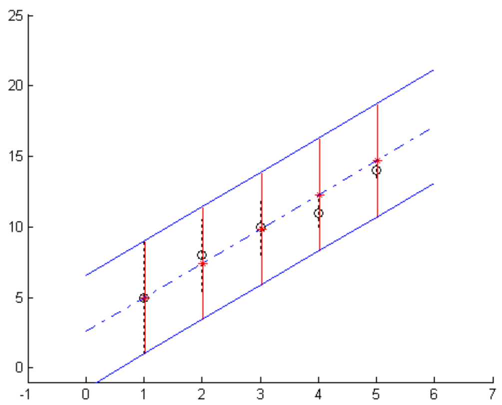

Figure 5: Representation of the identified model

More generally, one weakness of this method is the fact that the fuzziness of the model output varies in the same way than the absolute value of the inputs. In this case, it is impossible to have a decreasing (resp. increasing) spread of the model output for positive (resp. negative) inputs. This restriction is acceptable in a measurement context where it is usual to express percentage relative errors. However, when fuzziness is considered as an intrinsic characteristic of the system to be modeled, the assumption that the higher the input, the higher the fuzziness attached to the model output, is open to criticism. Finally, as classical fuzzy regression models are not able to represent any tendency of output spread, they become more imprecise than necessary in some situations. As a consequence, in piecewise fuzzy regression problems, in which collected data can have any kind of spread tendency, actual identification methods are clearly insufficient.

Let us now study the tendency output problem in order to release a suitable solution.

Form the model (11), the output modal value and spread can be determined. Indeed, as A_0 and A_1 are symmetrical triangular fuzzy intervals, and x a crisp input, $\hat{Y}(x)$ is also a symmetrical triangular one. In this case, the modal value $M(\hat{Y}(x))$ and the spread $R(\hat{Y}(x))$ are given by:

$$\begin{cases} M(\hat{Y}(x)) = K_{\hat{Y}(x)} = K_{A_0} + K_{A_1} \cdot x \\ R(\hat{Y}(x)) = R_{\hat{Y}(x)} = R_{A_0} + R_{A_1} \cdot |x| \end{cases} \quad (16)$$

As x is varying on D, the variation of (16) needs to be analyzed according to the sign of x.

From (16) it means that the variation of $M(\hat{Y}(x))$ depends on the sign of K_{A_1} and can be increasing or decreasing for any value of the input x.

According to (16), we see that the variation $R(\hat{Y}(x))$ depends on the sign of the input. As R_{A_1} is always positive, it can be stated that: if x is increasing being positive, the output radius will increase, whereas when x is negative, the output radius will decrease.

In this framework, it is possible to have any kind of variation of the output modal value, with an appropriate sign of K_{A_1}. However, the radius output variation is limited by the sign of the input x.

4 Revisited fuzzy linear regression

In order to deal with the two drawbacks discussed in the previous section (inclusion and tendency problems), two evolutionary concepts are introduced into the conventional fuzzy regression model identification problems.

4.1 Inclusion problem solution

In order to overcome the inclusion problem relating to the α-cut specification, the fuzzy model parameters A_0 and A_1 are assumed to be trapezoidal. In this case, it can be ensured that total inclusion of all observed inputs in the predicted ones at each level α is respected. As the fuzzy parameters are trapezoidal, the model output $\hat{Y}(x)$ is also a trapezoidal one.

In order to extend the Tanaka interval method and solving the inclusion problem, two inclusion constraints must be taken into account in the identification method:

$$[Y_j]_{\alpha=0} \subseteq [\hat{Y}_j]_{\alpha=0}, \text{ and : } [Y_j]_{\alpha=1} \subseteq [\hat{Y}_j]_{\alpha=1} \quad (17)$$

In this case, as a trapezoidal fuzzy interval shape is assumed, it is obvious that if (17) is respected, then the total inclusion is guaranteed for each level $\alpha \in [0, 1]$, i.e.,

$$\forall \alpha \in [0, 1], [Y_j]_\alpha \subseteq [\hat{Y}_j]_\alpha \quad (18)$$

