Non-adaptability measures with the branching property

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Abstract

In this paper we will deal with the measure of non-adaptability between partitions. This concept was especially defined in order to compare partitions in the context of pseudo-questionnaires, even though it can be used in any general context where the comparison of two partitions is required. In particular, we will characterize the family of non-adaptability measures which satisfy the branching property. A procedure to construct all the measures in this class will be given and some equivalent definitions presented.

Keywords: Pseudo-questionnaire, partition, non-adaptability measure, branching property.

1 Introduction

Pseudoquestionnaires are the formalization of inquiring processes, that is, the assignment of an element to a subset of an "a priori" given classification by means of a collection of indirect questions. The earlier version by Terrenoire ([17, 21]) was defined in terms of probabilities of questions and answers. However, in many situations no probability distribution is available and classifying processes have to be built up. Bertoluzza (see [1, 4]) proposed a generalization based on the fact that the answers, which are propositions, can be represented, via Stone's Theorem, by subsets of a suitable reference space. In this context questions are described by expected answers, that is, by a collection of subsets, which is a complete or an incomplete partition.

A classical example of pseudoquestionnaire can be found in the medical environment: the "a priori" given classification is the class of sets A_i, each representing the proposition "the disease is δ_i" (A_i are supposed to be pairwise disjointed), and questions can be clinical tests, radiographs, and so on. In general, they do not reach an unquestionable classification and they only give a sequence of partitions which approximate more and more to the right individuation of diseases.

As the previous example shows, it is very difficult to obtain a complete individuation process. Therefore, it is natural to ask until a partition which approximates to the "a priori" given one is obtained, and consequently, a measure of "fitting" is needed. Several proximity measures have been proposed in order to compare classifications derived from different clustering algorithms (for a review, see [8]) and this is still an ongoing search (see, for instance, [12, 20]). A solution especially adapted to our problem in the pseudo-questionnaire context was given in [2]. This solution is based on some previous papers of the authors about this topic (see, for instance, [15, 16]). In this work, such measure was axiomatically defined and named *non-adaptability measure*. It should also be noted, for practical purposes, that it is convenient to restrict this general class of measures to some particular subclasses, which allows us to build up the

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measure in a recurrent way. This restriction is very usual in Information Theory for Uncertainty Measures (see, among others, [3, 19]) and we have used a parallel scheme for non-adaptability-measures. Thus, we have considered two main methods in order to build the final classification:

- By successive aggregations: the classification is built by adding any possible result of it one by one. By means of this scheme we obtain the compositive non-adaptability measures. In [2] we have characterized left, right and totally compositive non-adaptability measures by means of t-norms and t-conorms ([9]).

- By successive divisions: Taking the population as a starting point, we split it into two parts, three parts and so on until we obtain the final classification. By means of this scheme we will obtain the branching non-adaptability measures. In this paper we have characterized the non-adaptability measures satisfying the branching property.

Thus, this communication is organized as follows: in Section 2 we will present a brief review of already-known concepts of partitions and non-adaptability measures. In Section 3 we will introduce the concept of branching non-adaptability measures and propose three possible definitions for this concept. We will also prove that all of them are equivalent and we will characterize this class of non-adaptability measures. This characterization will give us a method of construction for this kind of measures. We will finish with some concluding remarks and the presentation of some open problems.

2 Basic concepts

In this section, some definitions needed in the rest of the paper are recalled. For a more extended revision of these concepts, see [2].

2.1 Partitions

Fist of all, let us establish the space in which we are working.

Let $(\Omega, \mathcal{A})$ be a measurable space, where Ω is the set of elementary events and $\mathcal{A}$ the algebra of subsets of Ω formed by the observable events. An experience over $(\Omega, \mathcal{A})$ is defined (see, for instance, [7, 11, 19]) as a finite collection of incompatible events represented by a partition $\Pi_1 = \{A_{1i}\}_{i=1}^{n_1}$ such that $A_{1i} \in \mathcal{A}$, $i = 1, \ldots, n_1$. A partition Π_1 is said to be complete if, and only if, $Supp(\Pi_1) = \Omega$, where $Supp(\Pi_1)$ denotes the support of Π_1, that is, $Supp(\Pi_1) = \cup_{i=1}^{n_1} A_{1i}$. In other case, if $Supp(\Pi_1) \subset \Omega$, Π_1 is said to be incomplete. Both kind of partitions, complete and incomplete, made sense in the pseudo-questionnaires context, as it was explained in [2]. From now on, for simplicity, we will consider the following notation for the elements and support of a partition: the partition Π_j is formed by the elements $\{A_{ji}\}_{i=1}^{n_j}$ and its support is A_j.