Let us consider the j^{th} observed data, whose output is the triangular symmetrical fuzzy interval $Y_j = (K_{Y_j}, R_{Y_j})$. The corresponding predicted output is the trapezoidal fuzzy interval given by:

$$\hat{Y}_j = (K_{\hat{Y}_j}, S_{\hat{Y}_j}) = ([K^-_{\hat{Y}_j}, K^+_{\hat{Y}_j}], [S^-_{\hat{Y}_j}, S^+_{\hat{Y}_j}]) . \quad (19)$$

In this case, the constraints (17) can be written as:

- for α = 1:

$$[Y_j]_{\alpha=1} \subseteq [\hat{Y}_j]_{\alpha=1} \Leftrightarrow K_{Y_j} \in [K^-_{\hat{Y}_j}, K^+_{\hat{Y}_j}] \quad (20)$$

- for α = 0:

$$[Y_j]_{\alpha=0} \subseteq [\hat{Y}_j]_{\alpha=0} \Leftrightarrow$$
$$[K_{Y_j} - R_{Y_j}, K_{Y_j} + R_{Y_j}] \subseteq [S^-_{\hat{Y}_j}, S^+_{\hat{Y}_j}] \quad (21)$$

4.2 Tendency problem solution

As stated previously, the output model tendencies are not taken into account in the conventional method. In order to solve this problem, a modified model expression is proposed. In this case, the model output can have any kind of spread variation for any sign of x by introducing a shift on the original model input. Doing so, it is possible to obtain the desired sign for the shifted input variable, and so to influence the spread variation of the output.

In this case, the fuzzy linear model (11) defined on its domain D, becomes:

$$\hat{Y}(x) = A_0 \oplus A_1(x - shift) \quad (22)$$

where A_0 and A_1 are trapezoidal parameters.

In the model (22), the output spread is given by the support radius, i.e. $\forall x \in D$:

$$R([S_{\hat{Y}}]) = R([A_0]) + R([A_1])|x - shift| \quad (23)$$

According to (23) and by tuning the value of *shift*, the model output can have any spread variation on D. Indeed,

- if $x - shift \geq 0 \quad \forall x \in D$, i.e. $shift \leq x_{min}$, then the model output has an increasing spread on D.
- if $x - shift \leq 0 \quad \forall x \in D$, i.e. $shift \geq x_{max}$, then the model output has a decreasing spread on D.

For the sake of simplicity, the value $shift = x_{min}$ is chosen for a model whose output has an increasing radius. In the contrary, for decreasing radius output, $shift = x_{max}$ is taken (see Table 3).

Table 3 : The two models

output spread variation	↗	↘
Used model	$A_0 \oplus A_1(x - x_{min})$	$A_0 \oplus A_1(x - x_{max})$

5 The identification process

In this section, a modified identification methodology for linear regression models is proposed. Indeed, the latter exploits the concepts of inclusion and tendency discussed previously for determining the parameters of a fuzzy model in the form (22).

When considering:

- *a set of N observed data (x_j, Y_j) where x_j are crisp inputs, sorted in an increasing order, and Y_j the corresponding fuzzy triangular output*
- *a fuzzy model in the form (22), where its output is defined on the domain D,*

the identification statement lies in the answers given to the following questions:

1. *In order to ensure the inclusion of all observed data in the predicted ones for any* $\alpha \in [0, 1]$, *is it possible to identify the fuzzy trapezoidal parameters A_0 and A_1? In other words, what are the constraints to be taken in the optimization problem?*

2. *For a better representation of the observed data tendencies, is it possible to determine the parameter shift which allows the integration of any kind of spread in the model?*

So, two main steps have to be discussed: the choice of the value of *shift* and the parameters model identification

5.1 The *shift* value determination

The first step of the identification concerns the choice of the *shift* value according to the output radius tendency. The most appropriate tendency is determined from observed data, comparing the initial output radius R_{init} attached to minimal inputs with the final output radius R_{fin} attached to maximal inputs. If $R_{init} < R_{fin}$, an increasing tendency is chosen, otherwise a decreasing tendency is preferred. The corresponding *shift* value is defined as:

- *If* $R_{init} > R_{fin}$ *then* $shift = x_{max}$
- *If* $R_{init} \leq R_{fin}$ *then* $shift = x_{min}$

The R_{nit} and R_{fin} values are estimated by computing mean values from k data, that is R_{init} = mean(R_1, R_2, ..., R_k) and R_{fin} = mean(R_{N-k+1}, ..., R_{N-1}, R_N).
The next step of the identification concerns the optimization of the fuzzy coefficients A_0 and A_1.

5.2 The identification method

Like all linear regression identification methods, the proposed one is based on the minimization of a criterion under some constraints.

A. The used criterion

In the sequel, for the clarity and the simplicity of notations we take: $w_j = (x_j - shift)$. So w_j can be positive or negative, depending on the appropriate value of *shift* for the considered interval. Indeed, for positive inputs, the *shift* is chosen equal to x_{min} leads to w_{min} positive. In the opposite, when negative inputs are considered, the *shift* is taken as x_{max}, causing a w_{max} negative.
According to the model expression, the output of the fuzzy model is a trapezoidal interval given by:

$$\forall\, w \in D : \begin{cases} K_{\hat{Y}}^{-} = K_{A_0}^{-} + (M(K_{A_1}) - R(K_{A_1}) \cdot \Delta) \cdot w \\ K_{\hat{Y}}^{+} = K_{A_0}^{+} + (M(K_{A_1}) + R(K_{A_1}) \cdot \Delta) \cdot w \\ S_{\hat{Y}}^{-} = S_{A_0}^{-} + (M(S_{A_1}) - R(S_{A_1}) \cdot \Delta) \cdot w \\ S_{\hat{Y}}^{+} = S_{A_0}^{-} + (M(S_{A_1}) + R(S_{A_1}) \cdot \Delta) \cdot w \end{cases} \qquad (24)$$

where:

$$\Delta = sign(w_{min} + w_{max}) \qquad (25)$$

The choice of the criterion to be minimized is also an important issue. In conventional methods [10], the used criteria are only based on the available data, their minimization does not guarantee that the identified model has the least global fuzziness that could be achieved on the whole domain D. If the identified model is to be used on the whole domain D, it may be more judicious to prefer a model with a lower global fuzziness, i.e. a less imprecise model. Indeed, it has been shown in [1], that it is possible to minimize the whole spread of the identified model for fuzzy triangular output. In this case, the global fuzziness of the model is the area covered by its output on D, i.e. the integration of the wide of the output on D. Moreover, as the level 0 and 1 are considered, it is necessary to consider the vertical dimension. So, the global fuzziness of the model is now defined by a volume on D.
It can be stated that the output area represented by a trapezoidal fuzzy number [14] is given by the following expression:

$$area(w) = \frac{K_{\hat{Y}}^{+} + S_{\hat{Y}}^{+}}{2} - \frac{K_{\hat{Y}}^{-} + S_{\hat{Y}}^{-}}{2} \qquad (26)$$

In this case, the volume delimited by the model output on its whole domain D is given by:

$$volume = \int_{w_{min}}^{w_{max}} area(w)dw \qquad (27)$$

By substitution of equation (26) in (27) it yields:

$$\begin{aligned} volume &= (w_{max} - w_{min})(R(K_{A_0}) + R(S_{A_0})) \\ &+ \frac{1}{2}(w_{max}^2 - w_{min}^2)(R(K_{A_1}) + R(S_{A_1})) \cdot \Delta \end{aligned} \qquad (28)$$

B. Assumed constraints

In the optimization procedure, the constraints (20) and (21) must be respected.

- For $\alpha = 1$:

$$K_{Y_j} \in [K_{\hat{Y}_j}^{-}, K_{\hat{Y}_j}^{+}] \Leftrightarrow \left|M(K_{\hat{Y}_j}) - K_{Y_j}\right| \leq R(K_{\hat{Y}_j}) \qquad (29)$$

where

$$\begin{cases} M(K_{\hat{Y}_j}) = M(K_{A_0}) + M(K_{A_1}) \cdot w_j \\ R(K_{\hat{Y}_j}) = R(K_{A_0}) + R(K_{A_1}) \cdot w_j \cdot \Delta \end{cases} \qquad (30)$$