The collection of possible experiences is denoted by $\mathcal{E}$. For sake of simplicity, we suppose $\{A\} \in \mathcal{E}$ for all $A \in \mathcal{A}$. Moreover, we consider on $\mathcal{E}$ a partial order relation and the two usual operations, product and union, defined as follows.

Definition 2.1 *Let $(\Omega, \mathcal{A})$ a measurable space, let $\mathcal{E}$ be the collection of finite partitions on $(\Omega, \mathcal{A})$ and let Π_1 and Π_2 be two partitions in $\mathcal{E}$.*

- Π_1 *is a* subpartition *of* Π_2 *(* $\Pi_1 \sqsubseteq \Pi_2$*) if, and only if, any element of* Π_1 *is a subset of an element of* Π_2*, that is,*

$$\Pi_1 \sqsubseteq \Pi_2 \Leftrightarrow \\ \forall A_{1i} \in \Pi_1, \exists A_{2j} \in \Pi_2 \,|\, A_{1i} \subseteq A_{2j}.$$

- *The* product *of* Π_1 *and* Π_2 *(*$\Pi_1 \wedge \Pi_2$*) is the partition formed by the intersections between each element of* Π_1 *and each element of* Π_2*, that is,*

$$\Pi_1 \wedge \Pi_2 = \{A_{1i} \cap A_{2j} \,|\, A_{1i} \in \Pi_1, A_{2j} \in \Pi_2\}.$$

- *If $Supp(\Pi_1) \cap Supp(\Pi_2) = \emptyset$, the* union *of Π_1 and Π_2 ($\Pi_1 \vee \Pi_2$) is the partition obtained jointing the elements of Π_1 and the elements of Π_2, that is,*

$$\Pi_1 \vee \Pi_2 = \{A_i \,|\, A_i \in \Pi_1 \text{ or } A_i \in \Pi_2\}.$$

Remark 2.2 *Obviously, $Supp(\Pi_1) \subseteq Supp(\Pi_2)$ if $\Pi_1 \sqsubseteq \Pi_2$. In the particular case, $Supp(\Pi_1) = Supp(\Pi_2)$ a subpartition is a refinement. Thus, the concept of subpartition generalizes the classical concept of refinement.*

On the other hand, it is also obvious that $Supp(\Pi_1 \wedge \Pi_2) = Supp(\Pi_1) \cap Supp(\Pi_2)$ and $Supp(\Pi_1 \vee \Pi_2) = Supp(\Pi_1) \cup Supp(\Pi_2)$.

As we already commented, our purpose was to compare a partition, obtained by a sequence of questions with a reference partition. Thus, we are going to work into a subset of $\mathcal{E} \times \mathcal{E}$, which is formed by the pairs of experiences with ordered supports, by the content relation. More precisely,

$$\mathcal{E}^2 = \{(\Pi_1, \Pi_2) \in \mathcal{E}\times\mathcal{E} | Supp(\Pi_1) \subseteq Supp(\Pi_2)\}.$$

Thus, from now on, we are going to work into the space $(\Omega, \mathcal{A}, \mathcal{E}^2)$, which will be called *comparison measurable space.*

2.2 Non-adaptability measures

In the set $\mathcal{E}^2$ we have considered suitable, mainly in the pseudoquestionnaire environment, the following definition of non-adaptability measure. The idea behind this concept was that for any partition Π in $\mathcal{E}$, we are interested in classifying the partitions, so that the "closer" partitions are the best adapted to Π (smaller value of the non-adaptability measure) and the "more far away" ones the worst adapted to Π (greater value of the non-adaptability measure).