- For $\alpha = 0$:

$$\begin{aligned} &[K_{Y_j} - R_{Y_j}, K_{Y_j} + R_{Y_j}] \subseteq [S_{\hat{Y}_j}^{-}, S_{\hat{Y}_j}^{+}] \\ &\Leftrightarrow \left|M(S_{\hat{Y}_j}) - K_{Y_j}\right| \leq R(S_{\hat{Y}_j}) - R_{Y_j} \end{aligned} \qquad (31)$$

where:

$$\begin{cases} M(S_{\hat{Y}_j}) = M(S_{A_0}) + M(S_{A_1}) \cdot w_j \\ R(S_{\hat{Y}_j}) = R(S_{A_0}) + R(S_{A_1}) \cdot w_j \cdot \Delta \end{cases} \qquad (32)$$

- In order to obtain a fuzzy interval, another inclusion constraint must be verified, i.e., the inclusion of the kernel into the support:

$$\begin{aligned} &[K_{\hat{Y}_j}^{-}, K_{\hat{Y}_j}^{+}] \subseteq [S_{\hat{Y}_j}^{-}, S_{\hat{Y}_j}^{+}] \Leftrightarrow \\ &\left|M(S_{\hat{Y}_j}) - M(K_{\hat{Y}_j})\right| \leq R(S_{\hat{Y}_j}) - R(K_{\hat{Y}_j}) \end{aligned} \qquad (33)$$

To sum up, the identification method is performed by minimizing the criterion (28) under the constraints (29), (31) and (33).

6 A piecewise linear regression problem

In this section, the previous identification method is used to identify a piecewise fuzzy linear model of the form (34):

$$\hat{Y}(x) = \sum_{k=1}^{S\oplus} [A_{k0} \oplus A_{k1}(x - shift_k)] 1_{[x^k_{min}, x^k_{max}]} \quad (34)$$

where $shift_k \in \{x^k_{min}, x^k_{max}\}$, S is the number of segments which will compose the global model, $\sum^{\oplus}$ represents the sum of several fuzzy intervals. The coefficients A_{k0} and A_{k1} are fuzzy trapezoidal intervals. The function $1_{[x^k_{min}, x^k_{max}]}$ is equal to 1 on $[x^k_{min}, x^k_{max}]$ and to 0 otherwise.

At the beginning of the process, it is necessary to find a good segmentation of the data set, i.e. to determine in which intervals we have to identify the different sub-models. As for a given data set, output modal value tendency and output radius one have to be considered, the segmentation is made on both.

So, in order to finally get the different $[x^k_{min}, x^k_{max}]$, *k=1,...,S*, we apply on collected data the following method:

- first, we make a segmentation on observed outputs modal values;
- on each interval got, we make another segmentation on the corresponding observed outputs radius values.

Then, we apply the identification method presented in section 5 on each interval given by a segmentation process, in order to determine the best model on this domain for the volume criterion.

7 Application on several examples

The proposed identification method is applied for the first example presented in Table 1. The following model (35) defined on $D = [1, 5]$ is obtained:

$$\hat{Y}(x) = A_0 \oplus A_1(x - 1) \quad (35)$$

with:

$$\begin{cases} A_0 = ([4.45, 8], [2.25, 9.8]) \\ A_1 = ([1.95, 1.95], [1.95, 2.1]) \end{cases} \quad (36)$$

The model representation is illustrated in Figure 6. The optimal volume computed on $D = [1, 5]$ is 22.8.

In this case, it can be stated that all observed data and included in the predicted ones. For example,

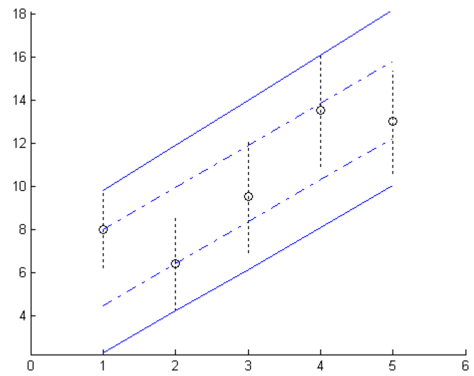

Figure 6: Trapezoidal identified model

when *j* = 1, the observed and the predicted output are respectively $Y_1 = [6.2, 9.8]$ and $\hat{Y}_1 =$ ([4.45,8], [2.25,9.8]) which proves that the inclusion of the observed output into the predicted one is ensured $\forall\alpha \in [0, 1]$ (see Figure 7).