Definition 2.3 [2] *Let $(\Omega, \mathcal{A}, \mathcal{E}^2)$ be a comparison measurable space. A map $\Delta : \mathcal{E}^2 \longrightarrow \mathbb{R}^+$ is a* non-adaptability measure *if, and only if, we have that*

(N1) $\Delta(\Pi_1, \Pi_1) = 0$, *for any $\Pi_1 \in \mathcal{E}$.*

(N2) $\Delta(\Pi_1, \Pi_3) \leq \Delta(\Pi_2, \Pi_3)$, *for any $(\Pi_2, \Pi_3) \in \mathcal{E}^2$ and any $\Pi_1 \in \mathcal{E}$ with $\Pi_1 \sqsubseteq \Pi_2$.*

(N3) $\Delta(\Pi_3, \Pi_1) \geq \Delta(\Pi_3, \Pi_2)$, *for any $(\Pi_3, \Pi_1) \in \mathcal{E}^2$ and any $\Pi_2 \in \mathcal{E}$ with $\Pi_1 \sqsubseteq \Pi_2$.*

These three axioms arise in a natural way and they can be easily interpreted. The first axiom is understood as follows: if the goal is reached, that is, if we have obtained the final partition, then the adaptability is maximum ($\Delta(\Pi_1, \Pi_1) = 0$). For the second one, if we do more questions, then the adaptability obtained is, at least, as good than if we stop at this moment ($\Delta(\Pi_1, \Pi_3) \leq \Delta(\Pi_2, \Pi_3)$). Moreover, at the third axiom we establish that if the final partition has more elements, then it is more difficult to reach to it ($\Delta(\Pi_3, \Pi_1) \geq \Delta(\Pi_3, \Pi_2)$).

Example 2.4 *As we proven in Example 10 of [2], for any space $(\Omega, \mathcal{A}, \mathcal{E}^2)$ and any probability measure P on $\mathcal{A}$, the map*

$$\Delta(\Pi_1, \Pi_2) = \mathop{S}_{i=1}^{n_1} \min_{j=1}^{n_2} P(A_{1i} - A_{2j})$$

is a non-adaptability measure, where S is a triangular conorm (see [9]), that is, an increasing, commutative and associative binary operation on $[0,1]$ with neutral element 0. Two of the most important triangular conorms are the Lukasiewicz t-conorm $S_L(x, y) = \min(x + y, 1)$ and the maximum operator $S_M(x, y) = \max(x, y)$. Thus, two different examples of non-adaptability measures are:

$$\Delta_1(\Pi_1, \Pi_2) = \sum_{i=1}^{n_1} \min_{j=1}^{n_2} P(A_{1i} - A_{2j}),$$

and

$$\Delta_2(\Pi_1, \Pi_2) = \max_{i=1}^{n_1} \min_{j=1}^{n_2} P(A_{1i} - A_{2j}).$$

As an immediate consequence of Definition 2.3 we had proven several properties for a non-adaptability measures, which are related in the following proposition.

Proposition 2.5 [2] *Let $(\Omega, \mathcal{A}, \mathcal{E}^2)$ be a comparison measurable space and let Δ be a non-adaptability measure on it. The following properties are satisfied:*

1. $\Delta(\Pi_1, \Pi_2) = 0, \forall\, \Pi_1, \Pi_2 \in \mathcal{E}$ *such that* $\Pi_1 \sqsubseteq \Pi_2$.

2. $\Delta(\Pi_1 \wedge \Pi_2, \Pi) \leq \min\{\Delta(\Pi_1, \Pi), \Delta(\Pi_2, \Pi)\}$, $\forall\, \Pi_1, \Pi_2, \Pi \in \mathcal{E}$ *with* $(\Pi_1, \Pi), (\Pi_2, \Pi) \in \mathcal{E}^2$.

3. $\Delta(\Pi, \Pi_1 \wedge \Pi_2) \geq \max\{\Delta(\Pi, \Pi_1), \Delta(\Pi, \Pi_2)\}$, $\forall\, \Pi_1, \Pi_2, \Pi \in \mathcal{E}$ *with* $(\Pi, \Pi_1 \wedge \Pi_2) \in \mathcal{E}^2$.

4. $\Delta(\Pi_1 \wedge \Pi_2, \Pi_i) = 0, i = 1, 2$, $\forall\, \Pi_1, \Pi_2 \in \mathcal{E}$.

5. $\max\{\Delta(\Pi_1, \Pi), \Delta(\Pi_2, \Pi)\} \leq \Delta(\Pi_1 \vee \Pi_2, \Pi)$, $\forall\, \Pi_1, \Pi_2, \Pi \in \mathcal{E}$ *with* $\mathit{Supp}(\Pi_1) \cap \mathit{Supp}(\Pi_2) = \emptyset$ *and* $(\Pi_1 \vee \Pi_2, \Pi) \in \mathcal{E}^2$.

6. $\min\{\Delta(\Pi, \Pi_1), \Delta(\Pi, \Pi_2)\} \geq \Delta(\Pi, \Pi_1 \vee \Pi_2)$, $\forall\, \Pi_1, \Pi_2, \Pi \in \mathcal{E}$ *with* $\mathit{Supp}(\Pi_1) \cap \mathit{Supp}(\Pi_2) = \emptyset$ *and* $(\Pi, \Pi_1), (\Pi, \Pi_2) \in \mathcal{E}^2$.

7. $\Delta(\Pi_i, \Pi_1 \vee \Pi_2) = 0, i = 1, 2$, $\forall\, \Pi_1, \Pi_2 \in \mathcal{E}$ *with* $\mathit{Supp}(\Pi_1) \cap \mathit{Supp}(\Pi_2) = \emptyset$.

8. $\Delta(\Pi_1, \Pi) = \Delta(\Pi_2, \Pi)$, $\forall\, \Pi_1, \Pi_2, \Pi \in \mathcal{E}$ *with* $\Pi_1 = \Pi_2 \vee \{\emptyset\}$ *and* $(\Pi_1, \Pi) \in \mathcal{E}^2$.

9. $\Delta(\Pi, \Pi_1) = \Delta(\Pi, \Pi_2)$, $\forall\, \Pi_1, \Pi_2, \Pi \in \mathcal{E}$ *with* $\Pi_1 = \Pi_2 \vee \{\emptyset\}$ *and* $(\Pi, \Pi_2) \in \mathcal{E}^2$.

We have developed a theory for "non-adaptability" rather than a theory for "adaptability" since this is the common way of quantifying the degree of similarity/difference in our context. Thus, for instance, the usual measure for comparing two probability distributions is a divergence measure (see, among many other, [10, 18]) instead of a similarity measure. In any case, a non-increasing function could transform a non-adaptability measure into an adaptability measure in those contexts in which intuition could pose a problem when dealing with non-adaptability measures.

3 Branching non-adaptability measures

As in a pseudoquestionnaire, by means of the questions, we split subsets in smaller ones, it seems reasonable to ask the non-adaptability measure to satisfy the branching property. In this case, since a new question supposes a refinement of the first argument of the non-adaptability measure, the partition to be split is precisely this one.

Definition 3.1 *A measure of non-adaptability Δ on $(\Omega, \mathcal{A}, \mathcal{E}^2)$ is said to be* type 1 branching *if there exists a map $H : \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{E} \longrightarrow \mathbb{R}^+$ such that, for any pair of partitions $(\Pi_1, \Pi_2) \in \mathcal{E}^2$ and any element A_{11} in Π_1, we have that*

$$\Delta(\Pi_1, \Pi_2) - \Delta(\{A'_{11}, A''_{11}, A_{12}, \ldots, A_{1n_1}\}, \Pi_2) = H(A'_{11}, A''_{11}, A_1, \Pi_2)$$

where $\{A'_{11}, A''_{11}\}$ is any partition of A_{11}.