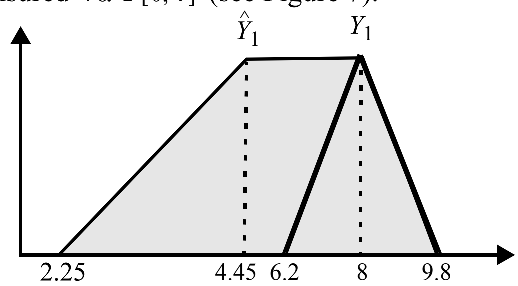

Figure 7 : Trapezoidal identified model representation (*j*=1).

The identification method is also applied on the second example, presented in Table 2 leads to the following model:

$$\hat{Y}(x) = A_0 \oplus A_1(x - 5) \quad (37)$$

with:

$$D = [1, 5] \text{ and } \begin{cases} A_0 = ([13, 14], [13, 14.6]) \\ A_1 = ([2, 2], [1.3, 3]) \end{cases} \quad (38)$$

A representation of the model is given in Figure 8.

The model output has a decreasing spread, and so it well represents the data tendency. So, the obtained model is less fuzzy than the one presented in Figure 5.

Finally, the piecewise identification process is applied on data proposed by Tanaka and Ishibuchi ([11], example 2).

Two different segments can be distinguished where the change point is $x = 11$. The observed outputs on first segment present a globally decreasing spread. On the second segment, the spread is globally increasing.

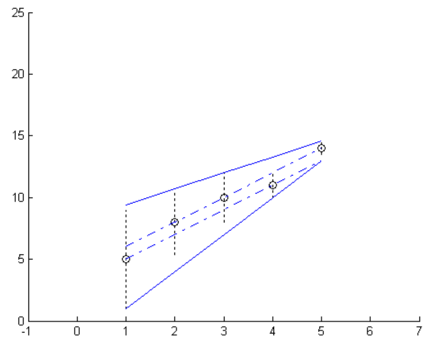

Figure 8: Trapezoidal identified model representation.

Identification method leads to the following piecewise model (39) (see Figure 9):

$$\hat{Y}(x) = (A_{01} + A_{11}(x-11))1_{[5,\,11]} + (A_{02} + A_{12}(x-11))1_{[11,\,17]} \quad (39)$$

with:

$$\begin{cases} A_{01} = ([10, 10.18], [9, 11]) \\ A_{11} = ([0.392, 0.5], [0, 1]) \\ A_{02} = ([9.84, 10], [9, 11]) \\ A_{12} = ([0.385, 0.5], [-0.83, 1.83]) \end{cases} \quad (40)$$

So, with a piecewise model and shifted inputs, a good data representation can be achieved, without using interactive coefficients as used by Tanaka and Ishibuchi in [11].

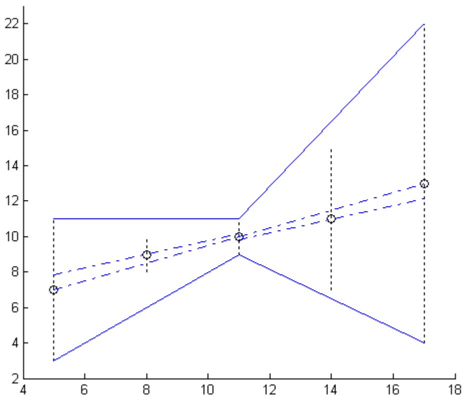

Figure 9: A representation of a piecewise model

8 Conclusion

The proposed methodology is based on the using of shifted models with trapezoidal fuzzy parameters. In this case, it becomes possible to represent output spreads either increasing or decreasing with respect to inputs. Moreover, a total inclusion of the observed data in the model output is ensured. Identifying such models leads to models whose fuzziness is possibly lower than usually.

Further works concern the extension of the proposed approach to multi-input problems and the comparison of our method with regard to other existing techniques, for example the granular clustering proposed by Pedrycz in [7]. Moreover, although the proposed method doesn't preserve the scalability, an extension can be find in order to take into account this propriety. Another point to be studied is the generalization of the identification procedure to fuzzy inputs, in order to manage uncertainties on collected data.

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