Example 3.2 *The non-adaptability measure Δ_1 defined in Example 2.4 is branching, since the difference $\Delta(\Pi_1, \Pi_2) - \Delta(\{A'_{11}, A''_{11}, A_{12}, \ldots, A_{1n_1}\}, \Pi_2)$ is equal to*

$$\min_{j=1}^{n_2} P((A'_{11} \cup A''_{11}) - A_{2j}) - \min_{j=1}^{n_2} P(A'_{11} - A_{2j}) - \min_{j=1}^{n_2} P(A''_{11} - A_{2j})$$

that is, this difference only depends on A'_{11}, A''_{11} and Π_2.

However, the non-adaptability measure Δ_2, introduced at the same example, is not branching in general, as it shows the following example. Let us consider $\Omega = \{x_1, x_2, x_3, x_4\}$, $\mathcal{A} = \mathcal{P}(\Omega)$, the probability measure $P(\{x_i\}) = 0.i, \forall x_i \in \Omega$ and the partitions $\Pi_1 = \{\{x_1, x_2\}, \{x_3, x_4\}\}$, $\Pi_2 = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}\}$ and $\Pi_3 = \{\{x_1, x_2\}, \{x_3\}, \{x_4\}\}$. We have that

$$\Delta_2(\Pi_1, \Pi_2) - \Delta_2(\{\{x_1\}, \{x_2\}, \{x_3, x_4\}\}, \Pi_2) = 0.3 - 0.3 = 0,$$
$$\Delta_2(\Pi_3, \Pi_2) - \Delta_2(\Pi_2, \Pi_2) = 0.1 - 0 = 0.1.$$

If Δ_2 is branching then, by Definition 3.1, both differences are equal to

$H(\{x_1\},\{x_2\},\Omega,\Pi_2)$, but it is not possible the map H assumes at the same time the values 0 and 0.1, which is a contradiction.

From Definition 3.1, it is easy to see the meaning of the branching property. However, this definition is not very operative from a mathematical point of view. Thus, we had to reconsider a new definition of the branching property, which is easily characterizable. At the end of this section, we will prove both definitions (types 1 and 2) are the same.

Definition 3.3 *A measure of non-adaptability Δ on $(\Omega,\mathcal{A},\mathcal{E}^2)$ is said to be* type 2 branching *if, and only if, there exists a map $F:\mathcal{A}\times\mathcal{A}\times\mathcal{E}\longrightarrow\mathbb{R}^+$ such that for any pair of partitions (Π_1,Π_2) in $\mathcal{E}^2$,*

$$\Delta(\Pi_1,\Pi_2)-\Delta(\{A_{11}\cap A_{21},\ldots,A_{11}\cap A_{2n_2}, A_{12},\ldots,A_{1n_1}\},\Pi_2)=F(A_{11},A_1,\Pi_2).$$

Firstly, we are going to present the properties of the map associated to a branching non-adaptability measure. After that, we will show these properties characterize the branching non-adaptability measures.

Theorem 3.4 *Let Δ be a type 2 branching non-adaptability measure on $(\Omega,\mathcal{A},\mathcal{E}^2)$. Then there exists a map $G:\mathcal{A}\times\mathcal{A}\times\mathcal{E}\longrightarrow\mathbb{R}^+$ fulfills the following conditions, for any $A,B,C\in\mathcal{A}$ and any $\Pi_1,\Pi_2\in\mathcal{E}$:*

1. *If $\{A\}\sqsubseteq\Pi_1$ then $G(A,B,\Pi_1)=0$.*
2. *If $B\subseteq C$, then $G(A,B,\Pi_1)\leq G(A,C,\Pi_1)$.*
3. *If $A\subseteq B$, then $G(A,B\cup C,\Pi_1)+G(B-A,B\cup C,\Pi_1)\leq G(B,B\cup C,\Pi_1)$.*
4. *If $\Pi_1\sqsubseteq\Pi_2$, then $G(A,A\cup B,\Pi_1)\geq G(A,A\cup B,\Pi_2)$.*

such that

$$\Delta(\Pi_1,\Pi_2)=\sum_{i=1}^{n_1}G(A_{1i},A_1,\Pi_2).$$

We have seen that any branching non-adaptability measure can be decomposed as the sum of G. Now, we are going to prove that this decomposition characterizes the measures with the branching property.

Theorem 3.5 *Let $G:\mathcal{A}\times\mathcal{A}\times\mathcal{E}\longrightarrow\mathbb{R}^+$ be a map such that*

1. *If $\{A\}\sqsubseteq\Pi_1$ then $G(A,B,\Pi_1)=0$.*
2. *If $B\subseteq C$, then $G(A,B,\Pi_1)\leq G(A,C,\Pi_1)$.*
3. *If $A\subseteq B$, then $G(A,B\cup C,\Pi_1)+G(B-A,B\cup C,\Pi_1)\leq G(B,B\cup C,\Pi_1)$.*
4. *If $\Pi_1\sqsubseteq\Pi_2$, then $G(A,A\cup B,\Pi_1)\geq G(A,A\cup B,\Pi_2)$.*

then the map $\Delta:\mathcal{E}^2\longrightarrow\mathbb{R}^+$ defined by

$$\Delta(\Pi_1,\Pi_2)=\sum_{i=1}^{n_1}G(A_{1i},A_1,\Pi_2)$$

is a type 2 branching non-adaptability measure.

The idea behind the concept of branching is that the difference only depends on the changed elements and the reference partition. Other way to formalize this idea, already used for uncertainty measures in Information Theory (see, for instance, [19]), allows us to define a third type of the branching property.

Definition 3.6 *A measure of non-adaptability Δ on $(\Omega,\mathcal{A},\mathcal{E}^2)$ is said to be* type 3 branching *if, and only if, for any three partitions Π_1,Π_2,Π_3 in $\mathcal{E}$ with $Supp(\Pi_1)\cap Supp(\Pi_2)=\emptyset$ and $(\Pi_1\vee\Pi_2,\Pi_3)\in\mathcal{E}^2$, we have that*

$$\Delta(\Pi_1\vee\Pi_2,\Pi_3)-\Delta(\{A_1\}\vee\Pi_2,\Pi_3)$$
$$=\Delta(\Pi_1\vee\{A_2\},\Pi_3)-\Delta(\{A_1\}\vee\{A_2\},\Pi_3).$$

We have collected the idea of branching by means of three different definitions (type 1, 2 and 3). However, the following theorem proves that the these definitions are exactly the same.

Theorem 3.7 *Let Δ be a non-adaptability measure on $(\Omega,\mathcal{A},\mathcal{E}^2)$. The following statements are equivalent:*

1. Δ *has the type 1 branching property.*
2. Δ *has the type 2 branching property.*
3. Δ *has the type 3 branching property.*

Thus, from now on we will simply say branching non-adaptability measure, without any specification about the type, for any non-adaptability measure fulfilling the requirement imposed in Definition 3.1, Definition 3.3 or Definition 3.6. The explicit form of this kind of measures is presented at the following corollary.

Corollary 3.8 *If a non-adaptability measure has the branching property then it can be decomposed as*

$$\Delta(\Pi_1, \Pi_2) = \sum_{i=1}^{n_1} \Delta(\{(A_1 - A_{1i}) \cap A_{21}, \ldots, (A_1 - A_{1i}) \cap A_{2n_2}, A_{1i} \cap A_2\}, \Pi_2)$$

for any $(\Pi_1, \Pi_2) \in \mathcal{E}^2$.

4 Conclusion

Since in a pseudoquestionnaire we split subsets in smaller ones by means of questions, it seems reasonable to ask the non-adaptability measure to satisfy the branching property. In this case, since a new question supposes a refinement of the first argument of the non-adaptability measure, the partition to be split is precisely this one. Non-adaptability measures are considered an essential tool in the analysis of pseudo-questionnaires, although it could also be considered in any field where a classification of a reference partition is required. In this work we have focused our attention on a particular family of non-adaptability measures: the branching non-adaptability measures. It is the family of measures with an appropriate behavior when a new question is made in the pseudo-questionnaire. We have presented three different definitions of branching property which try to convey this idea of good behavior for new questions and finally proven that they are actually the same. Moreover, we have completely characterized the branching non-adaptability measures as the sum, in any element of the first partition, of a function of three arguments: this element, the support of the first partition and the second partition.

In the future, other possible situations could be analyzed, such as the sense and meaning of a branching property for the second argument of the non-adaptability measure. Apart from that, we would also like to be able to define the non-adaptability measure between fuzzy partitions, where we would use our previous studies about fuzzy partitions ([6, 13, 14]) and about comparison between fuzzy sets ([5, 15]).

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References

[1] C. Bertoluzza. Variazione di informazione associata a rispostenon attendibili. *Calcolo*, 3:297–313, 1974.

[2] C. Bertoluzza, V. Doldi, J. Jiménez, and S. Montes. Non-adaptability measures in the pseudo-questionnaires context. *International Journal of Approximate Reasoning*, doi: 10.1016/j.ijar.2007.09.002, 2007.

[3] C. Bertoluzza, V. Doldi, and G. Naval. Uncertainty measure on fuzzy partitions. *Fuzzy Sets and Systems*, 142:105–116, 2004.

[4] C. Bertoluzza and A. Salas. Pseudo-questionnaires in a deterministic environment. In *Proc. European Symposium on Intelligent Techniques (ESIT'99)*, Crete, Greece, 1999.

[5] I. Couso and S. Montes. An axiomatic definition of fuzzy divergence measures. *Int. J. Uncertain. Fuzziness Knowledge-Based Systems*, 16(1):1–17, 2008.

[6] S. Díaz, P. Gil, J. Jiménez, and S. Montes. Compatibility of t-norms with

the concept of ϵ-partition. *Information Sciences*, 177:2925–2944, 2007.

[7] B. Forte and N. Pintacuda. Sull'informazione associata alle esperienze incomplete. *Ann. Mat. Pura Appl.*, 80:215–234, 1968.

[8] A.D. Gordon. *Classification*. Chapman & Hall/CRC, London, 1999.

[9] E.-P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*. Kluwer Academic Publishers, Dordrecht, 2000.

[10] S. Kullback. *Information Theory and Statistics*. Wiley, New York, 1959.

[11] M. Mastrangelo and V. Mastrangelo. Ordre et informations sur les expriences non disjointes. *Annales de l'Institut Henri Poincar (A)*, 32:257–275, 1980.

[12] R. Miglio and G. Soffritti. The comparison between classification trees through proximity measures. *Computational Statistics & Data Analysis*, 45(3):577–593, 2004.

[13] S. Montes, I. Couso, and P. Gil. One-to-one correspondences between ϵ-partitions, (1-ϵ)-equivalences and ϵ-pseudometrics. *Fuzzy Sets and Systems*, 124(1):87–95, 2001.

[14] S. Montes, I. Couso, and P. Gil. Fuzzy δ-ϵ-partitions. *Information Sciences*, 152:267–285, 2003.

[15] S. Montes, I. Couso, P. Gil, and C. Bertoluzza. Divergence measure between fuzzy sets. *International Journal of Approximate Reasoning*, 30(2):91–105, 2002.

[16] S. Montes, P. Miranda, J. Jiménez, and P. Gil. A measure of the difference between two fuzzy partitions. *Tatra Mt. Math. Publ.*, 21:133–152, 2001.

[17] N. Nicoloyannis, M. Terrenoire, and D. Tounissoux. *Pertinence for a classification*, pages 177–184. Stud. Classification Data Anal. Knowledge Organ. Springer, Berlin, 1998.

[18] L. Pardo. *Statistical Inference Based on Divergence Measures*. Chapman & Hall/CRC, Boca Raton, 2006.

[19] C. Poggi and M Divari. Measures of uncertainty totally compositive with the branching property. *Stochastica*, X(3):271–281, 1986.

[20] F. Tavares, J. Rui, V. Mousseau, and B. Roy. Comparing two territory partitions in districting problems: Indices and practical issues. *Socio-Economic Planning Sciences*, doi: 10.1016/j.seps.2007.04.001, 2007.

[21] M. Terrenoire and M. Simonet. An evaluation of algol 68 for interrogation process algorithms. *International Journal of Computer & Information Sciences*, 1(1):67–73, 1972.